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GENERALIZED q-BASKAKOV OPERATORS

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ABSTRACT. In the present paper we propose a generalization of the Baskakov operators, based on q integers. We also estimate the rate of convergence in the weighted norm. In the last section, we study some shape preserving properties and the property of monotonicity of q-Baskakov operators.

1. Introduction

Along with the prevalence of q-analysis methods in approximation theory, the study of operators sequence has attracted more and more attention. Currently it continues being an important subject of study. It has been shown that linear positive operators constructed by q-numbers are quite effective as far as the rate of convergence is concerned and we can have some unexpected results, which are not observed for classical case. This type of construction was first used to generate Bernstein operators. In 1987, Lupas [17] defined a q-analogue of Bernstein operators and studied some approximation properties of them. In 1997, Phillips [25] (see also [27]) introduced another generalization of Bernstein operators based on the q-integers called q-Bernstein operators. Research results show that q-Bernstein operators possess good convergence and approximation properties in $C[0,1]$. These operators have been studied by a number of authors, we mention the some due to II'inskii and Ostrovska [16], Oruc and Tuncer [21], Ostrovska [22], [23] and Videnskii [28] etc. Heping [14], Heping and Fanjun [15] discussed Voronovskaya-type formulas and saturation of convergence for q-Bernstein polynomials for arbitrary fixed q, $0 < q < 1$. Further results on certain q-Bernstein Durrmeyer operators are also discussed recently by Finta and Gupta [10] and Gupta and Heping [13].

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The q-analogue of Szász Mirakyan operators was introduced and approximation properties of them were obtained in [4] and [5].

This, along with the recent work in this area, motivated us to study further in this direction. The aim of this paper is to study the approximation properties of a new generalization of the Baskakov operator based on q-integers. First, we recall classical Baskakov operators [7], which for $f \in C\,[0,\infty)$ are defined as

$$B_n\,(f,x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k\,(1+x)^{-n-k}\, f\left(\frac{k}{n}\right) \tag{1.1}$$

for $x \in [0,\infty)$ and $n \in \mathbb{N}$. We mention here that some generalizations of the operators (1.1) are proposed and studied by several researchers see e.g. ([12], [11], [1], [2], [8] and [24]).

We denote

$$(x,q)_n = (1-x)\,(1-qx)\ldots\left(1-q^{n-1}x\right) = \prod_{j=0}^{n-1}(1-q^j x).$$

Let parameter q be a positive real number and n a non-negative integer. $[n]_q$ denotes a q-integer, defined by

$$[n]_q = \begin{cases} (1-q^n)\,/\,(1-q)\,, & q \neq 1, \\ n, & q = 1. \end{cases}$$

The factorial of q-number $[n]_q$, which is defined by

$$[n]_q! = \begin{cases} [n]_q[n-1]_q\ldots[1]_q, & n = 1,2,\ldots, \\ 1, & n = 0 \end{cases}$$

is called q-factorial of n.

The q-binomial coefficient $\left[\begin{array}{c} n \\ k \end{array}\right]_q$ which is the generating function for restricted partitions is defined by

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q := \frac{(q,q)_n}{(q,q)_k(q,q)_{n-k}} = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

for $n \geq k \geq 1$, and has the value 1 when $k = 0$ and value zero otherwise (see [26] and [3]).

We recall that the q-derivative operator $\mathscr{D}_q$ is given by

$$\mathscr{D}_q f\,(x) := \frac{f\,(qx) - f\,(x)}{(q-1)\,x}, \qquad x \neq 0 \tag{1.2}$$

and $\mathscr{D}_q f\,(x)\big|_{x=0} := f'\,(0)$. Also $\mathscr{D}_q^0 f := f$, $\mathscr{D}_q^n f := \mathscr{D}_q\left(\mathscr{D}_q^{n-1} f\right)$, $n = 1,2,3,\ldots$.

