Commun.Fac.Sci.Univ.Ank.Series A1
Volume 54, Number 2, Pages 33-37 (2005)
ISSN 1303-5991

LORENTZIAN CYLINDER LIE GROUP IN $\mathbf{R}_1^5$ AND ITS A C^∞-ACTION

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ABSTRACT. In this paper, two binary operations on Lorentzian sphere in $\mathbf{R}_1^4$ and on Lorentzian cylinder in $\mathbf{R}_1^5$ are defined. Also, it has been shown that Lorentzian sphere in $\mathbf{R}_1^4$ and Lorentzian cylinder in $\mathbf{R}_1^5$ with the corresponding binary operation form Lie groups. A C^∞-action on Lorentzian cylinder of arbitrary radius of Lorentzian cylinder of radius one is defined in $\mathbf{R}_1^5$ and its some properties are given. Finally, the action is expanded to $\mathbf{R}_1^5$.

1. INTRODUCTION

Let G be a group and also be differentiable manifold. G is called a Lie group if the group function on G as a manifold is differentiable[2].

Let G be a Lie group and M be a differentiable manifold. Then Lie group G is said to act on differentiable manifold M, if there is a mapping $\theta : G \times M \to M$ satisfying the following two conditions:

i) If $g_1, g_2 \in G$, then

$$\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x), \qquad \text{for all } x \in M.$$

ii) If e is the identity element of G then

$$\theta(e, x) = x \qquad \text{for all } x \in M.$$

When M is a C^∞-manifold and θ is a C^∞, then we speak of a C^∞–action[1].

If $p \in M$ the set $G_p = \{\theta(g,p) | \, g \in G\}$ is called the orbit of p under the C^∞-action θ of G [1].

G is said to act transitively on M if, given any two points $m_1, m_2 \in M$ there is an element $g \in G$ such that $m_2 = \theta(g, m_1)$. G is said to act effectively on M if e is the only element of G such that $\theta(g,m) = m$ for all $m \in M$[2].

Let $\mathbf{R}_1^n$ be the vector space $\mathbf{R}^n$ provided with Lorentzian inner product

$$\langle x, y \rangle = -x_1 y_1 + \sum_{i=2}^{n} x_i y_i \ , \quad \text{for } x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n).$$

$\mathbf{R}_1^n$ is called Lorentz spaces of n-dimension[4].

Let $d \in \mathbf{R}^+$,

$$S_1^n = \{(x_1, x_2, ..., x_{n+1}) \in \mathbf{R}_1^{n+1} | -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = 1\}$$

Received by the editors Nov. 10, 2005, Accepted: Dec.09, 2005.

Key words and phrases. Lorentzian space, Lie Group, C^∞-Action, Lorentzian Sphere, Lorentzian Cylinder.

$$S^{n-1} = \{(x_1, x_2, ..., x_n) \in \mathbf{R}^n | \sum_{i=1}^{n} x_i^2 = 1\}$$

$$\overline{S_1^1} = \{(x_1, x_2) \in S_1^2 | x_1 \geq 1\}$$

$$\overline{S^1} = \{(x_1, x_2) \in S^1 | x_1 \geq 0\}$$

$$LC_d^n = \{(x_1, x_2, ..., x_{n+1}) \in \mathbf{R}_1^{n+1} | -x_1^2 + \sum_{i=2}^{n} x_i^2 = d^2\}.$$

The function $\otimes$ Lorentz spherical product is defined by

$$\otimes : \overline{S_1^1} \times S^{n-1} \to S_1^n \ , \ \otimes((a,b),(x_1, x_2, ..., x_n)) = (ax_1, ax_2, ..., ax_n, b).$$

The function $\boxtimes$ Lorentz cylinderical product is defined by

$$\boxtimes : (\{d\} \times IR) \times S_1^{n-1} \to LC_d^n \ , \ \boxtimes((d,a),(x_1, x_2, ..., x_n)) = (dx_1, dx_2, ..., dx_n, a).$$

These functions are one-to-one and onto[5].

