TECHNOLOGY, Volume 7, (2004), Issue 1, 61-69

TECHNOLOGY

INVENTORY PROCUREMENT UNDER STOCHASTIC PRICES AN OPERATIONAL POLICY

Ali GÜZEL
Kırıkkale University, Faculty of Economics and Administrative Sciences, Kırıkkale, Turkey

ABSTRACT

This paper is an extension of our previous work "Optimal Commodity Procurement Under Stochastic Prices" published in this journal. In that article, we presented a new mode of analysis to inventory management systems when the unit-purchasing cost is allowed to be a general stochastic process. We now develop an heuristic policy based on computationally efficient bounds of the optimal cost function. A case study shows that the heuristic policy derived with our model saved approximately 25% of the maximum potential savings - the difference between the total inventory cost of forward buying when all the future prices are perfectly known and that of no forward buying. Our results are intuitive and provide managerial insights for purchasing in a stochastic environment by showing the savings achievable when the nature of the unit cost is explicitly considered in procurement decisions.

Key Words: Forward buying; Case study; Stochastic prices; Inventory management; Procurement; Supply chain prices

RASTSAL FİYAT ORTAMINDA ENVANTER ALIMI: OPERASYONEL POLİTİKA

ÖZET

Bu makale daha önce aynı dergide basılmış olan "Optimal Commodity Procurement under Stochastic Prices" adlı çalışmanın bir uzantısıdır. O çalışmamızda, birim sipariş maliyetleri rastsal olan envanter sistemlerinin analizine yeni bir analitik yaklaşım sunmuştuk. Bu çalışmada ise, optimal maliyet fonksiyonunun hesaplaması kolay alt ve üst limitlerini kullanan bir yaklaşık politika geliştirilmiştir. Örnek çalışmamızda, önceki modelimizden türetilen bu yaklaşık politika, maksimum potansiyel tasarrufun – gelecekteki bütün fiyatların bilindiği varsayıldığında, ileriye dönük satınalma politikasının uygulanması ile böyle bir politikanın uygulanmaması durumunda toplam envanter maliyetler arasındaki fark - yaklaşık 25% ini tasarruf etmiştir. Bulduğumuz sonuçlar insan sağduyusuna hitap etmekte ve birim maliyetlerinin doğası açık bir şekilde satınalma kararlarına dahil edildiği zaman elde edilebilecek tasarrufları göstererek yöneticilerin rastsal ortamlarda satınalma fonksiyonunu daha iyi anlamalarına yardımcı olmaktadır.

Anahtar Kelimeler: İleriye dönük satınalma, Örnek çalışma, Rastsal fiyatlar, Envanter yönetimi, Satınalma, ikame zinciri fiyatları

1. INTRODUCTION

In our previous work "Optimal Commodity Procurement Under Stochastic Prices," we studied a forward buying decision that can be classified as an inventory procurement problem under stochastic prices. It was motivated by an application involving the procurement of raw materials by a firm. The firm needed a systematic means of trading off critical cost and risk factors, such as the dynamics of future unit prices, inventory costs, and projections of upcoming production requirements. The work report here is a development of an operational policy incorporating stochastic prices into periodic review inventory replenishment. We finally demonstrate our findings on a case study.

2. PROCUREMENT MODEL AND ITS BOUNDS

In this section, we review the procurement model and some definitions of functions. We consider a single item, infinite horizon, stochastic price and deterministic demand commodity procurement problem. The ordering price per unit follows a nonnegative discrete time stochastic process. Constant lead-time, constant discount rate, and no fixed ordering cost are assumed. No backlogging is allowed. The procurement decision is made at discrete points in time and the procurement cost is incurred when an order is placed. The sequence of events for inventory control at each period is as described in our earlier work. For ease of reading, the notation used is reproduced below:

t = Period index

h = Holding cost per unit per period

IP_t = Inventory position in period t

L = Lead time

Q_t = Quantity ordered in period t

α = Discount rate

z_t = Realized price in period t

$z_{t,t+i}$ = Forecast made in period t for period t+i

u_t = Forecast error in period t based on forecast made in period t-1

D_t = Demand in period t

H_n = Cumulative discounted carrying costs, defined as:

$$H_n = \alpha^L h + \alpha^{L+1} h + \cdots + \alpha^{L+n-1} h = \alpha^L h \frac{1-\alpha^n}{1-\alpha}$$

$E_t[\cdot]$ = Expectation conditional on information known at time t

The procurement decision can be visualized as comparing the cost of buying now versus a nested expectation of minimums of discounted future costs. For example, the optimal decision is to buy in period t for period t+L+3 if

$$z_t + H_3 < \alpha E_t[\min(z_{t+1} + H_2, \alpha E_{t+1}[\min(z_{t+2} + H_1, \alpha E_{t+2}[z_{t+3}])])],$$

otherwise, wait.