The q-analogue of product and quotient rules are as follows:

$$\mathscr{D}_q\,(f\,(x)\,g\,(x)) = g\,(x)\,\mathscr{D}_q f\,(x) + f\,(qx)\,\mathscr{D}_q g\,(x)\,. \tag{1.3}$$

$$\mathscr{D}_q\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\,\mathscr{D}_q f(x)-f(x)\,\mathscr{D}_q g(x)}{g(x)\,g(qx)}. \tag{1.4}$$

For details see [9].

Very recently we [6] introduced the following q-analogues of the Baskakov operators. For $f\in C[0,\infty)$, $q\in(0,1)$ and each positive integer n, the q-Baskakov operators discussed in [6] are defined as

$$P_{n,q}(f;x)=\left(\frac{qx}{1+x},q\right)_n\sum_{k=0}^{\infty}f\left(\frac{[k]_q}{q^k[n]_q}\right)\begin{bmatrix} n+k-1\\ k\end{bmatrix}_q\left(\frac{qx}{1+x}\right)^k \tag{1.5}$$

and

$$P^*_{n,q}(f;x)=\left(\frac{q^2x}{1+x},q\right)_n\sum_{k=0}^{\infty}f\left(\frac{[k]_q}{q^{k+1}[n]_q}\right)\begin{bmatrix} n+k+1\\ k\end{bmatrix}_q\left(\frac{q^2x}{1+x}\right)^k. \tag{1.6}$$

The above q-analogues of Baskakov operators are defined for $q\in(0,1)$. For these analogues we are not able to study the q-derivatives and their applications. The new operators which we propose in the present paper are defined for $q>0$, also for the new generalization we obtain q-derivative and applications of q-derivative to them.

The present paper is organized as follows: in Section 2, we introduce a new generalization of Baskakov operators (1.1) by using q-integers which is different and improved from (1.5) and (1.6), we also establish moments using the q-derivatives. Also we propose a representation of the q-Baskakov operators in terms of divided differences. In Section 3, we deal with rate of convergence in the weighted norm. In the last section, we study some shape preserving properties and the property of monotonicity of q-Baskakov operators.

2. Construction of operators and some properties of them

For $f\in C[0,\infty)$, $q>0$ and each positive integer n, a new q-Baskakov operators can be defined as

$$\begin{aligned}\mathscr{B}_{n,q}(f,x) &= \sum_{k=0}^{\infty}\begin{bmatrix} n+k-1\\ k\end{bmatrix}_q q^{\frac{k(k-1)}{2}}x^k(-x,q)^{-1}_{n+k}\,f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\\ &= \sum_{k=0}^{\infty}\mathscr{P}^q_{n,k}(x)\,f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right).\end{aligned} \tag{2.1}$$

While for $q=1$ these polynomial coincide with the classical ones.

Definition 1. Let f be a function defined on an interval (a,b) and h be a positive real number. The q-forward differences ∇_h^r of f are defined recursively as

$$\begin{aligned} \nabla_q^0 f(x_j) &:= f(x_j), \\ \nabla_q^{r+1} f(x_j) &:= q^r \nabla_q^r f(x_{j+1}) - \nabla_q^r f(x_j) \end{aligned}$$

for $r \geq 0$.

Note that the above definition is different from definition given in [26, pp. 44].

As usual, we show divided differences with $f[x_0, x_1, \dots, x_n]$ at the abscissas $x_0, x_1, \dots, x_n$.

We now show following general relation that connect the divided differences $f[x_0, x_1, \dots, x_n]$ and q-forward differences.

Lemma 1. *For all $j, r \geq 0$, we have*

$$f[x_j, x_{j+1}, \dots, x_{j+r}] = q^{\frac{r(2j+r-1)}{2}} \frac{\nabla_q^r f(x_j)}{[r]_q!}, \tag{2.2}$$

where $x_j = \frac{[j]_q}{q^{j-1}}$.

