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Fixed point and common fixed point theorems on ordered cone metric spaces

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ABSTRACT

In the present work, some fixed point and common fixed point theorems for self-maps on ordered cone metric spaces, where the cone is not necessarily normal, are proved.

1. Introduction

Huang and Zhang [1] have introduced the concept of cone metric space, replacing the set of real numbers by an ordered Banach space. They have proved some fixed point theorems of contractive type mappings on cone metric spaces. Later, many authors generalized their fixed point theorems. The purpose of this work is to carry some of these theorems to ordered cone metric spaces.

We recall some definitions of cone metric spaces and some of their properties [1]. Let E be a real Banach space and P be a subset of E. By θ we denote the zero element of E and by $Int\,P$ the interior of P. The subset P is called a cone if and only if:

(i) P is closed, nonempty and $P \neq \{\theta\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Longrightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Longrightarrow x = \theta$.

A cone P is called solid if it contains interior points, that is, if $Int\,P \neq \emptyset$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in Int\,P$.

The cone P in a real Banach space E is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \quad \text{implies} \quad \|x\| \preceq K\,\|y\|\,. \tag{1.1}$$

The least positive number K satisfying (1.1) is called the normal constant of P. It is clear that $K \geq 1$.

In the following we always suppose that E is a Banach space, P is a cone in E with $Int\,P \neq \emptyset$ and $\preceq$ is a partial ordering with respect to P.

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Definition 1 ([1]). Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(d_1) $\theta \prec d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$,
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(d_3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is obvious that the cone metric spaces generalize metric spaces.

Example 2 ([1]). Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E \mid x, y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ be such that $d(x, y) = (|x - y|, \alpha |x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 3 ([1]). Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there is an N such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in E$ with $\theta \ll c$ there is an N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X. (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Lemma 4 ([1]). *Let (X, d) be a cone metric space, P be a normal cone and $\{x_n\}$ be a sequence in X. Then:*

(i) *$\{x_n\}$ converges to x if and only if $d(x_n, x) \to \theta$ $(n \to \infty)$,*
(ii) *$\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ $(n, m \to \infty)$.*

Let (X, d) be a cone metric space, $f : X \to X$ and $x_0 \in X$. Then the function f is continuous at x_0 if for any sequence $x_n \to x_0$ we have $fx_n \to fx_0$ [2].
The following theorem has been proved by Huang and Zhang.

Theorem 5 ([1]). *Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $f : X \to X$ satisfies the contractive condition*

$$d(fx, fy) \preceq kd(x, y) \quad \text{for all } x, y \in X,$$

where $k \in [0, 1)$ is a constant. Then f has a unique fixed point in X.

Also, Huang and Zhang [1] gave an example showing that Theorem 5 is a generalization of the Banach fixed point principle.
Rezapour and Hamlbarani [3] proved that there are no normal cones with a normal constant $K < 1$ and that for each $h > 1$ there are cones with normal constant $K > h$. Also, omitting the assumption of normality they obtain generalizations of some results of [1].
The following example shows that there are non-normal cones.

Example 6 ([3]). Let $E = C_R^2([0, 1])$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and consider the cone $P = \{f \in E : f \geq 0\}$. For each $K > 1$, put $f(x) = x$ and $g(x) = x^{2K}$. Then $0 \leq g \leq f$, $\|f\| = 2$ and $\|g\| = 2K + 1$. Since $K\|f\| < \|g\|$, K is not the normal constant of P. Therefore, P is a non-normal cone.

Since the definition of the concept of cone metric space, the fixed point theory on these spaces has been developing (see [4–18]).
On the other hand, the existence of fixed points in partially ordered sets has been considered recently in [19]. The usual contraction of the Banach fixed point principle is weakened but at the expense that the operator is monotone. Ran and Reurings [19] proved the following theorem for in ordered metric spaces.

Theorem 7 ([20,19]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a metric d in X such that the metric space (X, d) is complete. Let $f : X \to X$ be a continuous and nondecreasing mapping w.r.t. $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) *there exists $k \in (0, 1)$ such that $d(fx, fy) \leq kd(x, y)$ for each $x, y \in X$ with $y \sqsubseteq x$;*
(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.*

Then f has a fixed point $x^* \in X$.

Some generalizations and variants of the result of [19] are given in [21–27]. For example, in [20], the following theorem has been proved, by removing the continuity of f in Theorem 7.

