DOI: 10.2478/auom-2020-0017
An. Şt. Univ. Ovidius Constanţa **Vol. 28(2),2020, 15–34**

On Constraint Manifolds of Lorentz Sphere

Buşra Aktaş, Olgun Durmaz and Hal't Gündoğan

Abstract

The expression of the structure equation of a mechanism is significant to present the last position of the mechanism. Moreover, in order to attain the constraint manifold of a chain, we need to constitute the structure equation. In this paper, we determine the structure equations and the constraint manifolds of a spherical open-chain in the Lorentz space. The structure equations of spherical open chain with reference to the causal character of the first link are obtained. Later, the constraint manifolds of the mechanism are determined by means of these equations. The geometric constructions corresponding to these manifolds are studied.

1 Introduction

Kinematics is a branch of physics and a subsection of classical mechanics interested in the geometrically possible motion of a body or a system of bodies without consideration of the forces included, i.e., causes and effects of the motions. The study of kinematics can be summarized into purely mathematical representations that can be used to compute various situations of the motions such as acceleration, time, displacement, velocity, and trajectory. Kinematics aims to describe the spatial positions of bodies or the systems of material particles, the velocity of particles, and the rate at which their velocity changes ([1], [3] and [9]).

Key Words: Spherical open chain, Constraint manifold, Structure equation, Split quaternion
2010 Mathematics Subject Classification: 70B15, 76E07, 51A40, 93B17
Received: 18.09.2019.
Accepted: 15.12.2019.

In classical mechanism, an object on which the distance between two points never changes is called as a rigid body. The motion of a rigid body is represented as a continuous sequence of displacement $D(t) : F \to M$. The frame F is fixed, but the position of the moving frame M varies with the parameter t.

Spherical mechanism is a mechanical system in which pin-jointed spatial links are used to move the body through a 3-dimensional way. In spherical mechanism, the axis of revolute joint coincides with the center of sphere [12].

In a chain, the position of each link relative to its neighbor is defined by the coordinate transformations. The sequence of links has a corresponding sequence of the transformations. The position of the last link of the chain related to the base is determined by the product of the transformations. An equation of the chain is obtained by equating this product to a specification of the position of the end link. This equation is termed by the structure equation for this mechanism. Why is there a need for the equation? The structure equation assists to compute the last position of the mechanism in terms of the joint parameters or to identify the joint movement, which is required to determine the last position. On the contrary, there is no technique describing the structure equation on the Lorentz sphere. For this reason, the first goal of this study is to present how these structure equations are described in the Lorentz sphere and the last position for the mechanism in this sphere is designated.

A surface, hypersurface, or parameterized curve in the image space is represented as the structure equation for an open chain written in quaternion form. This manifold is called the constraint manifold for this mechanism. Geometrically, the constraint impressed in the last situations by the rest of the mechanism is expressed as constraint manifolds [12]. The constraint manifold defines the set of all movements that the mechanism can make according to the joints used and the lengths of the links. It is used in explaining robotic motion, manipulation planning, kinodynamic planning and physical motion of human skeleton ([2] and [4]). McCarthy [12] investigated the constraint manifolds of the 2R and 3R planar and spherical open chains. In 2018, we [6] gave the constraint manifolds of the 2R and 3R planar open chains on the Lorentz plane. Another aim of this paper is to study the constraint manifolds of the 2R and 3R open chains on the Lorentz sphere.

2 Preliminaries

A link is a nominally rigid body that possesses at least two loops which are attachment points to other links via joints. Some common types of links are binary link (one with two loops), ternary link (one with three loops), quaternary link (one with four loops), etc. These joints are a connection between two or more links that satisfy a certain or potential motion between the connected links. There are joints different from one another such as revolute (R), prismatic (P), cylindrical (C), and spherical or ball (S) joints. The joint parameters can parameterize the rigid motion of a body part.

A combination of the links and joints interlocked in a path to supply a controlled output motion in response to a provided input motion is called a kinematic chain. In mechanical engineering, biomechanics, and robotics, kinematic chains help to characterize the motion of systems obtained from attached bodies such as a robotic arm, human skeleton, or an engine. The kinematic chains are either open or closed. An open kinematic chain is a mechanism in which the last link has a part, such as the head or hand. If the series chain closes on itself, the chain is called a closed kinematic chain. For more details, we refer the readers to ([9], [12] and [16]).

Throughout this paper, we will make some computations by using open chains with binary links and revolute joints.

Lorentz inner product on $\mathbb{R}^3$ is symbolized as

$$\begin{array}{rcl} \langle , \rangle_L & : & \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \\ (x,y) & \to & \langle x,y\rangle_L = -x_1y_1 + x_2y_2 + x_3y_3. \end{array}$$

This space is called as the 3-dimensional Lorentz space and denoted by $\mathbb{R}^3_1$ or L^3. A vector $x = (x_1, x_2, x_3) \in L^3$ is said to be timelike if $\langle x,x\rangle_L < 0$, spacelike if $\langle x,x\rangle_L > 0$ and lightlike (null) if $\langle x,x\rangle_L = 0$ and $x \neq 0$. The vector $x = 0$ is said to be spacelike. The norm of x is identified to be $\|x\|_L = \sqrt{\langle x,x\rangle_L}$. There exist three situations for $\|x\|_L$: *i*) positive, *ii*) zero, *iii*) positive imaginary. In case $\|x\|_L$ is positive imaginary, the expression $|\|x\||$ is written instead of $\|x\|_L$. In L^3, Lorentz and Hyperbolic unit spheres are defined as

$$\begin{array}{rcl} S^2_1(1) & = & \left\{x \in L^3 \mid \langle x,x\rangle_L = 1\right\} \\ H^2_0(1) & = & \left\{x \in L^3 \mid \langle x,x\rangle_L = -1\right\}, \end{array}$$

respectively. For any $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in L^3$, Lorentzian cross product is defined by

$$x \times_L y = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

Let V be a vector subspace of L^3. V is called spacelike if every vector in V is spacelike, timelike if V has a timelike vector, or lightlike otherwise.