Proof. Let us use induction on r. By Definition 1, the result is obvious for $r = 0$. Let us assume that the equality (2.2) is true for some $r \geq 0$ and all $j \geq 0$. Since

$$x_{j+r+1} - x_j = \frac{[r+1]_q}{q^{j+r}}$$

we have

$$\begin{aligned} & f[x_j, x_{j+1}, \dots, x_{j+r+1}] \\ &= \frac{f[x_{j+1}, \dots, x_{j+r+1}] - f[x_j, \dots, x_{j+r}]}{x_{j+r+1} - x_j} \\ &= \frac{q^{j+r}}{[r+1]_q} \left(q^{\frac{r(2j+r+1)}{2}} \frac{\nabla_q^r f(x_{j+1})}{[r]_q!} - q^{\frac{r(2j+r-1)}{2}} \frac{\nabla_q^r f(x_j)}{[r]_q!} \right) \\ &= q^{\frac{r(2j+r-1)}{2} + j + r} \left(\frac{q^r \nabla_q^r f(x_{j+1}) - \nabla_q^r f(x_j)}{[r+1]_q!} \right) \\ &= q^{\frac{(r+1)(2j+r)}{2}} \frac{\nabla_q^{r+1} f(x_j)}{[r+1]_q!}. \end{aligned}$$

□

Lemma 2. *For $n, k \geq 0$, we have*

$$\mathscr{D}_q \left[x^k (-x,q)_{n+k}^{-1} \right] = [k]_q x^{k-1} (-x,q)_{n+k}^{-1} - q^k x^k [n+k]_q (-x,q)_{n+k+1}^{-1} \tag{2.3}$$

P r o o f. First, we prove that $\mathscr{D}_q\left(-x,q\right)_n=[n]_q\left(-qx,q\right)_{n-1}$. Using q-derivative operator (1.2) we have

$$\begin{aligned}
\mathscr{D}_q\left(-x,q\right)_n &= \frac{1}{(q-1)\,x}\left(\prod_{j=0}^{n-1}\left(1+q^{j+1}x\right)-\prod_{j=0}^{n-1}\left(1+q^{j}x\right)\right)\\
&= \frac{1}{(q-1)\,x}\prod_{j=0}^{n-2}\left(1+q^{j+1}x\right)\left(\left(1+q^{n}x\right)-\left(1+x\right)\right)\\
&= \frac{q^n-1}{q-1}\prod_{j=0}^{n-2}\left(1+q^{j+1}x\right)\\
&= [n]_q\left(-qx,q\right)_{n-1}.
\end{aligned}$$

The q-derivative formula for a quotient (1.4) imply that

$$\begin{aligned}
\mathscr{D}_q\left(-x,q\right)_{n+k}^{-1} &= \frac{-\left[n+k\right]_q\left(-qx,q\right)_{n+k-1}}{\left(-x,q\right)_{n+k}\left(-qx,q\right)_{n+k}}\\
&= -\left[n+k\right]_q\left(-x,q\right)_{n+k+1}^{-1}. \qquad (2.4)
\end{aligned}$$

Also it is obvious that

$$\mathscr{D}_q x^k=[k]_q\,x^{k-1}. \qquad (2.5)$$

Then using (2.5) and (2.4), the result follows by (1.3)

$$\mathscr{D}_q\left[x^k\left(-x,q\right)_{n+k}^{-1}\right]=[k]_q\,x^{k-1}\left(-x,q\right)_{n+k}^{-1}-q^kx^k\left[n+k\right]_q\left(-x,q\right)_{n+k+1}^{-1}.$$

□

We wish to calculate the moments. For this purpose we give q-derivative of $B_{n,q}$. Next Theorem gives a representation of the rth derivative of $B_{n,q}$ in terms of q-forward differences.