In general,

$$R_{t,n} = \alpha E_t[\min(z_{t+1} + H_{n-1}, R_{t+1,n-1})],\ t \ge 1, n \ge 1.$$

Then, the optimal decision is to buy in period t for period $t + L + n$ if the expected savings

$$S_{t,n} = R_{t,n} - z_t - H_n$$

is positive. A lower bound that gives minimum quantity to buy is

$$L_{t,n} = \alpha E_t\left[\min_{1 \le i \le n} \alpha^{i-1}(z_{t+i} + H_{n-i})\right], \quad t \ge 1, n \ge 1.$$

The minimum expected savings function is given as:

$$S_{t,n}^{Min} = L_{t,n} - z_t - H_n.$$

Then, $S_{t,n}^{Min}$ provides a lower bound for $S_{t,n}$. An upper bound for $S_{t,n}$ is given by

$$U_{t,n} = \alpha \min_{1 \le i \le n} \left\{\alpha^{i-1}(E_t[z_{t+i}] + H_{n-i})\right\} \quad n \ge 1.$$

And finally the maximum expected savings function, $S_{t,n}^{Max}$ is given as

$$S_{t,n}^{Max} = U_{t,n} - z_t - H_n.$$

As the important definitions in our previous work have been reviewed, let us derive our new policy, namely operational policy.

3. OPERATIONAL ORDERING POLICY

$S_{t,n}^{Min}$ and $S_{t,n}^{Max}$ are used to compute the maximum and minimum number of periods to forward buy, k_t^{Min} and k_t^{Max}. In the case study, it was found that k_t^{Max} could be more than three times as large as k_t^{Min}. To be operational, we have to choose a value for k_t from the set $\{k_t^{Min}, k_t^{Min}+1, ..., k_t^{Max}\}$. For $0 \le \beta \le 1$, let us define the β-expected cost function,

$$C_{t,n}(\beta) = \alpha E_t\left[\min_{1\le i\le n} \alpha^{i-1}(\beta z_{t+i} + (1-\beta)E_t[z_{t+i}] + H_{n-i})\right] \quad \text{for } \forall\ t \ge 1, n \ge 1.$$

$C_{t,n}(.)$ has both upper bound and lower bound characteristics. Proposition 1 shows that the $C_{t,n}(\beta)$ non-increasing in β; for which the following lemma and its corollary is used.

Lemma 1: For any random variables $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$,

$$\min\{X_1, X_2, \cdots, X_n\} + \min\{Y_1, Y_2, \cdots, Y_n\} \le \min_{\substack{1\le i\le n \\ 1\le j\le n}}\{X_i + Y_j\}.$$

Proof: It is easy to see that the left hand side is less than or equal to the sum of any given pair $X_i + Y_j$, $\forall i, j \in \{1, 2, \cdots, n\}$ and hence, less than or equal to the minimum of all such pairs. ■

The following corollary follows from Lemma 1.

Corollary 1: For any random variables $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$,

$$\min\{X_1, X_2, \cdots, X_n\} + \min\{Y_1, Y_2, \cdots, Y_n\} \le \min\{X_1 + Y_1, X_2 + Y_2, \cdots, X_n + Y_n\}.$$

Proposition 1: $C_{t,n}(\beta)$ monotonically decreases in $0 \le \beta \le 1$ for $\forall\ t \ge 1, n \ge 1$.