Theorem 8 ([20]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a metric d in X such that the metric space (X, d) is complete. Let $f : X \to X$ be a nondecreasing mapping w.r.t. $\sqsubseteq$. Suppose that the following three assertions hold:*

(i) *there exists $k \in (0, 1)$ such that $d(fx, fy) \leq kd(x, y)$ for each $x, y \in X$ with $y \sqsubseteq x$;*
(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$;*
(iii) *if an increasing sequence $\{x_n\}$ converges to x in X, then $x_n \sqsubseteq x$ for all n.*

Then f has a fixed point $x^* \in X$.

In Theorem 7 [19] it is proved that if

$$\text{every pair of elements has a lower bound and an upper bound,} \tag{1.2}$$

then for every $x \in X$, $\lim_{n\to\infty} f^n x = y$, where y is the fixed point of f such that $y = \lim_{n\to\infty} f^n x_0$ and hence f has a unique fixed point. If condition (1.2) fails, it is possible to find examples of functions f with more than one fixed point. In [20] some examples are presented to illustrate this fact.

The following example shows the differences between the fixed point theorems working on metric and ordered metric spaces.

Example 9. Let $X = \mathbb{R}$ and consider a relation on X as follows:

$$x \sqsubseteq y \iff \{(x = y) \text{ or } (x, y \in [1, 4] \text{ with } x \le y)\}.$$

It is easy to see that $\sqsubseteq$ is a partial order on X. Let d be usual metric on X. Now define a self-map of X as follows: $fx = 2x$ if $x < 1$, $fx = \frac{x+5}{3}$ if $1 \le x \le 4$ and $fx = 2x - 5$ if $x > 4$. Now, it is easy to see that all conditions of Theorem 7 are satisfied and so f has a fixed point on X. Also note that the contractive condition of the Banach fixed point principle is not satisfied.

In [6], Altun and Durmaz have proved the following theorem using the ideas of Theorems 5 and 7.

Theorem 10 ([6]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete and let P be a normal cone with normal constant K. Let $f : X \to X$ be a continuous and nondecreasing mapping w.r.t. $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) *there exists $k \in (0, 1)$ such that $d(fx, fy) \preceq kd(x, y)$ for each $x, y \in X$ with $y \sqsubseteq x$;*
(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.*

Then f has a fixed point $x^* \in X$.

The following example was given in [6]; we can apply Theorem 10 to this example, but we cannot apply Theorems 5 and 7 because the contractive condition of Theorem 5 is not satisfied and that the conditions of Theorem 7 are not satisfied if d is the usual metric in $\mathbb{R}^2$.

Example 11 ([6]). Let $E = \mathbb{R}^2$, the Euclidean plane, and $P = \{(x, y) \in \mathbb{R}^2 : x, y \ge 0\}$, a normal cone in E. Let $X = \{(x, 0) \in \mathbb{R}^2 : x \ge 0\} \cup \{(0, x) \in \mathbb{R}^2 : x \ge 0\}$ and consider a relation on X as follows: for $(x, y), (z, w) \in X$,

$$(x, y) \trianglerighteq (z, w) \iff \{x \sqsubseteq z \text{ and } y \sqsubseteq w\},$$

where $\sqsubseteq$ is a partial order on $\mathbb{R}$ as follows: for $x, y \in \mathbb{R}$,

$$x \sqsubseteq y \iff \{(x = y) \text{ or } (x, y \in [0, 1] \text{ with } x \le y)\}.$$

We can show that $\trianglerighteq$ is a partial order on X. Let $d : X \times X \to E$ be defined by (as in [1]) $d((x, 0), (y, 0)) = (\frac{4}{3}|x - y|, |x - y|)$, $d((0, x), (0, y)) = (|x - y|, \frac{2}{3}|x - y|)$ and $d((x, 0), (0, y)) = d((0, y), (x, 0)) = (\frac{4}{3}x + y, x + \frac{2}{3}y)$. Then (X, d) is a complete cone metric space.

Let a mapping $f : X \to X$ be defined with $f(x, 0) = (0, x)$, $f(0, x) = (2x - \frac{3}{2}, 0)$ if $x > 1$ and $f(0, x) = (\frac{x}{2}, 0)$ if $0 \le x \le 1$. It is easy to see that f is continuous and nondecreasing w.r.t. $\trianglerighteq$. Also f satisfies the condition (i) of Theorem 10 with $k = \frac{3}{4}$. Again, since $(0, 0) \trianglerighteq f(0, 0)$, then the condition (ii) of the theorem is satisfied. Therefore, we can apply Theorem 10 to this example. Note that the contractive condition of Theorem 5 is not satisfied and that the conditions of Theorem 7 are not satisfied if d is the usual metric in $\mathbb{R}^2$.