Let ϕ be an angle between the vectors x and y. There exist the following equalities:

1. For the spacelike vectors x, y in L^3 that span spacelike vectors subspace

$$\begin{array}{rcl} \langle x,y\rangle_L & = & \|x\|_L \|y\|_L \cos\phi, \\ \|x \times_L y\|_L & = & \|x\|_L \|y\|_L \sin\phi. \end{array}$$

2. For the timelike vectors x, y in L^3,

$$\begin{aligned} \langle x, y\rangle_L &= -\,|||x|||\,|||y|||\cosh\phi, \\ \|x \times_L y\|_L &= |||x|||\,|||y|||\sinh\phi. \end{aligned}$$

3. For the spacelike vectors x, y in L^3 that span a timelike vector subspace

$$\begin{aligned} |\langle x, y\rangle_L| &= \|x\|_L\,\|y\|_L\cosh\phi, \\ \|x \times_L y\|_L &= \|x\|_L\,\|y\|_L\sinh\phi \end{aligned}$$

([13] and [15]).

Lorentzian inner product in L^n is expressed as follows:

$$\langle x, y\rangle_L = -x_1 y_1 + \sum_{v=2}^{n} x_v y_v,$$

where $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ are the vectors in L^n.

Suppose that $\mathbb{R}^{m\times n}$ is the set of matrices of m rows and n columns. For $A = [a_{gh}] \in \mathbb{R}^{m\times n}$ and $B = [b_{hr}] \in \mathbb{R}^{n\times l}$, Lorentzian matrix multiplication of the matrices A and B is determined as below:

$$A \cdot_L B = \left[-a_{g1}b_{1r} + \sum_{h=2}^{n} a_{gh}b_{hr}\right].$$

Note that $A \cdot_L B$ is an $m \times l$ matrix. The $n \times n$ L-identity matrix with reference to the Lorentzian matrix multiplication denoted by I_n is defined by

$$I_n = \begin{bmatrix} -1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ 0 & 0 & ... & 1 \end{bmatrix}_{n\times n}.$$

If the condition $A \cdot_L B = B \cdot_L A = I_n$ is satisfied, an $n \times n$ matrix A is termed by $L-$invertible, where B is the $n \times n$ matrix. Then, B is called $L-$inverse and is symbolized as A^{-1}. $A^T = [a_{hg}] \in \mathbb{R}^{n\times m}$ is denoted by the transpose of a matrix $A = [a_{gh}] \in \mathbb{R}^{m\times n}$. In case of $A^T = A^{-1}$, $A \in \mathbb{R}^{n\times n}$ is called an $L-$orthogonal matrix. Consider that $L-$orthogonal matrix A satisfies $\det A = \mp 1$. We call $L-$orthogonal matrices with $\det A = -1$ rotations and $\det A = 1$ reflections ([8] and [10]).

Assume that the transformation f is determined by $f : L^n \to L^n,\ f(x) = A \cdot_L x + d$ where A is an $n \times n$ matrix and d is an $n-$dimensional vector. When $A^T = A^{-1}$, f preserves the distances which are measured between the points [8].

Definition 1. *Let α be a curve in L^3. The causal character of α is the same as the causal character of α', where α' is the first derivative of the curve α [11].*

A split quaternion is determined by the base $\{1, i, j, k\}$, where i, j, k satisfy the qualities

$$\begin{array}{ccc} i^2 = -1 & j^2 = 1 & k^2 = 1 \\ ij = k & jk = -i & ki = j \\ ji = -k & kj = i & ik = -j. \end{array}$$

Then, a split quaternion p can be represented as $p = q_4 + q_1 i + q_2 j + q_3 k$, where q_1, q_2, q_3, q_4 are real scalars. The conjugate of p is symbolized as $\overline{p}$. It is $\overline{p} = q_4 - q_1 i - q_2 j - q_3 k$. The norm of p is expressed as follows:

$$N(p) = \sqrt{q_4^2 + q_1^2 - q_2^2 - q_3^2}.$$

A rotation about the origin is defined by $A \cdot_L x = X$, where A is an 3×3 $L-$orthogonal matrix and $x \in L^3$. The $L-$Cayley formula is defined as

$$A = (I - B)^{-1} \cdot_L (I + B)$$

or its equivalent form

$$A = (I + B) \cdot_L (I - B)^{-1},$$

where B is a skew-symmetric matrix. A 3×3 skew-symmetric matrix B has only three independent elements, i.e.,

$$B = \begin{bmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & -b_1 \\ b_2 & b_1 & 0 \end{bmatrix}.$$

These elements can be assembled into the vector $b = (b_1, b_2, b_3)$. Here, the matrix B satisfies the below equality:

$$B \cdot_L y = b \times_L y, \tag{1}$$

where y is an arbitrary vector of L^3. Given the $L-$orthogonal matrix A, we have the following equality by considering the $L-$Cayley formula:

$$X - x = B \cdot_L (X + x).$$

If we use the equality (1), we can write

$$X - x = b \times_L (X + x).$$

This equation is called the $L-$Rodrigues equation for rotations, and the vector b is termed by the $L-$Rodrigues vector.

Theorem 1. *In L^3, the rotation matrix A transforms the timelike vectors to timelike vectors, the spacelike vectors to spacelike vectors, and the null vectors to null vectors [14].*

Let us take the $L-$Rodrigues equation $X-x=b\times_L(X+x)$. There exist two situations with regard to the causal characters of the vectors b and x:

Situation I. Let b be a timelike vector and x be any vector in L^3. Assume that x^* and X^* are projections of x and X onto a plane perpendicular to b. In this case, it is seen that

$$\|b\|_L=\tan\frac{\phi}{2},$$

where ϕ is the angle between x^* and X^* spacelike vectors in spacelike subspace. Let the unit vector in the direction of b be $s=(s_x,s_y,s_z)$. Then the components of the vector b, or equivalently the components of the skew-symmetric matrix B are

$$\begin{aligned} b_1 &= \left(\tan\frac{\phi}{2}\right)s_x, \\ b_2 &= \left(\tan\frac{\phi}{2}\right)s_y, \\ b_3 &= \left(\tan\frac{\phi}{2}\right)s_z. \end{aligned}$$

These are called timelike $L-$Rodrigues parameters. The $L-$Cayley formula for the $L-$orthogonal matrix A can be written in terms of the rotation angle ϕ and the unit timelike vector s determined from

$$B=\left(\tan\frac{\phi}{2}\right)S.$$

The result is:

$$A=\left(\left(\cos\frac{\phi}{2}\right)I-\left(\sin\frac{\phi}{2}\right)S\right)^{-1}\cdot_L\left(\left(\cos\frac{\phi}{2}\right)I+\left(\sin\frac{\phi}{2}\right)S\right).$$

If $C=\left(\left(\cos\frac{\phi}{2}\right)I+\left(\sin\frac{\phi}{2}\right)S\right)$ is taken, the constants

$$\begin{aligned} c_0 &= \cos\frac{\phi}{2}, \\ c_1 &= \left(\sin\frac{\phi}{2}\right)s_x, \\ c_2 &= \left(\sin\frac{\phi}{2}\right)s_y, \end{aligned}$$

and

$$c_3 = \left(\sin\frac{\phi}{2}\right)s_z$$

are called timelike $L-$Euler parameters of A. Therefore, the rotations in L^3 can be defined by the split quaternion

$$q = \cos\frac{\phi}{2} + s_x\left(\sin\frac{\phi}{2}\right)i + s_y\left(\sin\frac{\phi}{2}\right)j + s_z\left(\sin\frac{\phi}{2}\right)k$$

[10].