Proof: Without loss of generality, assume $0 \le \beta_1 \le \beta_2 \le 1$. Then

$$C_{t,n}(\beta_2) - C_{t,n}(\beta_1) = \alpha E_t\left[\min_{1\le i\le n} \alpha^{i-1}(\beta_2 z_{t+i} + (1-\beta_2)E_t[z_{t+i}] + H_{n-i})\right]$$

$$-\alpha E_t\left[\min_{1\le i\le n} \alpha^{i-1}(\beta_1 z_{t+i} + (1-\beta_1)E_t[z_{t+i}] + H_{n-i})\right]$$

$$= \alpha E_t\left[\min_{1\le i\le n} \alpha^{i-1}(\beta_2 z_{t+i} + (1-\beta_2)E_t[z_{t+i}] + H_{n-i})\right.$$

$$\left. + \min_{1\le i\le n} -\alpha^{i-1}(\beta_1 z_{t+i} + (1-\beta_1)E_t[z_{t+i}] + H_{n-i})\right]$$

$$\le \alpha E_t\left[\min_{1\le i\le n} \alpha^{i-1}\{\beta_2 z_{t+i} + (1-\beta_2)E_t[z_{t+i}] + H_{n-i} - (\beta_1 z_{t+i} + (1-\beta_1)E_t[z_{t+i}] + H_{n-i})\}\right] \qquad \text{(by Corollary 1)}$$

$$= \alpha E_t\left[\min_{1\le i\le n} \alpha^{i-1}\{(\beta_2 - \beta_1)z_{t+i} + (\beta_1 - \beta_2)E_t[z_{t+i}]\}\right]$$

$$= \alpha(\beta_2 - \beta_1)E_t\left[\min_{1\le i\le n} \alpha^{i-1}\{z_{t+i} - E_t[z_{t+i}]\}\right]$$ since the expectation of a minimum is less than or equal to the minimum of the expectations,

$$\le \alpha(\beta_2 - \beta_1)\min_{1\le i\le n} \alpha^{i-1}E_t[z_{t+i} - E_t[z_{t+i}]] = 0$$

Since $\beta_1 \le \beta_2$ are arbitrarily chosen on [0,1], the result follows ■

Next proposition establishes that $C_{t,n}(\beta)$ is bounded above by $U_{t,n}$ and below by $L_{t,n}$.

Proposition 2: $L_{t,n} \le C_{t,n}(\beta) \le U_{t,n}$ for $0 \le \beta \le 1$ and $\forall\ t \ge 1, n \ge 1$.

Proof: Notice that for $\beta = 0$ and $\forall\ t \ge 1, n \ge 1$,

$$C_{t,n}(0) = \alpha E_t\left[\min_{1\le i\le n} \alpha^{i-1}\left(E_t[z_{t+i}] + H_{n-i}\right)\right] \text{ since all the terms are constants,}$$

$$= \alpha \min_{1\le i\le n} \alpha^{i-1}\left(E_t[z_{t+i}] + H_{n-i}\right) \equiv U_{t,n}.$$

Also, for $\beta = 1$ and $\forall\ t \ge 1, n \ge 1$,

$$C_{t,n}(1) = \alpha E_t\left[\min_{1\le i\le n} \alpha^{i-1}(z_{t+i} + H_{n-i})\right] \equiv L_{t,n}.$$

Then, the result follows by using Proposition 1 ■

As in previous sections, let us define the β-expected savings function, $S_{t,n}(\beta)$, as follows:

$S_{t,n}(\beta) = C_{t,n}(\beta) - z_t - H_n$, $\forall\ t \ge 1, n \ge 1$.

$S_{t,n}(\beta)$ is non-increasing in n just as $S_{t,n}$, $S_{t,n}^{Min}$, and $S_{t,n}^{Max}$.

Proposition 3: $S_{t,n}(\beta) \ge S_{t,n+1}(\beta)$ for $0 \le \beta \le 1$ and $\forall\ t \ge 1, n \ge 1$.

Proof: Follows the same line of reasoning as in Proposition 3 in Güzel (2004) if z_{t+i} is replaced by $\beta z_{t+i} + (1-\beta)E_t[z_{t+i}]$ ■

Next proposition shows $S_{t,n}(\beta)$ also takes the same limiting value as $S_{t,n}$, $S_{t,n}^{Min}$, and $S_{t,n}^{Max}$.

Proposition 4: Suppose $\lim_{n\to\infty} \alpha^n E_t[z_{t+n}] = 0\ \forall t$ for $0 \le \alpha < 1$. Then

$\lim_{n\to\infty} S_{t,n}(\beta) = -(z_t + H_\infty)\ \forall t$.