Theorem 1. *Let $r\geq 0$. Then the rth derivative of q-Baskakov operator has the representation*

$$\mathscr{D}_q^rB_{n,q}(f,x)=\frac{[n+r-1]_q!}{[n-1]_q!}\sum_{k=0}^{\infty}q^{rk}\mathscr{P}^q_{n+r,k}\left(x\right)\nabla_q^rf\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \qquad (2.6)$$

P r o o f. We use induction on r. Equality (2.3),

$$\begin{bmatrix} n+k\\ k+1\end{bmatrix}_q[k+1]_q=[n]_q\begin{bmatrix} n+k\\ k\end{bmatrix}_q$$

and

$$\begin{bmatrix} n+k-1\\ k\end{bmatrix}_q[n+k]_q=[n]_q\begin{bmatrix} n+k\\ k\end{bmatrix}_q$$

imply that

$$
\begin{aligned}
&\mathscr{D}_q B_{n,q}(f,x)\\
&= \sum_{k=1}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} [k]_q \, x^{k-1} \, (-x,q)_{n+k}^{-1} \, f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\\
&\quad - \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} q^k x^k \, [n+k]_q \, (-x,q)_{n+k+1}^{-1} \, f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\\
&= [n]_q \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}+k} x^k \, (-x,q)_{n+k+1}^{-1} \left(f\left(\frac{[k+1]_q}{q^k[n]_q}\right) - f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\right)\\
&= [n]_q \sum_{k=0}^{\infty} q^k \mathscr{P}_{n+1,k}^q \, (x) \, \nabla_q^1 f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right).
\end{aligned}
$$

It is clear that (2.6) holds for $r=1$. Let us assume that (2.6) holds for some $r \geq 2$. Applying q-derivative operator to (2.6) we have

$$
\begin{aligned}
&\mathscr{D}_q^{r+1} \left(B_{n,q}(f,x)\right)\\
&= \frac{[n+r-1]_q!}{[n-1]_q!} \sum_{k=1}^{\infty} \begin{bmatrix} n+k+r-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}+rk} [k]_q\\
&\quad \times x^{k-1} \, (-x,q)_{n+k+r}^{-1} \, \nabla_q^r f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\\
&\quad - \frac{[n+r-1]_q!}{[n-1]_q!} \sum_{k=0}^{\infty} \begin{bmatrix} n+k+r-1 \\ k \end{bmatrix}_q [n+k+r]_q q^k q^{\frac{k(k-1)}{2}+rk}\\
&\quad \times x^k \, (-x,q)_{n+k+r+1}^{-1} \, \nabla_q^r f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\\
&= \frac{[n+r]_q!}{[n-1]_q!} \sum_{k=0}^{\infty} \begin{bmatrix} n+k+r \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}+(r+1)k} x^k \, (-x,q)_{n+k+r+1}^{-1}\\
&\quad \times \left(q^r \nabla_q^r f\left(\frac{[k+1]_q}{q^k[n]_q}\right) - \nabla_q^r f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\right)\\
&= \frac{[n+r]_q!}{[n-1]_q!} \sum_{k=0}^{\infty} q^{(r+1)k} \mathscr{P}_{n+r+1,k}^q \, (x) \, \nabla_q^{r+1} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)
\end{aligned}
$$

This completes the proof by induction. □

Corollary 1. *q-Baskakov operators can be represent as*

$$B_{n,q}(f,x) = \sum_{r=0}^{\infty} \frac{[n+r-1]_q!}{[n-1]_q!} \nabla_q^r f\,(0) \, \frac{x^r}{[r]_q!}.$$

P r o o f. By Theorem 1 we have

$$\begin{aligned} \mathscr{D}_q^r \left(B_{n,q}(f,x)\right)\big|_{x=0} &= \frac{[n+r-1]_q!}{[n-1]_q!} \mathscr{P}^q_{n+r,0}\,(0)\, \nabla_q^r f\,(0) \\ &= \frac{[n+r-1]_q!}{[n-1]_q!} \nabla_q^r f\,(0) \end{aligned}$$

for $r \geq 1$. By using above equality in q-Taylor formula given in [9], we get

$$B_{n,q}(f,x) = \sum_{r=0}^{\infty} \frac{[n+r-1]_q!}{[n-1]_q!} \nabla_q^r f\,(0)\, \frac{x^r}{[r]_q!}. \tag{2.7}$$

□

From Lemma 1 and Corrollary 1, we have following corollary.