Situation II. Let b be a spacelike vector and x be a timelike vector in L^3. Suppose that b is perpendicular to x. In this case, the following equality is obtained:

$$\|b\|_L = \tanh\frac{\phi}{2},$$

where ϕ is the angle between x and X timelike vectors. Let the unit vector in the direction of b be $s = (s_x, s_y, s_z)$. Then the components of the vector b are

$$\begin{aligned} b_1 &= \left(\tanh\frac{\phi}{2}\right)s_x, \\ b_2 &= \left(\tanh\frac{\phi}{2}\right)s_y, \\ b_3 &= \left(\tanh\frac{\phi}{2}\right)s_z. \end{aligned}$$

These are called $L-$Rodrigues parameters. Similar to the first situation, the constants that are called $L-$Euler parameters of A are obtained as follows:

$$\begin{aligned} c_0 &= \cosh\frac{\phi}{2}, \\ c_1 &= \left(\sinh\frac{\phi}{2}\right)s_x, \\ c_2 &= \left(\sinh\frac{\phi}{2}\right)s_y, \\ c_3 &= \left(\sinh\frac{\phi}{2}\right)s_z. \end{aligned}$$

Thus, the rotations in L^3 can be identified by the split quaternion

$$q = \cosh\frac{\phi}{2} + s_x\left(\sinh\frac{\phi}{2}\right)i + s_y\left(\sinh\frac{\phi}{2}\right)j + s_z\left(\sinh\frac{\phi}{2}\right)k$$

[14].

Let $Z = Z_4 + Z_1 i + Z_2 j + Z_3 k$ be a split quaternion. Each split quaternion can be identified with a 4-dimensional vector $Z = (Z_1, Z_2, Z_3, Z_4)$. For $W = W_4 + W_1 i + W_2 j + W_3 k$, the product of two split quaternions W and Z is given by the matrix product

$$WZ = W^+ \cdot_L Z$$

or

$$WZ = Z^- \cdot_L W,$$

where

$$W^+ = \begin{bmatrix} -W_4 & W_3 & -W_2 & W_1 \\ -W_3 & W_4 & -W_1 & W_2 \\ W_2 & W_1 & W_4 & W_3 \\ W_1 & W_2 & W_3 & W_4 \end{bmatrix},$$

$$Z^- = \begin{bmatrix} -Z_4 & -Z_3 & Z_2 & Z_1 \\ Z_3 & Z_4 & Z_1 & Z_2 \\ -Z_2 & -Z_1 & Z_4 & Z_3 \\ Z_1 & Z_2 & Z_3 & Z_4 \end{bmatrix}.$$

W^+ and Z^- are defined as follows: i, j, k being the basis vectors of split quaternions, each column of the matrix W^+ is the result of the product of W on the right with basis bivectors $-i, j, k, 1$ and each column of the matrix Z^- is the result of the product of Z on the left with the basis bivectors $-i, j, k$ and 1 ([10] and [14]).

Theorem 2. *Suppose that $P(x_1, x_2, x_3)$ is a point on the Lorentzian revolution ellipsoid specified by the number $b_1 > 0$, and this ellipsoid has the foci $F_1(0, 0, b_3)$ and $F_2(0, 0, -b_3)$ which are on the spacelike axis. Hence, the below expression exists*

$$-\frac{x_1^2}{b_2^2} + \frac{x_2^2}{b_2^2} + \frac{x_3^2}{b_1^2} = 1,$$

where b_1, b_2, b_3 satisfy the condition $b_1^2 = b_2^2 + b_3^2$ [7].

Theorem 3. *Assume that $P(x_1, x_2, x_3)$ is a point on the Lorentzian revolution hyperboloid identified by the number $b_1 > 0$, and this hyperboloid has the foci $F_1(0, 0, b_3)$ and $F_2(0, 0, -b_3)$ which are on the spacelike axis. Thus, there is the following equality*

$$\frac{x_1^2}{b_2^2} - \frac{x_2^2}{b_2^2} + \frac{x_3^2}{b_1^2} = 1,$$

where b_1, b_2, b_3 satisfy $b_3^2 = b_1^2 + b_2^2$ [7].

Acknowledgement 1. *The Lorentzian revolution ellipsoid and hyperboloid are indicated by $L-$ellipsoid and $L-$hyperboloid in the study, respectively.*

Note that the mechanism is designated by an open chain of rigid bodies A_v, $v = 1, 2, ..., n$. Each body of the chain expects the last has two joints attaching it to the bodies before and after it. Consider that these joints on A_v are symbolized as o_v and t_v. In this case, the mechanism is constituted by attaching the joint t_{v-1} of the body A_{v-1} to o_v of A_v. A_1 is on the ground in order that o_1 coincides with the ground point t_o. The link A_n has only one joint o_n.

Assume that T_v is a reference frame connected to the body A_v at the joint t_v. Then, the transformation $S_v : T_{v-1} \to T_v$ is determined by the position of A_v relative to A_{v-1}. The frame T_0 is located on the ground frame with its origin at t_0. The transformation $H : O_n \to M$ is described by the position of the reference frame M in the last link. The beginning position of the chain is indicated by $G : F \to T_0$. With these conventions, a sequence of coordinate transformations between M and the base frame F is attained. This sequence is named by the structure equation of the mechanism. The structure equation is identified by

$$D = GS_1S_2...S_nH.$$

The equation determines the position of M relative to F with regard to the relative positions of each body of the mechanism.

Consider the coordinate frame O_v at every point o_v. This transformation S_v consists of the product $S_v = Z_vX_v$, where $Z_v : T_{v-1} \to O_v$ and $X_v : O_v \to T_v$ are called joint and link transformations, respectively. In this case, the displacement $D : F \to M$ can be stated by

$$D = GZ_1X_1Z_2X_2...Z_{n-1}X_{n-1}Z_nH,$$

where G, H and X_v are constant [12].