Proof: Immediately follows from Proposition 7 in Güzel (2004) since prices are nonnegative and $0 \le C_{t,n}(\beta) \le U_{t,n}\ \forall n, \forall t$ ■

Now, we can define the operational policy to follow in deciding how much to procure each period, if any. Specifically, if $S_{t,1}(\beta) \le 0$, then the operational number of periods to forward buy, $k_t(\beta)$, is zero. Otherwise, $k_t(\beta)$ is the unique integer such that $S_{t,k_t(\beta)}(\beta) > 0$ but $S_{t,k_t(\beta)+1}(\beta) \le 0$. Furthermore, the operational order quantity,

$$Q_t(\beta) = \max\left\{\sum_{i=t}^{t+L+k_t(\beta)} D_i - IP_t, 0\right\}$$

where IP_t is on-hand plus on-order inventory in period t.

Propositions 3 and 4 guarantee that $k_t(\beta)$, if exists, is unique. If the assumption of Proposition 4 does not hold, then it is possible that $k_t(\beta) = \infty$. The next proposition shows that the order quantity based on $C_{t,n}(\beta)$, lies within the interval that bounds the optimal order quantity.

Proposition 5: $Q_t^{Min} \le Q_t(\beta) \le Q_t^{Max}$ for $\forall t \ge 1$, $0 \le \beta \le 1$.

Proof: $0 \le S_{t,k_t^{Min}}^{Min} \le S_{t,k_t^{Min}}(\beta) \Rightarrow k_t^{Min} \le k_t(\beta)$, and

$$S_{t,k_t^{Max}+1}(\beta) \le S^{Max}_{t,k_t^{Max}+1} \le 0 \;\Rightarrow k_t^{Max} \ge k_t(\beta) \qquad \blacksquare$$

In order to identify the specific procurement quantity, it remains to find $\beta^* \in [0,1]$ such that $C_{t,n}(\beta^*)$ best approximates the $R_{t,n}$. We estimate β^* by computing the total inventory costs for a number of β values using sample paths generated from the price model and then fitting a quadratic polynomial function to the total costs. The first order condition for a minimum of the polynomial determines an estimate for β^*. We chose the sample path and response surface methodology since the range of β is small and the development time is shorter. The next section presents this approach and evaluates the above policies.

4. CASE STUDY

We have tested the operational as well as the upper and lower bound policies using a specific commodity, kiln-dried 2x8-16 foot framing lumber, with a recent 5 year history of weekly prices. An ARIMA(2,1,{2,11}) model that is two autoregressive terms with lags 1 and 2, two moving average terms with lags 2 and 11, and a first differencing, was selected as the best fit for this data set. Using the price models of each year (1996, 1997, …, 2001), 20 historical samples of length 50 are generated. For each historical sample, the inventory ordering decisions are simulated along the sample by observing prices one period at a time. $C_{t,n}(\beta)$ is estimated using the 20 observations and is then used to determine $k_t(\beta)$ at each period t of the historical sample. Note that different price models are allowed for different years and the price model at the end of each year has been fit using prices only up to the end of that year.

The physical holding cost, which excludes opportunity cost of capital, is assumed to be one dollar per unit per week for this illustration. The cost of capital invested in inventory is 15%. For brevity, a zero lead-time is assumed, but the extension to include a constant non-zero lead-time is straightforward.

In order to explore the quality of the solution procedure, we compare the total cost resulting from the procurement policy for this bench item as generated by the solution methodology with two benchmark policies. Specifically, we compare our costs with these resulting from procurements made under <u>perfect information</u> (i.e., prices are known at the beginning of the simulation period), by following a <u>myopic buying policy</u> (i.e., purchase each period; one period's worth of demand). In each case, the cost per unit per period is computed by taking the average of each of these costs calculated over all historical samples for all years. These two benchmark costs are used to represent the extremes – perfect information would lead to a minimum cost procurement schedule, whereas a myopic policy provides a "you should be able to do at least as good as this" maximum cost. The "Percentage Saved" by using the optimal policy is computed as:

$$\frac{\text{Myopic Cost per unit} - \text{Model Cost per unit}}{\text{Myopic Cost per unit} - \text{Perfect Cost per unit}} * 100 .$$

Figure 1 plots the model cost as a function of β and fitted quadratic polynomial function where goodness of fit is summarized by the R^2 value. Because of the frequent use of quadratic response functions for relatively flat output variables in the simulation literature, we do not consider other types of response functions. Notice that Y-axis scale exaggerates the differences of cost function across beta values. On an actual scale that includes zero, this function would look very flat. We evaluated the operational policy for ten different β values. As shown in this figure, the measure of the goodness of fit is high, which means the model costs of generated "historical" sample paths for all years is well approximated by this polynomial function. By the first order condition on the response function, we estimate the minimum of the model cost per unit to be at $\beta^* = 0.6098$.