Corollary 2. *q-Baskakov operators can be represent as*

$$B_{n,q}(f,x) = \sum_{r=0}^{\infty} \frac{[n+r-1]_q!}{[n-1]_q!} q^{-\frac{r(r-1)}{2}} f\left[0, \frac{1}{[n]_q}, \frac{[2]_q}{q[n]_q}, \ldots, \frac{[r]_q}{q^{r-1}[n]_q}\right] \frac{x^r}{[n]_q^r}.$$

We are now in a position to give the moments of the first and second order of the operators $B_{n,q}$.

Lemma 3. *For $B_{n,q}(t^m,x)$, $m = 0, 1, 2$, one has*

$$\begin{aligned} B_{n,q}(1,x) &= 1. \\ B_{n,q}(t,x) &= x, \\ B_{n,q}(t^2,x) &= x^2 + \frac{x}{[n]_q}\left(1 + \frac{1}{q}x\right). \end{aligned}$$

P r o o f. It is well known [26, pp. 10] that

$$f\left[x_0, x_1, \ldots, x_r\right] = \frac{f^{(r)}\,(\xi)}{r!}, \tag{2.8}$$

where $\xi \in (x_0, x_r)$. We also see from Lemma 1 and (2.8)

$$\frac{q^{\frac{r(r-1)}{2}} \nabla_q^r f\,(x_0)}{[r]_q!} [n]_q^r = \frac{f^{(r-1)}\,(\xi)}{r!}.$$

Thus it is observed that rth q-forward differences of x^m, $m > r$, are zero. From (2.7), we have

$$B_{n,q}(1,x) = 1. \tag{2.9}$$

For $f\,(x) = x$ we have $\nabla_q^0 f\,(0) = f\,(0) = 0$ and $\nabla_q^1 f\,(0) = f\left(\frac{1}{[n]_q}\right) - f\,(0) = \frac{1}{[n]_q}$ and it follows from (2.7)

$$B_{n,q}(t,x) = x \tag{2.10}$$

For $f(x)=x^2$ we have $\nabla_q^0 f(0)=f(0)=0$ and $\nabla_q^1 f(0)=f\left(\frac{1}{[n]_q}\right)-f(0)=\frac{1}{[n]_q^2}$ and $\nabla_q^2 f(0)=qf\left(\frac{[2]_q}{q[n]_q}\right)-(1+q)f\left(\frac{1}{[n]_q}\right)-f(0)$

$$\begin{aligned} B_{n,q}(t^2,x) &= \frac{[n+1]_q}{[n]_q}\left(\frac{1}{q}[2]_q-1\right)x^2+\frac{x}{[n]_q} \\ &= \frac{q[n]_q+1}{[n]_q}\left(\frac{1}{q}(1+q)-1\right)x^2+\frac{x}{[n]_q} \\ &= x^2+\frac{1}{q[n]_q}x^2+\frac{x}{[n]_q} \\ &= x^2+\frac{x}{[n]_q}\left(1+\frac{1}{q}x\right). \end{aligned} \tag{2.11}$$

□

The following proposition is the another application of q-derivatives, which enables us to give the estimation of moments:

Proposition 1. *If we define*

$$U_{n,m}^q(x):=B_{n,q}(t^m,x)=\sum_{k=0}^{\infty}\mathscr{P}_{n,k}^q(x)\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)^m,$$

then $U_{n,0}^q(x)=1$, $U_{n,1}^q(x)=x$ *and there holds the following recurrence relation:*

$$[n]_q U_{n,m+1}^q(qx)=qx(1+x)D_qU_{n,m}^q(x)+qx[n]_qU_{n,m}^q(qx), \qquad m>1.$$