3 The $L-$Structure Equations on $S_1^2(1)$ for an Open Chain

Let us consider the unit Lorentz sphere

$$S_1^2(1) = \left\{x = (x_1, x_2, x_3) \mid -x_1^2 + x_2^2 + x_3^2 = 1\right\} \subset L^3$$

and a geodesic μ of this sphere.

i) if μ is timelike, it is a parametrization of one branch of a hyperbola in L^3,

ii) if μ is spacelike, it is a periodic parametrization of an ellipse in L^3 (see Fig.1.) [13].

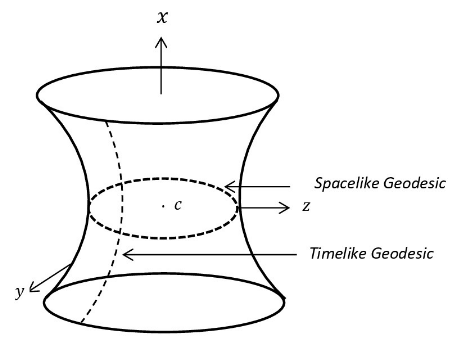

Fig. 1. The Lorentz Sphere

Now, let us investigate the structure equations for an open chain by means of the spacelike and timelike geodesics on the Lorentz sphere. The structure equation on the Lorentz sphere is called $L-$structure equation throughout this paper. There exist two situations according to the first link considered.

Case I. Let the first link be on the spacelike geodesic. Any link on this chain is the link A_v with two coordinate frames O_v and T_v at points o_v and t_v. Suppose that the centre of the Lorentz sphere $S_1^2(1)$ is c. In order to calculate the motion on this sphere, let us consider that the $x-$axis is normal to the plane in the direction $(o_v - c) \times_L (t_v - c)$, and the length of the link A_v is the angle ψ_v between the $z - axes$ of the frames O_v and T_v. Hence, the matrix form of the rotation motion between the above mentioned frames is obtained in the following way [5]:

$$[X_v] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\psi_v & -\sin\psi_v \\ 0 & \sin\psi_v & \cos\psi_v \end{bmatrix}.$$

This rotation motion is on the same spacelike geodesic. The rotation matrix with the hyperbolic angle ϕ_v about $z-$axis is defined as [5]

$$[Z_v] = \begin{bmatrix} -\cosh\phi_v & \sinh\phi_v & 0 \\ -\sinh\phi_v & \cosh\phi_v & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where ϕ_v is the hyperbolic angle between the frames T_{v-1} and O_v. This rotation motion is a movement from a spacelike geodesic to another spacelike geodesic.

The motion of this link is determined by the following matrix form:

$$\begin{aligned}
[S_v] &= [Z_v] \cdot_L [X_v] \\
&= \begin{bmatrix} -\cosh\phi_v & \sinh\phi_v & 0 \\ -\sinh\phi_v & \cosh\phi_v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot_L \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\psi_v & -\sin\psi_v \\ 0 & \sin\psi_v & \cos\psi_v \end{bmatrix} \\
&= \begin{bmatrix} -\cosh\phi_v & \sinh\phi_v \cos\psi_v & -\sinh\phi_v \sin\psi_v \\ -\sinh\phi_v & \cosh\phi_v \cos\psi_v & -\cosh\phi_v \sin\psi_v \\ 0 & \sin\psi_v & \cos\psi_v \end{bmatrix}.
\end{aligned}$$

Using the above mentioned matrix multiplication, the $L-$structure equation for the spherical open chain is obtained by the equality

$$[D] = [S_1] \cdot_L \ldots \cdot_L [S_n],$$

where $[D]$ is the orientation of the moving frame M.

Case II. Assume that the link A_v with the frames O_v and T_v at points o_v and t_v is on the timelike geodesic. Take into consideration the $y-$axis being normal to the plane in the direction $(o_v - c) \times_L (t_v - c)$, where the centre of $S_1^2(1)$ is c. Let us determine the hyperbolic angle α_v between the $z-$axes of the frames O_v and T_v. The matrix form of the rotation motion between the above mentioned frames is written as below:

$$[Y_v] = \begin{bmatrix} -\cosh\alpha_v & 0 & \sinh\alpha_v \\ 0 & 1 & 0 \\ -\sinh\alpha_v & 0 & \cosh\alpha_v \end{bmatrix}.$$

Note that this rotation motion with the hyperbolic angle α_v is on the same timelike geodesic. On the other hand, the rotation matrix with the angle ϕ_v between the frames T_{v-1} and O_v about $z-$axis is

$$[Z_v] = \begin{bmatrix} -\cosh\phi_v & \sinh\phi_v & 0 \\ -\sinh\phi_v & \cosh\phi_v & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A movement from a timelike geodesic to another timelike geodesic is determined by this rotation motion.

The below mentioned matrix form expresses the motion of this link:

$$\begin{aligned}
[S_v] &= [Z_v] \cdot_L [Y_v] \\
&= \begin{bmatrix} -\cosh\phi_v & \sinh\phi_v & 0 \\ -\sinh\phi_v & \cosh\phi_v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot_L \begin{bmatrix} -\cosh\alpha_v & 0 & \sinh\alpha_v \\ 0 & 1 & 0 \\ -\sinh\alpha_v & 0 & \cosh\alpha_v \end{bmatrix} \\
&= \begin{bmatrix} -\cosh\phi_v \cosh\alpha_v & \sinh\phi_v & \cosh\phi_v \sinh\alpha_v \\ -\sinh\phi_v \cosh\alpha_v & \cosh\phi_v & \sinh\phi_v \sinh\alpha_v \\ -\sinh\alpha_v & 0 & \cosh\alpha_v \end{bmatrix}.
\end{aligned}$$

Similarly, the $L-$structure equation for this situation is calculated by the equality

$$[D] = [S_1] \cdot_L \ldots \cdot_L [S_n] \, .$$

Case I and II show that if the link established on the Lorentz sphere is on the spacelike (resp. timelike) geodesic, then this link makes all the movements on the spacelike (resp. timelike) geodesic.

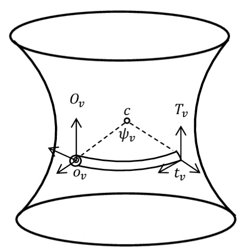

Fig. 2. A spacelike link with frame $\boldsymbol{O_v}$ *and* $\boldsymbol{T_v}$.