The aim of the work is to give some generalized versions of Theorem 10 in ordered cone metric spaces, where a cone P is not necessarily normal.

2. Main results

Theorem 12. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete. Let $f : X \to X$ be a continuous and nondecreasing mapping w.r.t. $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) *there exist $\alpha, \beta, \gamma \ge 0$ with $\alpha + 2\beta + 2\gamma < 1$ such that*

$$d(fx, fy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)]$$

for all $x, y \in X$ with $y \sqsubseteq x$,

(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.*

Then f has a fixed point $x^* \in X$.

Proof. If $fx_0 = x_0$, then the proof is finished. Suppose that $fx_0 \ne x_0$. Since $x_0 \sqsubseteq fx_0$ and f is nondecreasing w.r.t. $\sqsubseteq$, we obtain by induction that

$$x_0 \sqsubseteq fx_0 \sqsubseteq f^2x_0 \sqsubseteq \cdots \sqsubseteq f^n x_0 \sqsubseteq f^{n+1}x_0 \sqsubseteq \cdots.$$

Now, we have

$$d(f^{n+1}x_0, f^n x_0) \preceq \alpha d(f^n x_0, f^{n-1}x_0) + \beta[d(f^n x_0, f^{n+1}x_0) + d(f^{n-1}x_0, f^n x_0)] + \gamma d(f^{n-1}x_0, f^{n+1}x_0)$$
$$\preceq \alpha d(f^n x_0, f^{n-1}x_0) + \beta[d(f^n x_0, f^{n+1}x_0) + d(f^{n-1}x_0, f^n x_0)] + \gamma[d(f^{n-1}x_0, f^n x_0) + d(f^n x_0, f^{n+1}x_0)]$$

and so

$$d(f^{n+1}x_0, f^n x_0) \preceq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right) d(f^n x_0, f^{n-1}x_0),$$

for all $n \geq 1$. Repeating this relation we get

$$d(f^{n+1}x_0, f^n x_0) \preceq k^n d(fx_0, x_0), \tag{2.1}$$

where $k = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} < 1$. Let $m > n$; then from (2.1), we have

$$\begin{aligned} d(f^m x_0, f^n x_0) &\preceq d(f^m x_0, f^{m-1}x_0) + \cdots + d(f^{n+1}x_0, f^n x_0) \\ &\preceq (k^{m-1} + \cdots + k^n) d(fx_0, x_0) \\ &\preceq \frac{k^n}{1-k} d(fx_0, x_0). \end{aligned}$$

Therefore, we get

$$d(f^m x_0, f^n x_0) \preceq \frac{k^n}{1-k} d(fx_0, x_0). \tag{2.2}$$

Now we show that $\{f^n x_0\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, d). Let $\theta \ll c$ be arbitrary. Since $c \in Int\, P$, there is a neighborhood of θ:

$$N_\delta(\theta) = \{y \in E : ||y|| < \delta\}, \quad \delta > 0,$$

such that $c + N_\delta(\theta) \subseteq Int\, P$. Choose a natural number N_1 such that $\left|\left| -\frac{k^{N_1}}{1-k} d(fx_0, x_0) \right|\right| < \delta$. Then $-\frac{k^n}{1-k} d(fx_0, x_0) \in N_\delta(\theta)$ for all $n \geq N_1$. Hence $c - \frac{k^n}{1-k} d(fx_0, x_0) \in c + N_\delta(\theta) \subseteq Int\, P$. Thus we have

$$\frac{k^n}{1-k} d(fx_0, x_0) \ll c \quad \text{for all } n \geq N_1.$$

Therefore, from (2.2) we get

$$d(f^m x_0, f^n x_0) \preceq \frac{k^n}{1-k} d(fx_0, x_0) \ll c \quad \text{for all } m > n \geq N_1$$

and hence

$$d(f^m x_0, f^n x_0) \ll c \quad \text{for all } m > n \geq N_1.$$

Hence we conclude that $\{f^n x_0\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete cone metric space, there exists $x^* \in X$ such that $f^n x_0 \to x^*$ as $n \to \infty$. Finally, continuity of f and $f(f^n x_0) = f^{n+1}x_0 \to x^*$ imply that $fx^* = x^*$. Thus we proved that x^* is a fixed point of f. □

If we use the condition (iii) instead of the continuity of f in Theorem 12 we have the following result.