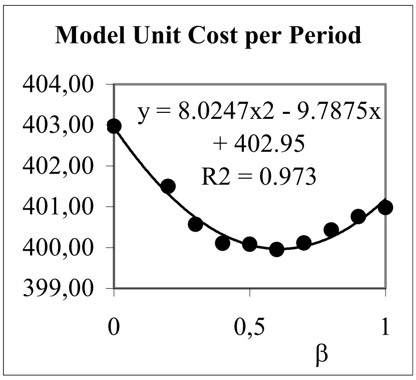

Figure 1. Quadratic Fit

4.1. POLICY EVALUATIONS

Using the β^*, we now simulate the inventory system using the actual prices to compare/evaluate the policies. We use the price model fitted from the prices of the previous years to simulate the inventory procurement decisions as new prices are observed. We update the price model at the end of each year. Note that if the price model is actually updated at each period as new prices are observed, the model may perform better and produce larger savings. However, this may result in model instability. A good price model is expected to be stable for some

period of time and will require fewer updates. At the end of this section, we discuss some experimental results with fewer price model updates. However, since our inventory procurement model is price model independent, the update frequency is left to the user.

Figure 2 shows actual prices plotted on the left Y-axis while inventory position, corresponding to

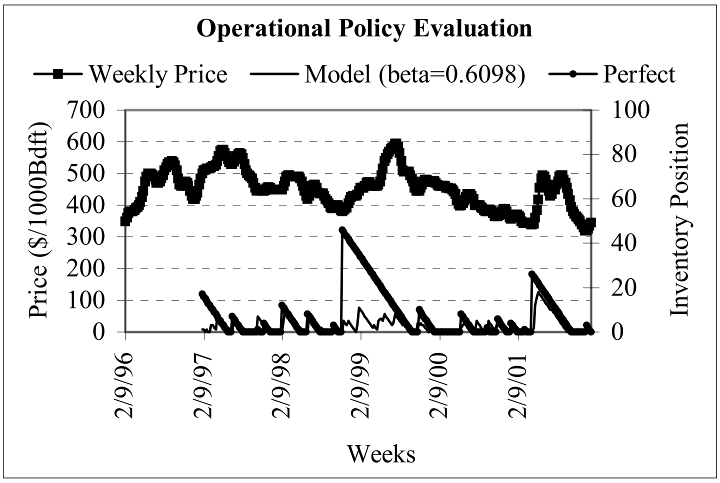

Figure 2. Evaluation of operational policy

$\beta = 0.6098$, for years 1997 – 2001 is plotted on the right Y-axis. Table 1 summarizes the average unit cost on a yearly basis, also the overall average for upper/lower bound, and operational policies (□=0, 1, 0.6098 respectively) as well as myopic and perfect information case.

Table 1. Policy Evaluations, $/1000Bdft-Week

Years		1997	1998	1999	2000	2001	Average
Perfect		463.90	376.61	441.27	374.39	331.56	389.15
Myopic		470.57	408.48	460.54	379.93	371.97	418.48
Model	β=0	475.79	407.16	462.03	372.26	364.44	416.30
	β=1	468.87	403.66	456.46	378.94	353.27	412.16
	β=0.6098	469.39	402.50	459.91	379.84	346.11	411.27

The operational policy saved
Myopic – Operational Policy = $418.48 – $411.27 = $7.21 per 1000 BdFt/week out of maximum potential savings of
Myopic – Perfect = $418.48 – $389.15 = $29.33.

Hence, use of the model resulted in purchases that achieved a 24.58% of the maximum potential savings. Typically, a medium sized manufactured homes firm uses approximately 135 thousand Bdft of 2X8-16 foot per week. This means that the operational policy saved 135*7.21*52 = $50,614 per year over the myopic policy on the average during the five year period.