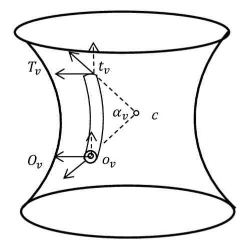

Fig. 3. A timelike link with frame $\boldsymbol{O_v}$ *and* $\boldsymbol{T_v}$.

4 $L-$Constraint Manifolds on $S_1^2\,(1)$

In this chapter, we will calculate the constraint manifolds for cases I and II, and make geometric comments. The constraint manifold on the Lorentz sphere is symbolized as the $L-$constraint manifold.

Consider that the first link of the chain is on the spacelike geodesic. There exist two axes on this link which intersects at a point c and intersects the unit Lorentz sphere at points o_1 and t_1. Furthermore, the angle between these axes is ψ_1. We know that the x-axis is normal to the plane in the direction $(o_1 - c) \times_L (t_1 - c)$. In this case, let us connect the moving body M at t_1, and attach o_1 to t_0 which is fixed in the base. Assume that the rotation of the first link and the rotation of the moving body are symbolized as ϕ_1 and ϕ_2, respectively. Taking advantage of these conventions, the $L-$structure equation for the 2R spherical open chain in case I is briefly stated by

$$[D] = [Z_1] \cdot_L [X_1] \cdot_L [Z_2] \, .$$

The displacement $[D]$ of the chain can be written in the matrix form as below:

$$\begin{aligned}[D] &= \begin{bmatrix} -\cosh\phi_1 & \sinh\phi_1 & 0 \\ -\sinh\phi_1 & \cosh\phi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot_L \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\psi_1 & -\sin\psi_1 \\ 0 & \sin\psi_1 & \cos\psi_1 \end{bmatrix} \\ &\cdot_L \begin{bmatrix} -\cosh\phi_2 & \sinh\phi_2 & 0 \\ -\sinh\phi_2 & \cosh\phi_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.\end{aligned}$$

The split quaternion form of this $L-$structure equation is easily attained as below:

$$D = Z_1 X_1 Z_2.$$

The right side of the above mentioned expression is the product

$$\begin{aligned} D &= \left(0, 0, \sinh\left(\frac{\phi_1}{2}\right), \cosh\left(\frac{\phi_1}{2}\right)\right)\left(\sin\left(\frac{\psi_1}{2}\right), 0, 0, \cos\left(\frac{\psi_1}{2}\right)\right) \\ &\left(0, 0, \sinh\left(\frac{\phi_2}{2}\right), \cosh\left(\frac{\phi_2}{2}\right)\right). \end{aligned} \tag{2}$$

If we widen the split quaternion product, the split quaternion form is found as below:

$$D(\phi_1, \phi_2) = (D_1(\phi_1, \phi_2), D_2(\phi_1, \phi_2), D_3(\phi_1, \phi_2), D_4(\phi_1, \phi_2)),$$

where

$$\begin{aligned} D_1(\phi_1, \phi_2) &= \sin\left(\frac{\psi_1}{2}\right)\cosh\left(\frac{\phi_1 - \phi_2}{2}\right), \\ D_2(\phi_1, \phi_2) &= \sin\left(\frac{\psi_1}{2}\right)\sinh\left(\frac{\phi_1 - \phi_2}{2}\right), \\ D_3(\phi_1, \phi_2) &= \cos\left(\frac{\psi_1}{2}\right)\sinh\left(\frac{\phi_1 + \phi_2}{2}\right), \\ D_4(\phi_1, \phi_2) &= \cos\left(\frac{\psi_1}{2}\right)\cosh\left(\frac{\phi_1 + \phi_2}{2}\right). \end{aligned}$$

If ϕ_1 and ϕ_2 are eliminated, the algebraic equation of the parameterized surface is obtained in the following way:

$$\cos^2\left(\frac{\psi_1}{2}\right)D_1^2 - \cos^2\left(\frac{\psi_1}{2}\right)D_2^2 + \sin^2\left(\frac{\psi_1}{2}\right)D_3^2 - \sin^2\left(\frac{\psi_1}{2}\right)D_4^2 = 0. \tag{3}$$

The positions $D = (D_1, D_2, D_3, D_4)$ that satisfy this equation are reachable by the end link of the 2R chain.

Assume that the coordinates of points in L^4 is presented by the vector $X = (x_1, x_2, x_3, x_4)$. Thus, (3) can be expressed as the canonical form of a quadric hypercone:

$$X^T \cdot_L [Q] \cdot_L X = 0,$$

where

$$[Q] = \begin{bmatrix} \cos^2\left(\frac{\psi_1}{2}\right) & 0 & 0 & 0 \\ 0 & -\cos^2\left(\frac{\psi_1}{2}\right) & 0 & 0 \\ 0 & 0 & \sin^2\left(\frac{\psi_1}{2}\right) & 0 \\ 0 & 0 & 0 & -\sin^2\left(\frac{\psi_1}{2}\right) \end{bmatrix}.$$

Thus, the equation (3) is written as below:

$$x_1^2 - x_2^2 + \tan^2\left(\frac{\psi_1}{2}\right) x_3^2 - \tan^2\left(\frac{\psi_1}{2}\right) x_4^2 = 0. \tag{4}$$

If the intersection of the equality (4) with the hyperplane $x_4 = 1$ is computed, it is clear that the set of $L-$hyperboloids

$$x_1^2 - x_2^2 + \tan^2\left(\frac{\psi_1}{2}\right) x_3^2 = \tan^2\left(\frac{\psi_1}{2}\right)$$

or

$$\frac{x_1^2}{\tan^2\left(\frac{\psi_1}{2}\right)} - \frac{x_2^2}{\tan^2\left(\frac{\psi_1}{2}\right)} + x_3^2 = 1$$

that have the foci $F_1 = \left(0, 0, \sqrt{1 + \tan^2\left(\frac{\psi_1}{2}\right)}\right)$ and $F_2 = \left(0, 0, -\sqrt{1 + \tan^2\left(\frac{\psi_1}{2}\right)}\right)$ is found.