P r o o f. Obviously $\sum\limits_{k=0}^{\infty}\mathscr{P}_{n,k}^q(x)=1$, thus by this identity and (2.1), the values of $U_{n,0}^q(x)$ and $U_{n,1}^q(x)$ easily follows. From Lemma 2, it is obvious that $x(1+q^{n+k}x)D_q\mathscr{P}_{n,k}^q(x)=\left([k]_q-q^k[n]_qx\right)\mathscr{P}_{n,k}^q(x)$, which implies that

$$x(1+x)\mathscr{D}_q\mathscr{P}_{n,k}^q(x)=\left(\frac{[k]_q}{q^{k-1}[n]_q}-qx\right)\frac{[n]_q}{q}\mathscr{P}_{n,k}^q(qx).$$

Thus using this identity, we have

$$\begin{aligned} qx(1+x)D_qU_{n,m}^q(x) &= \sum_{k=0}^{\infty}qx(1+x)D_q\mathscr{P}_{n,k}^q(x)\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)^m \\ &= [n]_q\sum_{k=0}^{\infty}\left(\frac{[k]_q}{q^{k-1}[n]_q}-qx\right)\mathscr{P}_{n,k}^q(qx)\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)^m. \\ &= [n]_qU_{n,m+1}^q(qx)-qx[n]_qU_{n,m}^q(qx). \end{aligned}$$

This completes the proof of recurrence relation. □

3. Approximation properties

We set

$$E_2(\mathbb{R}_+) := \left\{ f \in C(\mathbb{R}_+) : \lim_{x\to\infty} \tfrac{f(x)}{1+x^2} \text{ exist} \right\}$$

and

$$B_2(\mathbb{R}_+) := \left\{ f : |f(x)| \le B_f \left(1+x^2\right) \right\}$$

where B_f is a constant depending on f, endowed with the norm $\|f\|_2 := \sup\limits_{x\ge 0} \frac{|f(x)|}{1+x^2}$. As a consequence of Lemma 3, the operators (2.1) map $E_2(\mathbb{R}_+)$ into $E_2(\mathbb{R}_+)$. Since for a fixed value of q with $q > 0$,

$$\lim_{n\to\infty} [n]_q = \frac{1}{1-q},$$

$B_{n,q}(t^2, x)$ does not converge to x^2 as $n \to \infty$. According to well known Bohman-Korovkin theorem, relations (2.9), (2.10) and (2.11) don't guarantee that $\lim\limits_{n\to\infty} B_{n,q_n} f = f$ uniformly on compact subset of $\mathbb{R}_+$ for every $f \in E_2(\mathbb{R}_+)$. To ensure this type convergence properties of (2.1) we replace $q = q_n$ as a sequence such that $q_n \to 1$ as $n \to \infty$ for $q_n > 0$ and so that $[n]_{q_n} \to \infty$ as $n \to \infty$. Also, $B_{n,q_n} f$ are linear and positive operators for $q_n > 0$. In this situation, we can apply Bohman-Korovkin theorem to B_{n,q_n}. That is:

Theorem 2. *Let (q_n) be a sequence of real numbers such that $q_n > 0$ and $\lim\limits_{n\to\infty} q_n = 1$. Then for every $f \in E_2(\mathbb{R}_+)$*

$$\lim_{n\to\infty} B_{n,q_n} f = f$$

uniformly on compact subset of $\mathbb{R}_+$.