Notably, for this particular price data set, the operational policy saved $5.03 per 1000 Bdft/week ($416.30 – $411.27) more than the upper bound policy and $0.89 ($412.16 – $411.27) more than the lower bound policy. These results are in agreement with Figure 2 where the upper bound policy performed the worst and the operational policy slightly better than the lower bound policy. We should also mention that a better price model, if there were any, would possibly result in more savings.

Figure 3 shows the end of period inventory positions for all the policies under consideration. The upper bound policy consistently buys more than the others and the operational policy buys somewhere in between the upper and lower bound policies.

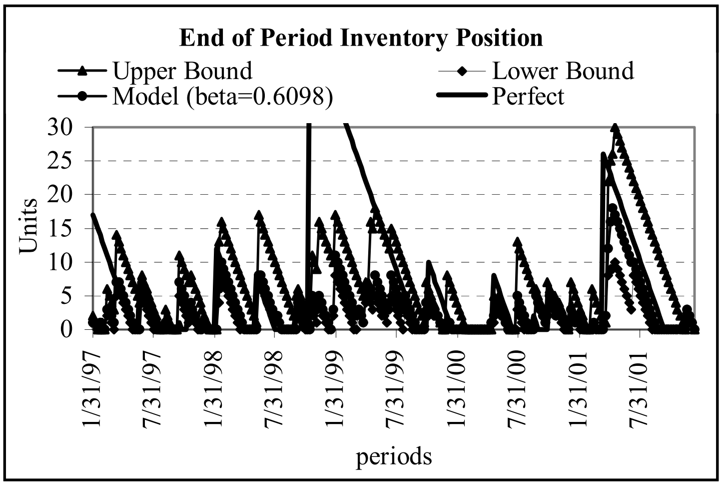

Figure 3. End of period inventory positions under the policies

As mentioned at the beginning of this section, we also evaluated the policies with fewer or no price model updates. We applied the initial, first year price model (i.e., for 1996) ARIMA(2,1,{2}), to the rest of the years without refitting. All of the polices performed relatively poorly compared to this base case as discussed above where the price model is refitted every year. Then, we kept the first year's model the same and applied ARIMA(2,1,{2,11}) to the rest of the years while fitting only once at the end of the second year. This resulted in slightly better results for all the policies compared to the base case. This result is reasonable since the price model parameters are stable after the second year as shown in Figure 4. We conclude that, for this particular price history and its model, refitting the model after the second year may not be needed. Finally, the performance of the policies using a covariance stationary price model, ARMA(1,4), is investigated while refitting this model at the end of each year. All of the policies performed better compared to the results for the nonstationary ARIMA(2,1,{2,11}) price model considered above. This result supports the mean-reverting behavior of prices reported in commodity literature. Although the classical time series methodology favored the nonstationary model, the covariance stationary price model resulted in better performance in terms of the expected costs of the policies. A more involved price model development scheme that includes expected policy cost as an additional model selection criterion may be desirable. We leave this extension to future research.

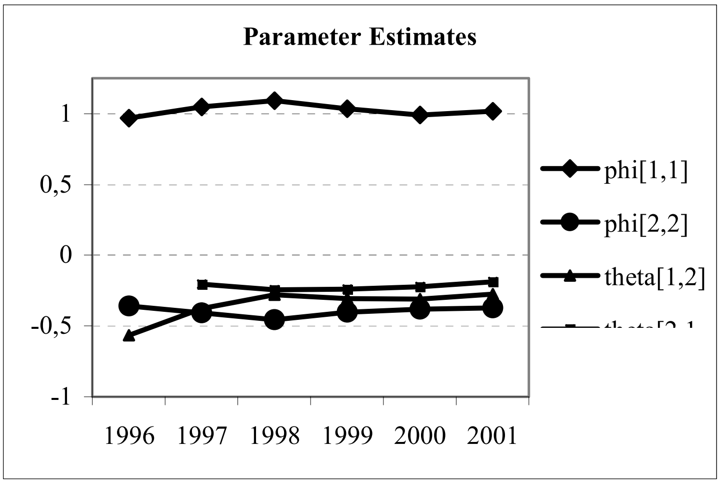

Figure 4. ARIMA(2,1,{2,11}) parameter estimates at the end of each year

Given β^*, one makes the procurement decision by utilizing the last updated price model and the most recent data and disturbances from model fitting to obtain the operational number of periods to forward buy, $k_t(\beta)$.