Presently, we will calculate the $L-$constraint manifold of the chain for case II. Taking the given basic information into consideration, the $L-$structure equation of the 2R spherical chain becomes

$$[D] = [Z_1] \cdot_L [Y_1] \cdot_L [Z_2].$$

The right side of this expression is the following matrix product:

$$\begin{aligned} [D] &= \begin{bmatrix} -\cosh\phi_1 & \sinh\phi_1 & 0 \\ -\sinh\phi_1 & \cosh\phi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot_L \begin{bmatrix} -\cosh\alpha_1 & 0 & \sinh\alpha_1 \\ 0 & 1 & 0 \\ -\sinh\alpha_1 & 0 & \cosh\alpha_1 \end{bmatrix} \\ &\quad \cdot_L \begin{bmatrix} -\cosh\phi_2 & \sinh\phi_2 & 0 \\ -\sinh\phi_2 & \cosh\phi_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

The split quaternion form of this $L-$structure equation is written as below:

$$D = Z_1 Y_1 Z_2.$$

The right side of this expression is the product

$$\begin{aligned} D &= \left(0, 0, \sinh\left(\frac{\phi_1}{2}\right), \cosh\left(\frac{\phi_1}{2}\right)\right)\left(0, \sinh\left(\frac{\alpha_1}{2}\right), 0, \cosh\left(\frac{\alpha_1}{2}\right)\right) \\ & \left(0, 0, \sinh\left(\frac{\phi_2}{2}\right), \cosh\left(\frac{\phi_2}{2}\right)\right). \end{aligned} \quad (5)$$

Expanding the split quaternion product, the following parametrized surface

$$D(\phi_1, \phi_2) = (D_1(\phi_1, \phi_2), D_2(\phi_1, \phi_2), D_3(\phi_1, \phi_2), D_4(\phi_1, \phi_2))$$

is attained, where

$$\begin{aligned} D_1(\phi_1, \phi_2) &= \sinh\left(\frac{\alpha_1}{2}\right)\sinh\left(\frac{\phi_1 - \phi_2}{2}\right), \\ D_2(\phi_1, \phi_2) &= \sinh\left(\frac{\alpha_1}{2}\right)\cosh\left(\frac{\phi_1 - \phi_2}{2}\right), \\ D_3(\phi_1, \phi_2) &= \cosh\left(\frac{\alpha_1}{2}\right)\sinh\left(\frac{\phi_1 + \phi_2}{2}\right), \\ D_4(\phi_1, \phi_2) &= \cosh\left(\frac{\alpha_1}{2}\right)\cosh\left(\frac{\phi_1 + \phi_2}{2}\right). \end{aligned}$$

Eliminating ϕ_1 and ϕ_2, the algebraic equation parameterized by α_1

$$\cosh^2\left(\frac{\alpha_1}{2}\right) D_1^2 - \cosh^2\left(\frac{\alpha_1}{2}\right) D_2^2 - \sinh^2\left(\frac{\alpha_1}{2}\right) D_3^2 + \sinh^2\left(\frac{\alpha_1}{2}\right) D_4^2 = 0 \quad (6)$$

is obtained. To analyze this equation as the canonical form of the quadric hypercone, consider that the points of L^4 is denoted $X = (x_1, x_2, x_3, x_4)$ in order that (6) becomes

$$x_1^2 - x_2^2 - \tanh^2\left(\frac{\alpha_1}{2}\right) x_3^2 + \tanh^2\left(\frac{\alpha_1}{2}\right) x_4^2 = 0. \quad (7)$$

The quadratic form of this equation is given by

$$X^T \cdot_L [Q] \cdot_L X = 0,$$

with the coefficient matrix

$$[Q] = \begin{bmatrix} \cosh^2\left(\frac{\alpha_1}{2}\right) & 0 & 0 & 0 \\ 0 & -\cosh^2\left(\frac{\alpha_1}{2}\right) & 0 & 0 \\ 0 & 0 & -\sinh^2\left(\frac{\alpha_1}{2}\right) & 0 \\ 0 & 0 & 0 & \sinh^2\left(\frac{\alpha_1}{2}\right) \end{bmatrix}.$$

Similarly, the set of $L-$ellipsoids

$$-x_1^2+x_2^2+\tanh^2\left(\frac{\alpha_1}{2}\right)x_3^2=\tanh^2\left(\frac{\alpha_1}{2}\right)$$

or

$$-\frac{x_1^2}{\tanh^2\left(\frac{\alpha_1}{2}\right)}+\frac{x_2^2}{\tanh^2\left(\frac{\alpha_1}{2}\right)}+x_3^2=1$$

arises from the projection of this form onto the hyperplane $x_4=1$.
These $L-$ellipsoids have the foci $F_1=\left(0,0,\sqrt{1-\tanh^2\left(\frac{\alpha_1}{2}\right)}\right)$ and $F_2=\left(0,0,-\sqrt{1-\tanh^2\left(\frac{\alpha_1}{2}\right)}\right)$.

Now, we will obtain the $L-$constraint manifold of a 3R spherical open chain by using the given structure equations. Consider that coordinate frames are appointed as in the same way defined for the 2R open chain. Thus, the joints for the second body are indicated by o_2 and t_2. Note that the angle between these axes is ψ_2 or α_2 for cases I and II, respectively. The moving frame M is attached at joint t_2. Its angle related to T_2 is shown by ϕ_3 in both situations.

In case I, we will investigate the $L-$constraint manifold for the 3R spherical open chain. The position of the workpiece now depends on the parameters $\phi_1,\ \phi_2$, and ϕ_3. If these conventions are taken into consideration, the product of the split quaternion form generates the below parameterized surface:

$$D\left(\phi_1,\phi_2,\phi_3\right)=Z_1X_1Z_2X_2Z_3, \tag{8}$$

where $Z_3=\left(0,0,\sinh\left(\frac{\phi_3}{2}\right),\cosh\left(\frac{\phi_3}{2}\right)\right),\ X_2=\left(\sin\left(\frac{\psi_2}{2}\right),0,0,\cos\left(\frac{\psi_2}{2}\right)\right)$ and $Z_1,\ X_1$, and Z_2 are given by (2). If the above expression (8) is expanded, it is seen that

$$D\left(\phi_1,\phi_2,\phi_3\right)=\left(D_1\left(\phi_1,\phi_2,\phi_3\right),D_2\left(\phi_1,\phi_2,\phi_3\right),D_3\left(\phi_1,\phi_2,\phi_3\right),D_4\left(\phi_1,\phi_2,\phi_3\right)\right),$$