2. Lie Group Structure of Lorentzian Sphere S_1^3

We consider the Lorentz spherical product in $\mathbf{R}_1^4$ by

$$\otimes : \overline{S^1} \times S_1^2 \to S_1^3 \ , \ \otimes((a,b),(x_1, x_2, x_3)) = (ax_1, ax_2, ax_3, b).$$

We define a binary operation on S_1^3 by

$$\odot : S_1^3 \times S_1^3 \to S_1^3,$$

$$x \odot y = \begin{pmatrix} \left(\sqrt{1-x_4^2}\sqrt{1-y_4^2} - x_4 y_4\right) \left(\dfrac{\sqrt{1-\frac{x_3^2}{1-x_4^2}}\sqrt{1-\frac{y_3^2}{1-y_4^2}} - \frac{x_3 y_3}{\sqrt{1-x_4^2}\sqrt{1-y_4^2}}}{\sqrt{1-\frac{x_3^2}{1-x_4^2}}\sqrt{1-\frac{y_3^2}{1-y_4^2}}\sqrt{1-x_4^2}\sqrt{1-y_4^2}} \right) \\ (x_1 y_2 + x_2 y_1, x_1 y_1 + x_2 y_2), \\ \left(\sqrt{1-x_4^2}\sqrt{1-y_4^2} - x_4 y_4\right) \left(\frac{x_3}{\sqrt{1-x_4^2}}\sqrt{1-\frac{y_3^2}{1-y_4^2}} + \frac{y_3}{\sqrt{1-y_4^2}}\sqrt{1-\frac{x_3^2}{1-x_4^2}} \right), \\ \sqrt{1-x_4^2}\,y_4 + \sqrt{1-y_4^2}\,x_4 \end{pmatrix}$$

for $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4)$, where the function $\odot$ is defined by the function $\otimes$.

The function $\odot$ is associative. The identity element e of S_1^3 according to the function $\odot$ is $(0,1,0,0)$. Also, the inverse element of all $x = (x_1, x_2, x_3, x_4) \in S_1^3$ is $(-x_1, x_2, -x_3, -x_4)$. Consequently, $(S_1^3, \odot)$ is a group.

Let

$$U_i = \left\{ (x_1, x_2, x_3, x_4) \in S_1^3 \middle| \begin{array}{l} x_{i+1} > 0, 1 \leq i \leq 3 \\ x_{i-2} < 0, 4 \leq i \leq 6 \end{array} \right\}, 1 \leq i \leq 6.$$

$$\varphi_i : U_i \to \mathbf{R}_1^3 \ , \ \varphi_i(x_1, x_2, x_3, x_4) = \begin{cases} (x_1, ..., \widehat{x}_{i+1}, ..., x_4) & \text{if } 1 \leq i \leq 3 \\ (x_1, ..., \widehat{x}_{i-2}, ..., x_4) & \text{if } 4 \leq i \leq 6 \end{cases},$$

then S_1^3 is a differentiable manifold together with its C^∞-structure $\{(U_i, \varphi_i)\}_{1 \leq i \leq 6}$.

$$\begin{array}{ccc} S_1^3 \times S_1^3 & \xrightarrow{\quad \odot \quad} & S_1^3 \\ \downarrow \varphi_i \times \varphi_j & & \downarrow \varphi_k \quad 1 \leq i, j, k \leq 6. \\ \mathbf{R}_1^3 \times \mathbf{R}_1^3 & \xrightarrow{\quad \phi \quad} & \mathbf{R}_1^3 \end{array}$$

From the above diagram, the function ϕ, which is the coordinate representative of the function $\odot$ is differentiable. Consequently, S_1^3 is a Lie group.