As a final note on this section, we simply state that $S_{t,n}^{Max}$ is insensitive to underlying noise level (i.e., the σ^2, the variance of the residuals from the price model fit) since $S_{t,n}^{Max}$ is based on a minimum of price forecasts and these forecasts are obtained by setting all future disturbances to their mean values. On the other hand, $S_{t,n}^{Min}$ is very sensitive to the underlying noise level of the price model since it is based on the expectation at time t of the minimum of random future prices. As σ^2 approaches zero, $S_{t,n}^{Min}$ approaches to $S_{t,n}^{Max}$ resulting in tighter bounds. As σ^2 gets larger, $S_{t,n}^{Min}$ moves away from $S_{t,n}^{Max}$, hence the bounds become loose. Also, as $\sigma^2 \to \infty$, $S_{t,n}^{Min} \to -\infty \;\forall n > 1$. In this case, the lower bound policy effectively becomes a myopic policy and results in forward buying of one period at the most. Since, the underlying noise of the price model is data driven and not a control variable, we do not attempt to prove these observations formally.

5. EXTENSION TO SUPPLY CHAIN CONTRACTS

Our model can also be used in a supply chains context. Manufacturing firms who buy commodities from suppliers can use an extension of our model in negotiating commodity contracts. In these contracts, parties mainly negotiate prices of commodities such as lumber, steel, core metals, etc. so that the manufacturer has a reliable source for raw materials at a fixed price over the duration of the contract. Let U(.) be the manufacturer's utility function and

$\tilde{C} =$ total cost (a random variable) incurred by the manufacturer over the contract duration, T, if the supplier's offer is not accepted.

Then, the manufacturer chooses the contract price, $\bar{p}$, if

$$E\left[U(\tilde{C})\right] > U\left(\bar{p}\sum_{t=1}^{T} D_t\right)$$

where $\sum_{t=1}^{T} D_t$ is the total requirements over T periods. Since utility functions are monotonic, we can solve this inequality to obtain the break-even contract price,

$$\bar{p} = \frac{U^{-1}\left(E\left[U(\tilde{C})\right]\right)}{\sum_{t=1}^{T} D_t}.$$

After some slight modifications, our model can be used to compute the only unknown quantity, the expected utility of not accepting the supplier's offer,

$$E\left[U(\tilde{C})\right] = \frac{\sum_{i=1}^{N} U(C_i)}{N}$$

where N is the number of sample paths generated and C_i is the total cost on the sample path i.

6. CONCLUSION

This paper has investigated the forward buying problem in a stochastic price environment. Although some simplifying assumptions are made concerning the inventory procurement problem, we have allowed prices to be any general stochastic process. We have developed an operational policy for such an inventory system. We have applied these policies on the actual prices in a case study and showed that the operational policy outperformed the upper bound and lower bound policies while all of the policies outperformed the myopic policy.

REFERENCE

1. Fabian, T., Fisher, J. L., Sasieni, M. W., Yardeni, A., "**Purchasing Raw Material on a Fluctuating Market**," Oper. Res., 7, (1959), 107-122.
2. Golabi, K., "**Optimal Inventory Policies when Ordering Prices are Random**," Oper. Res., 33, (1985), 575-588.
3. Güzel, A., "**Optimal Commodity Procurement Under Stochastic Prices,**" **Technology**, Vol 7, (2004), Issue 1, 29-39.
4. Kalymon, B. A., "**Stochastic Prices in a Single Item Purchasing Model**," Oper. Res., 19, (1971), 1434-1458.
5. Kingsman, B. G., "**Commodity Purchasing**," Oper. Res. Quarterly, 20, (1969), 59-65.
6. Scarf, H. "**The optimality of (s,S) policies in the dynamic inventory problem**," in K. Arrow, S. Karlin, & P. Suppes (Eds.), Mathematical Methods in the Social Sciences (pp. 196–202). Stanford: Stanford University Press, (1960).
7. Shastry, K. A., "**Inventory Policies under Stochastic Prices**," Ph.D. Dissertation. Tuscaloosa: The University of Alabama at Tuscaloosa, (1993).