where

$$\begin{aligned}
D_1&=\cosh\left(\tfrac{\phi_1-\phi_3}{2}\right)\cosh\left(\tfrac{\phi_2}{2}\right)\sin\left(\tfrac{\psi_1+\psi_2}{2}\right)-\sinh\left(\tfrac{\phi_1-\phi_3}{2}\right)\sinh\left(\tfrac{\phi_2}{2}\right)\sin\left(\tfrac{\psi_1-\psi_2}{2}\right),\\
D_2&=\sinh\left(\tfrac{\phi_1-\phi_3}{2}\right)\cosh\left(\tfrac{\phi_2}{2}\right)\sin\left(\tfrac{\psi_1+\psi_2}{2}\right)-\cosh\left(\tfrac{\phi_1-\phi_3}{2}\right)\sinh\left(\tfrac{\phi_2}{2}\right)\sin\left(\tfrac{\psi_1-\psi_2}{2}\right),\\
D_3&=\cosh\left(\tfrac{\phi_1+\phi_3}{2}\right)\sinh\left(\tfrac{\phi_2}{2}\right)\cos\left(\tfrac{\psi_1-\psi_2}{2}\right)+\sinh\left(\tfrac{\phi_1+\phi_3}{2}\right)\cosh\left(\tfrac{\phi_2}{2}\right)\cos\left(\tfrac{\psi_1+\psi_2}{2}\right),\\
D_4&=\sinh\left(\tfrac{\phi_1+\phi_3}{2}\right)\sinh\left(\tfrac{\phi_2}{2}\right)\cos\left(\tfrac{\psi_1-\psi_2}{2}\right)+\cosh\left(\tfrac{\phi_1+\phi_3}{2}\right)\cosh\left(\tfrac{\phi_2}{2}\right)\cos\left(\tfrac{\psi_1+\psi_2}{2}\right).
\end{aligned}$$

Eliminating the variables ϕ_1 and ϕ_3, the algebraic equation of this solid is found as follows:

$$D_1^2-D_2^2+D_3^2-D_4^2=-\cos\rho(\phi_2), \tag{9}$$

where $\cos\rho(\phi_2) = \cos\psi_1\cos\psi_2 - \sin\psi_1\sin\psi_2\cosh\phi_2$.

Let the coordinates of points in L^4 be represented by the vector $X = (x_1, x_2, x_3, x_4)$. Thus, the parameterized set of the hypercones is denoted by the equality (9):

$$X^T \cdot_L [Q] \cdot_L X = 0,$$

where

$$[Q] = \begin{bmatrix} \cos^2\dfrac{\rho(\phi_2)}{2} & 0 & 0 & 0 \\ 0 & -\cos^2\dfrac{\rho(\phi_2)}{2} & 0 & 0 \\ 0 & 0 & \sin^2\dfrac{\rho(\phi_2)}{2} & 0 \\ 0 & 0 & 0 & -\sin^2\dfrac{\rho(\phi_2)}{2} \end{bmatrix}.$$

In such a way, the following equation can be written

$$x_1^2 - x_2^2 + \tan^2\frac{\rho(\phi_2)}{2}x_3^2 - \tan^2\frac{\rho(\phi_2)}{2}x_4^2 = 0. \tag{10}$$

When the $L-$constraint manifold is projected onto the hyperplane $x_4 = 1$ along the lines of the hypercone, it is done by placing $x_4 = 1$ in (10). With these conventions, the set of $L-$hyperboloids is found as follows:

$$\frac{x_1^2}{\tan^2\dfrac{\rho(\phi_2)}{2}} - \frac{x_2^2}{\tan^2\dfrac{\rho(\phi_2)}{2}} + x_3^2 = 1,$$

where $F_1 = \left(0, 0, \sqrt{1 + \tan^2\dfrac{\rho(\phi_2)}{2}}\right)$ and $F_2 = \left(0, 0, -\sqrt{1 + \tan^2\dfrac{\rho(\phi_2)}{2}}\right)$ are the foci of $L-$hyperboloids.

Now, we try to demonstrate the $L-$constraint manifold of the mechanism for case II in a similar way. The position of this mechanism depends on three parameters as ϕ_1, ϕ_2, and ϕ_3. The split quaternion form of the structure equation of the chain is as in the following equality:

$$D(\phi_1, \phi_2, \phi_3) = Z_1 Y_1 Z_2 Y_2 Z_3, \tag{11}$$

where Z_1, Y_1, and Z_2 are obtained from (5) and $Y_2 = \left(0, \sinh\left(\frac{\alpha_2}{2}\right), 0, \cosh\left(\frac{\alpha_2}{2}\right)\right)$ and $Z_3 = \left(0, 0, \sinh\left(\frac{\phi_3}{2}\right), \cosh\left(\frac{\phi_3}{2}\right)\right)$. Expanding (11), the displacement D in the split quaternion form is computed as follows:

$$D(\phi_1, \phi_2, \phi_3) = (D_1(\phi_1, \phi_2, \phi_3), D_2(\phi_1, \phi_2, \phi_3), D_3(\phi_1, \phi_2, \phi_3), D_4(\phi_1, \phi_2, \phi_3)), \tag{12}$$

where

$$\begin{aligned}
D_1 &= \sinh\left(\tfrac{\phi_1-\phi_3}{2}\right)\cosh\left(\tfrac{\phi_2}{2}\right)\sinh\left(\tfrac{\alpha_1+\alpha_2}{2}\right)-\cosh\left(\tfrac{\phi_1-\phi_3}{2}\right)\sinh\left(\tfrac{\phi_2}{2}\right)\sinh\left(\tfrac{\alpha_1-\alpha_2}{2}\right),\\
D_2 &= \cosh\left(\tfrac{\phi_1-\phi_3}{2}\right)\cosh\left(\tfrac{\phi_2}{2}\right)\sinh\left(\tfrac{\alpha_1+\alpha_2}{2}\right)-\sinh\left(\tfrac{\phi_1-\phi_3}{2}\right)\sinh\left(\tfrac{\phi_2}{2}\right)\sinh\left(\tfrac{\alpha_1-\alpha_2}{2}\right),\\
D_3 &= \cosh\left(\tfrac{\phi_1+\phi_3}{2}\right)\sinh\left(\tfrac{\phi_2}{2}\right)\cosh\left(\tfrac{\alpha_1-\alpha_2}{2}\right)+\sinh\left(\tfrac{\phi_1+\phi_3}{2}\right)\cosh\left(\tfrac{\phi_2}{2}\right)\cosh\left(\tfrac{\alpha_1+\alpha_2}{2}\right),\\
D_4 &= \sinh\left(\tfrac{\phi_1+\phi_3}{2}\right)\sinh\left(\tfrac{\phi_2}{2}\right)\cosh\left(\tfrac{\alpha_1-\alpha_2}{2}\right)+\cosh\left(\tfrac{\phi_1+\phi_3}{2}\right)\cosh\left(\tfrac{\phi_2}{2}\right)\cosh\left(\tfrac{\alpha_1+\alpha_2}{2}\right).
\end{aligned}$$

The variables ϕ_1 and ϕ_3 can be eliminated from (12) to yield the algebraic equation

$$D_1^2 - D_2^2 + D_3^2 - D_4^2 = -\cosh\rho(\phi_2), \tag{13}$$

where $\cosh\rho(\phi_2) = \cosh\alpha_1\cosh\alpha_2 + \sinh\alpha_1\sinh\alpha_2\cosh\phi_2$. In case $D = (D_1, D_2, D_3, D_4)$ satisfies the above mentioned algebraic equation, the $3R$ chain can arrive at this position D.