3. Lie Group Structure of Lorentzian Cylinder LC_d^4 and Its A C^∞-Action

We define a binary operation on Lorentzian cylinder LC_d^4 in $\mathbf{R}_1^5$ by

$$\boxdot : LC_d^4 \times LC_d^4 \to LC_d^4,$$

$$x\boxdot y = \begin{pmatrix} \left(\frac{\sqrt{d^2-x_4^2}\sqrt{d^2-y_4^2}-x_4y_4}{d}\right)\left(\frac{\sqrt{1-\frac{x_3^2}{d^2-x_4^2}}\sqrt{1-\frac{y_3^2}{d^2-y_4^2}}-\frac{x_3y_3}{\sqrt{d^2-x_4^2}\sqrt{d^2-y_4^2}}}{\sqrt{1-\frac{x_3^2}{d^2-x_4^2}}\sqrt{1-\frac{y_3^2}{d^2-y_4^2}}\sqrt{d^2-x_4^2}\sqrt{d^2-y_4^2}}\right) \\ (x_1y_2+x_2y_1, x_1y_1+x_2y_2), \\ \left(\frac{\sqrt{d^2-x_4^2}\sqrt{d^2-y_4^2}-x_4y_4}{d}\right)\left(\frac{x_3}{\sqrt{d^2-x_4^2}}\sqrt{1-\frac{y_3^2}{d^2-y_4^2}}+\frac{y_3}{\sqrt{d^2-y_4^2}}\sqrt{1-\frac{x_3^2}{d^2-x_4^2}}\right), \\ \frac{\sqrt{d^2-x_4^2}y_4+\sqrt{d^2-y_4^2}x_4}{d}, x_5+y_5 \end{pmatrix}$$

for $x=(x_1,x_2,x_3,x_4,x_5)$, $y=(y_1,y_2,y_3,y_4,y_5)$, where the function $\boxdot$ is defined by Lorentz cylinderical product and by Lorentz spherical product in section 2.

The function $\boxdot$ is associative. The identity element e of LC_d^4 according to the function $\boxdot$ is $(0,d,0,0,0)$. Also, the inverse element of all $x=(x_1,x_2,x_3,x_4,x_5)\in LC_d^4$ is $(-x_1,x_2,-x_3,-x_4,-x_5)$. Consequently, $\left(LC_d^4,\boxdot\right)$ is a group.

The function π is defined by

$$\pi : \{d\}\times\mathbf{R}\to\mathbf{R}\ ,\ \pi(d,a)=a.$$

$\{d\}\times\mathbf{R}$ is a differentiable manifold together with its C^∞-structure $\{\{d\}\times\mathbf{R},\pi\}$.

Let $V_i=(\{d\}\times\mathbf{R})\boxtimes U_i$, $1\le i\le 6$, where $\{(U_i,\varphi_i)\}_{1\le i\le 6}$ is C^∞-structure of S_1^3. Then, LC_d^4 is a differentiable manifold together with its C^∞-structure $\{(V_i,(\pi\times\varphi_i)\circ\boxtimes^{-1})\}_{1\le i\le 6}$.

$$\begin{array}{lcl} LC_d^4\times LC_d^4 & \xrightarrow{\quad\boxdot\quad} & LC_d^4 \\ \downarrow\left((\pi\times\varphi_i)\circ\boxtimes^{-1}\right)\times\left((\pi\times\varphi_j)\circ\boxtimes^{-1}\right) & & \downarrow(\pi\times\varphi_k)\circ\boxtimes^{-1} \quad 1\le i,j,k\le 6. \\ \mathbf{R}_1^4\times\mathbf{R}_1^4 & \xrightarrow{\quad\psi\quad} & \mathbf{R}_1^4 \end{array}$$

From the above diagram, the function ψ, which is the coordinate representative of the function $\boxdot$ is differentiable. Consequently, LC_d^4 is a Lie group.