Theorem 3. *Let $q = q_n$ satisfies $q_n > 0$ and let $q_n \to 1$ as $n \to \infty$. For every $f \in B_2(\mathbb{R}_+)$*

$$\lim_{n\to\infty} \sup_{x\ge 0} \frac{|B_{n,q_n}(f;x) - f(x)|}{(1+x^2)^3} = 0. \tag{3.1}$$

P r o o f. Since f is continuous, it is also uniformly continuous, on any closed interval, there exist a number $\delta > 0$, depending on ε and f, for $|t-x| < \delta$ we have

$$|f(t) - f(x)| < \varepsilon.$$

Since $f \in B_2(\mathbb{R}_+)$, we can write for $|t-x| \ge \delta$

$$|f(t) - f(x)| < A_f(\delta) \left\{ (t-x)^2 + \left(1+x^2\right)|t-x| \right\},$$

where $A_f(\delta)$ is a positive constant depending on f and δ.

On combining above results, we obtain

$$|f(t)-f(x)|<\varepsilon+A_f(\delta)\left\{(t-x)^2+\left(1+x^2\right)|t-x|\right\},$$

where $t,x\in\mathbb{R}_+$. Thus, we have

$$\begin{aligned}&|B_{n,q_n}(f;x)-f(x)|\\&<\varepsilon+A_f(\delta)\left\{B_{n,q_n}\left((t-x)^2;x\right)+\left(1+x^2\right)B_{n,q_n}(|t-x|;x)\right\}\end{aligned}$$

and from Lemma 3

$$\sup_{x\geq 0}\frac{|B_{n,q_n}(f;x)-f(x)|}{1+x^2}<\varepsilon+A_f(\delta)\left\{\frac{1}{[n]_{q_n}}\left(1+\frac{1}{q_n}\right)+\sqrt{\frac{1}{[n]_{q_n}}\left(1+\frac{1}{q_n}\right)}\right\},$$

and this completes the proof. □

Remark 1. Using the similar method given in [1, p. 301], we have

$$|B_{n,q_n}(f;x)-f(x)|\leq M\omega_2\left(f;\sqrt{\frac{x}{[n]_q}\left(1+\frac{1}{q}x\right)}\right),$$

where $\omega_2(f;\delta)$ is classical second modulus of smoothness of f and f is bounded uniformly continuous function on $\mathbb{R}_+$. Thus we say that the rate of convergence of $B_{n,q_n}(f)$ to f in any closed subinterval of $\mathbb{R}_+$ is $\frac{1}{\sqrt{[n]_{q_n}}}$, which is at lest as fast as $\frac{1}{\sqrt{n}}$ which is the rate of convergence of classical Baskakov operators.

4. Shape preserving properties

Definition 2. ([24], [18], [19]) Let f be continuous and non-negative function such that $f(0)=0$. A function f is called star-shaped in $[0,a]$, a is a positive real number, if

$$f(\alpha x)\leq\alpha f(x)$$

for each α, $\alpha\in[0,1]$ and $x\in(0,a]$.

From the definition of q-derivative (1.2), the following lemma is obvious.

Lemma 4. *The function f is star-shaped if and only if $x\mathscr{D}_q(f)(x)\geq f(x)$ for each $q\in(0,1)$ and $x\in[0,a]$.*

Theorem 4. *If f is star-shaped, then $B_{n,q}(f)$ is star-shape.*

Proof. From Theorem 1, we can write

$$\begin{aligned}&\mathscr{D}_q\left(B_{n,q}(f,x)\right)-\frac{B_{n,q}(f,x)}{x}\\&=\ [n]_q\sum_{k=0}^{\infty}q^k\,\nabla_q^1 f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\begin{bmatrix}n+k\\k\end{bmatrix}_q q^{\frac{k(k-1)}{2}}x^k(-x,q)_{n+k+1}^{-1}\end{aligned}$$

$$-\sum_{k=1}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_q q^{\frac{k(k-1)}{2}} x^{k-1}(-x,q)_{n+k}^{-1}$$

$$= \quad [n]\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k \\ k \end{array}\right]_q q^{\frac{k(k-1)}{2}} q^k x^k (-x,q)_{n+k+1}^{-1}$$

$$\left(f\left(\frac{[k+1]_q}{q^k[n]_q}\right)-f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)-\frac{1}{[k+1]_q} f\left(\frac{[k+1]_q}{q^k[n]_q}\right)\right).$$