Let the points of L^4 be indicated by $X = (x_1, x_2, x_3, x_4)$. Thus, the parameterized set of the hypercones is identified as follows:

$$x_1^2 - x_2^2 - \tanh^2\frac{\rho(\phi_2)}{2}x_3^2 + \tanh^2\frac{\rho(\phi_2)}{2}x_4^2 = 0.$$

The canonical form of the quadric hypercone

$$X^T \cdot_L [Q] \cdot_L X = 0$$

can be observed by the equality (13), where

$$[Q] = \begin{bmatrix} \cosh^2\dfrac{\rho(\phi_2)}{2} & 0 & 0 & 0 \\ 0 & -\cosh^2\dfrac{\rho(\phi_2)}{2} & 0 & 0 \\ 0 & 0 & -\sinh^2\dfrac{\rho(\phi_2)}{2} & 0 \\ 0 & 0 & 0 & \sinh^2\dfrac{\rho(\phi_2)}{2} \end{bmatrix}.$$

This quadric form might be taken as a quadric equation expressed as the homogeneous coordinates of 3-dimensional projective space. We project this solid onto the hyperplane $x_4 = 1$ to obtain the parameterized set of $L-$ellipsoids:

$$-\frac{x_1^2}{\tanh^2\dfrac{\rho(\phi_2)}{2}} + \frac{x_2^2}{\tanh^2\dfrac{\rho(\phi_2)}{2}} + x_3^2 = 1.$$

It is clear that each $L-$ellipsoid has the foci $F_1 = \left(0, 0, \sqrt{1-\tanh^2\dfrac{\rho(\phi_2)}{2}}\right)$ and $F_2 = \left(0, 0, -\sqrt{1-\tanh^2\dfrac{\rho(\phi_2)}{2}}\right)$.

Conclusion 1. *The rotation movement of each link relative to the axis is represented by the matrices on the sphere and the multiplication of these matrices forms the structure equations. These equations define the last position for the mechanism. Throughout the study, each of the movements of the spacelike or timelike links placed on the geodesics of the Lorentz sphere is shown. After that, by using these movements, it is demonstrated that if the first link is placed on the spacelike (resp. timelike) geodesic of the Lorentz sphere, then the* $L-$*constraint manifolds of the* $2R$ *and* $3R$ *spherical open chains are the set of* $L-$*hyperboloids (resp.* $L-$*ellipsoids).*

Acknowledgement 2. *The first and second authors would like to thank TUBITAK (The Scientific and Technological Research Council of Turkey) for their financial supports during their Ph.D. studies.*

References

[1] J. S. Beggs, Kinematics, ISBN: 0891163557: Taylor & Francis p.1 (1983).

[2] D. Berenson, S. S. Srinivasa, D. Ferguson and J. J. Kuffner, Manipulation Planning on Constraint Manifolds, in IEEE International Conference on Robotics and Automation (2009).

[3] A. Biewener, Animal Locomotion, Oxford University Press (2003).

[4] R. Bordalba, L. Ros and J. M. Porta, Kinodynamic Planning on Constraint Manifolds, arXiv preprint arXiv: 1705.07637 (2017).

[5] O. Durmaz, B. Aktaş and H. Gündoğan, The Derivative and Tangent Operators of a Motion in Lorentzian Space, *International Journal of Geometric Methods in Modern Physics* **14** (2017).

[6] O. Durmaz, B. Aktaş and H. Gündoğan, Structure Equations and Constraint Manifolds on Lorentz Plane, *Mathematical Methods in Applied Sciences* DOI: 10.1002/mma.5275 (2018) 1-16.

[7] A.Ç. Gözütok, S. OzkaldıKarakuş and H. Gündoğan, Conics and Quadrics in Lorentz Space, *Mathematical Sciences and Applications E-Notes* **6**(1) (2018) 58-63.

[8] H. Gündoğan and O. Keçilioğlu, Lorentzian Matrix Multiplication and the Motion on Lorentzian Plane, *Glass. Mat.* **41** (2006) 329-334.

[9] R. C. Hibbeler, Kinematics and Kinetics of a Particle, Engineering Mechanics:Dynamics, Singapore Prentice Hall. (2009).

[10] O. Keçilioglu, S. Ozkaldı and H. Gündoğan, Rotations and Screw Motion with Timelike Vector in 3-Dimensional Lorentzian Space, *Adv. Appl. Clifford Algebra.* **22** (2012) 1081-1091.

[11] R. Lopez, Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space, *Int. Electron. J. Geom.* **7** (2014), 44-107.

[12] J. M. McCharty, An Introduction to Theoretical Kinematics, The MIT Press, Cambridge, Massachusetts, London, England (1990).

[13] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, Inc, New York (1983).

[14] S. Ozkaldı and H. Gündoğan, Cayley Formula, Euler Parameters and Rotations in 3-Dimensional Lorentzian Space, *Adv. Appl. Clifford Algebra.* **20** (2010) 367-377.

[15] R. G. Ratcliffe, Foundations of Hyperbolic Manifolds, Springer-Verlag, New York (1994).

[16] J. M. Winter, Terminology and Foundations of Movement Science. In: Winter J.M., Crago P.E. (eds) Biomechanics and Neural Control of Posture and Movement, Springer, New York, (2000).

Buşra Aktaş,
Department of Mathematics,
Faculty of Science and Arts,
University of Kırıkkale,
71450-Yahşihan, Kırıkkale, Turkey.
Email: baktas6638@gmail.com

Olgun Durmaz,
Department of Mathematics,
Faculty of Science,
University of Atatürk
25240-Yakutiye, Erzurum, Turkey
Email: olgun.durmaz@atauni.edu.tr

Hal˙t Gündoğan,
Department of Mathematics,
Faculty of Science and Arts,
University of Kırıkkale,
71450-Yahşihan, Kırıkkale, Turkey.
Email: hagundogan@hotmail.com