Theorem 13. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete. Let $f : X \to X$ be a nondecreasing mapping w.r.t. $\sqsubseteq$. Suppose that the following three assertions hold:*

(i) *there exist $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + 2\gamma < 1$ such that*

$$d(fx, fy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)],$$

for all $x, y \in X$ with $y \sqsubseteq x$,

(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$,*

(iii) *if an increasing sequence $\{x_n\}$ converges to x in X, then $x_n \sqsubseteq x$ for all n.*

Then f has a fixed point $x^* \in X$.

Proof. If we take $x_n = f^n x_0$ in the proof of Theorem 12, then we have $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots$, that is, $\{x_n\}$ is an increasing sequence. Also this sequence converges to x^*. Now the condition (iii) implies $x_n \sqsubseteq x^*$ for all n. Therefore, we can use the condition (i) and so we have

$$d(fx_n, fx^*) \preceq \alpha d(x_n, x^*) + \beta[d(x_n, fx_n) + d(x^*, fx^*)] + \gamma[d(x_n, fx^*) + d(x^*, fx_n)].$$

Taking $n \to \infty$, we have $d(x^*, fx^*) \preceq (\beta + \gamma)d(x^*, fx^*)$. Then by the condition (i) we have $d(x^*, fx^*) \preceq (1/2)\, d(x^*, fx^*)$ and hence $(1/2)\, d(x^*, fx^*) \preceq \theta$. Therefore, $-d(x^*, fx^*) \in P$ and so, as also $d(x^*, fx^*) \in P$, we have $d(x^*, fx^*) = \theta$. Hence $x^* = fx^*$. □

Now we give two common fixed point theorems on ordered cone metric spaces. We need the following definition.

Definition 14. Let $(X, \sqsubseteq)$ be a partially ordered set. Two mappings $f, g : X \to X$ are said to be weakly increasing if $fx \sqsubseteq gfx$ and $gx \sqsubseteq fgx$ hold for all $x \in X$.

Note that the two weakly increasing mappings need not be nondecreasing.

Example 15. Let $X = [0, \infty)$, endowed with usual ordering. Let $f, g : X \to X$ be defined by $fx = x$ if $0 \le x \le 1$, $fx = 0$ if $1 < x < \infty$, $gx = \sqrt{x}$ if $0 \le x \le 1$ and $gx = 0$ if $1 < x < \infty$; then it is obvious that $fx \le gfx$ and $gx \le fgx$ for all $x \in X$. Thus f and g are weakly increasing mappings. Note that both f and g are not nondecreasing.

Example 16. Let $X = [1, \infty) \times [1, \infty)$ be endowed with the usual ordering, that is, $(x, y) \sqsubseteq (z, w) \Leftrightarrow x \le z$ and $y \le w$. Let $f, g : X \to X$ be defined by $f(x, y) = (2x, 3y)$ and $g(x, y) = (x^2, y^2)$; then $f(x, y) = (2x, 3y) \sqsubseteq gf(x, y) = g(2x, 3y) = (4x^2, 9y^2)$ and $g(x, y) = (x^2, y^2) \sqsubseteq fg(x, y) = f(x^2, y^2) = (2x^2, 3y^2)$. Thus f and g are weakly increasing mappings.

Example 17. Let $X = \mathbb{R}^2$, endowed with lexicographical ordering, that is, $(x, y) \lesssim (z, w) \Leftrightarrow (x < z)$ or (if $x = z$, then $y \le w$). Let $f, g : X \to X$ be defined by $f(x, y) = (\max\{x, y\}, \min\{x, y\})$ and $g(x, y) = (\max\{x, y\}, \frac{x+y}{2})$; then f and g are weakly increasing mappings. Note that $(1, 4) \lesssim (2, 3)$ but $f(1, 4) = (4, 1) \not\lesssim (3, 2) = f(2, 3)$; then f is not nondecreasing. Similarly g is not nondecreasing.

Let us remark that in Theorem 18 we do not assume explicitly the condition "there exists an $x_0 \in X$ with $x_0 \sqsubseteq f(x_0)$" of Theorem 12.

Now we prove the following common fixed point theorem.

Theorem 18. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete. Let $f, g : X \to X$ be two weakly increasing mappings w.r.t. $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) *there exist $\alpha, \beta, \gamma \ge 0$ with $\alpha + 2\beta + 2\gamma < 1$ such that*

$$d(fx, gy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, gy)] + \gamma[d(x, gy) + d(y, fx)], \tag{2.3}$$

for all comparative $x, y \in X$,

(ii) *f or g is continuous.*

Then f and g have a common fixed point $x^* \in X$.