Let us consider the function

$$\theta : LC_1^4\times LC_d^4\to LC_d^4,$$

which is defined by

$$\theta(x,y) = \begin{pmatrix} \left(\sqrt{1-x_4^2}\sqrt{d^2-y_4^2}-x_4y_4\right)\left(\frac{\sqrt{1-\frac{x_3^2}{1-x_4^2}}\sqrt{1-\frac{y_3^2}{d^2-y_4^2}}-\frac{x_3y_3}{\sqrt{1-x_4^2}\sqrt{d^2-y_4^2}}}{\sqrt{1-\frac{x_3^2}{1-x_4^2}}\sqrt{1-\frac{y_3^2}{d^2-y_4^2}}\sqrt{1-x_4^2}\sqrt{d^2-y_4^2}}\right) \\ (x_1y_2+x_2y_1, x_1y_1+x_2y_2), \\ \left(\sqrt{1-x_4^2}\sqrt{d^2-y_4^2}-x_4y_4\right)\left(\frac{x_3}{\sqrt{1-x_4^2}}\sqrt{1-\frac{y_3^2}{d^2-y_4^2}}+\frac{y_3}{\sqrt{d^2-y_4^2}}\sqrt{1-\frac{x_3^2}{1-x_4^2}}\right), \\ \sqrt{1-x_4^2}y_4+\sqrt{d^2-y_4^2}x_4, x_5+y_5 \end{pmatrix}$$

for every $x = (x_1, x_2, x_3, x_4, x_5) \in LC_1^4$ and every $y = (y_1, y_2, y_3, y_4, y_5) \in LC_d^4$.

Thus the following theorem can be given.

Theorem 3.1. *The function θ is a C^∞-action of the Lie group LC_1^4 on the differentiable manifold LC_d^4.*

Proof. i) The differentiability of the function θ can be shown analogously to the differentiability of the function $\boxdot$.

ii) $\theta(x, \theta(y, p)) = \theta(x \boxdot y, p)$ for every $x, y \in LC_1^4$ and every $p \in LC_d^4$.

iii) $\theta(e, p) = p$ for $e = (0, 1, 0, 0, 0) \in LC_1^4$ and every $p \in LC_d^4$. □

Theorem 3.2. *The C^∞-action θ is transitive.*

Proof. For any $p, q \in LC_d^4$, there exists $x \in LC_1^4$ such that $p = \theta(x, q)$. □

Corollary 1. *Let $(LC_1^4)_{\{p\}}$ for any $p \in LC_d^4$ denotes the orbit of p with respect to the action θ. Then*

$$(LC_1^4)_{\{p\}} = LC_d^4.$$

Theorem 3.3. *The Lie group LC_1^4 acts effectively on the differentiable manifold LC_d^4.*

Proof. For any $m \in LC_d^4$ the equality $\theta(g, m) = m$ is satisfied only for $g = e$. Let $(\mathbf{R}_1^5)_B = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}_1^5 \,|\, |x_1| < \sqrt{x_2^2 + x_3^2 + x_4^2}\}$. $LC_d^4 \subset (\mathbf{R}_1^5)_B$ for any $d \in \mathbf{R}^+$. Let us define the function θ' by

$$\theta' : LC_1^4 \times (\mathbf{R}_1^5)_B \to (\mathbf{R}_1^5)_B \ , \ \theta'(x, y) = \theta(x, y).$$

The function θ' is a C^∞-action.

Let the orbit under the action θ' of any point $p = (p_1, p_2, p_3, p_4, p_5) \in (IR_1^5)_B$ is denoted by $(LC_1^4)'_{\{p\}}$ and $d^2 = -p_1^2 + p_2^2 + p_3^2 + p_4^2$. Then

$$(LC_1^4)'_{\{p\}} = LC_d^4.$$

□

ÖZET: Bu çalışmada R_1^4 de Lorentz birim küresi ve R_1^5 de Lorentz silindiri üzerinde birer grup işlemi tanımlandı ve bu işlemlerle birlikte bunların birer Lie grubu olduğu gösterildi. R_1^5 de 1-yarıçaplı Lorentz silindir Lie grubunun, keyfi yarıçaplı Lorentz silindir manifoldu üzerine bir C^∞-etkisi tanımlanarak bu etkinin bazı özellikleri incelendi. Ayrıca bu etki R_1^5 üzerine genişletildi.

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