Proof. Let x_0 be an arbitrary point of X and define a sequence $\{x_n\}$ in X as follows: $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \ge 0$. Note that, since f and g are weakly increasing, we have $x_1 = fx_0 \sqsubseteq gfx_0 = gx_1 = x_2$, and $x_2 = gx_1 \sqsubseteq fgx_1 = fx_2 = x_3$ and continuing this process we have $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots$. That is, the sequence $\{x_n\}$ is nondecreasing. Now, since x_{2n} and x_{2n+1} are comparative, we can use the inequality (2.3), and then we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(fx_{2n}, gx_{2n+1}) \\ &\preceq (\alpha + \beta + \gamma)d(x_{2n}, x_{2n+1}) + (\beta + \gamma)d(x_{2n+1}, x_{2n+2}) \end{aligned}$$

which implies that $d(x_{2n+1}, x_{2n+2}) \preceq kd(x_{2n}, x_{2n+1})$, where $k = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} < 1$. Similarly, it can be shown that $d(x_{2n+3}, x_{2n+2}) \preceq kd(x_{2n+2}, x_{2n+1})$. Therefore,

$$d(x_{n+1}, x_{n+2}) \preceq kd(x_n, x_{n+1}) \preceq k^2 d(x_{n-1}, x_n) \preceq \cdots \preceq k^{n+1} d(x_0, x_1)$$

for all $n \ge 1$. Let $m > n$; then we have

$$\begin{aligned} d(x_m, x_n) &\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\preceq \frac{k^n}{1-k} d(x_0, x_1). \end{aligned}$$

Therefore, we get

$$d(x_m, x_n) \le \frac{k^n}{1-k} d(x_0, x_1).$$

Hence, like in the proof of Theorem 12, one can prove that $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $x^* \in X$ such that $x_n \to x^* (n \to \infty)$.

Suppose that f is continuous. Then it is clear that x^* is a fixed point of f. We must show that x^* is also a fixed point of g. Since $x^* \sqsubseteq x^*$ we can use the inequality (2.3) for $x = y = x^*$; then we have

$$d(fx^*, gx^*) \preceq \alpha d(x^*, x^*) + \beta[d(x^*, fx^*) + d(x^*, gx^*)] + \gamma[d(x^*, gx^*) + d(x^*, fx^*)]$$

and so

$$d(x^*, gx^*) \preceq (\beta + \gamma) d(x^*, gx^*). \tag{2.4}$$

The condition (2.4) means that $(\beta + \gamma)d(x^*, gx^*) - d(x^*, gx^*) \in P$, that is, $-(1 - (\beta + \gamma))\, d(x^*, gx^*) \in P$. Hence, as $(1 - (\beta + \gamma))^{-1} > 0$, we have

$$-(1 - (\beta + \gamma))^{-1} (1 - (\beta + \gamma))\, d(x^*, gx^*) = -d(x^*, gx^*) \in P.$$

Since also $d(x^*, gx^*) \in P$, then $d(x^*, gx^*) \in P \cap (-P) = \theta$. Thus $d(x^*, gx^*) = \theta$ and hence $gx^* = x^*$. Similarly, if g is continuous, again we have $fx^* = x^*$. Therefore, we have proved that f and g have a common fixed point. □

The following theorem is a variant of Theorem 18.

Theorem 19. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete. Let $f, g : X \to X$ be two weakly increasing mappings w.r.t. $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) *there exist $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + 2\gamma < 1$ such that*

$$d(fx, gy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, gy)] + \gamma[d(x, gy) + d(y, fx)]$$

for all comparative $x, y \in X$,

(ii) *if an increasing sequence $\{x_n\}$ converges to x in X, then $x_n \sqsubseteq x$ for all n.*

Then f and g have a common fixed point $x^* \in X$.

Remark 20. We note that in our theorems we do not assume that a cone P is normal. Therefore, Theorems 18 and 19 are ordered versions and generalizations of the following recent result of Abbas and Rhoades.

Theorem 21 (*Abbas and Rhoades [5], Theorem 2.1*). *Let (X, d) be a complete cone metric space, and P a normal cone with normal constant K. Suppose that the mappings f and g are two self-maps of X satisfying*

$$d(fx, gy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, gy)] + \gamma[d(x, gy) + d(y, fx)]$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then f and g have a unique common fixed point in X; moreover, any fixed point of f is a fixed point of g, and conversely.

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