Since

$$1-\frac{1}{[k+1]_q}=\frac{q[k]_q}{[k+1]_q}$$

we have

$$\mathscr{D}_q\left(B_{n,q}(f,x)\right)-\frac{B_{n,q}(f,x)}{x}$$

$$= \quad [n]\sum_{k=0}^{\infty} q^k \mathscr{P}_{n+1,k}^q\left(\frac{q[k]_q}{[k+1]_q} f\left(\frac{[k+1]_q}{q^k[n]_q}\right)-f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\right). \quad (4.1)$$

Since f is star-shaped, we have

$$\frac{q[k]_q}{[k+1]_q} f\left(\frac{[k+1]_q}{q^k[n]_q}\right) \geq f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right).$$

From this inequality and (4.1), we have desired result. □

Now we give a certain monotonicity property of the q-Baskakov operators defined by (2.1). Similar results for classical Baskakov operators was given in [8].

Theorem 5. *Suppose $f(x)$ is defined on $(0,\infty)$ and $f(x)\geq 0$ for $x\in(0,\infty)$. If $\frac{f(x)}{x}$ is decreasing for all $x\in(0,\infty)$, then $\mathscr{D}_q\left(\frac{B_{n,q}(f;x)}{x}\right)\leq 0$ for $x\in(0,\infty)$ and for all $q\in(0,\infty)$.*

Proof. From (2.1) we get

$$\frac{B_{n,q}(f;x)}{x}$$

$$=\sum_{k=1}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_q q^{\frac{k(k-1)}{2}} x^{k-1}(-x,q)_{n+k}^{-1}+\frac{f(0)}{x}(-x,q)_n^{-1}.$$

If we take q-derivative of above equality and using Lemma 2, then we have

$$\mathscr{D}_q\left(\frac{B_{n,q}(f;x)}{x}\right)$$

$$=\sum_{k=2}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_q q^{\frac{k(k-1)}{2}}[k-1]_q x^{k-2}(-x,q)_{n+k}^{-1}$$

$$-\sum_{k=1}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_q q^{\frac{k(k-1)}{2}}[n+k]_q q^{k-1}x^{k-1}(-x,q)_{n+k+1}^{-1}$$
$$+\mathscr{D}_q\left(\frac{f(0)}{x}(-x,q)_n^{-1}\right).$$

Also using (1.3) and (1.4), we get

$$\mathscr{D}_q\left(\frac{f(0)}{x}(-x,q)_n^{-1}\right)=-\frac{f(0)}{qx^2}(-x,q)_n^{-1}-[n]_q\frac{f(0)}{x}(-x,q)_{n+1}^{-1}$$

Therefore

$$\begin{aligned}
&\mathscr{D}_q\left(\frac{B_{n,q}(f;x)}{x}\right)\\
=\ &\sum_{k=1}^{\infty} f\left(\frac{[k+1]_q}{q^k[n]_q}\right)\left[\begin{array}{c} n+k \\ k+1 \end{array}\right]_q q^{\frac{k(k-1)}{2}}q^k[k]_q x^{k-1}(-x,q)_{n+k+1}^{-1}\\
&-\sum_{k=1}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_q q^{\frac{k(k-1)}{2}}[n+k]_q q^{k-1}x^{k-1}(-x,q)_{n+k+1}^{-1}\\
&-\frac{f(0)}{qx^2}(-x,q)_n^{-1}-[n]_q\frac{f(0)}{x}(-x,q)_{n+1}^{-1}.
\end{aligned}$$

Using the identities

$$\begin{aligned}
\left[\begin{array}{c} n+k \\ k+1 \end{array}\right]_q &= \left[\begin{array}{c} n+k \\ k \end{array}\right]_q \frac{[n]_q}{[k+1]_q}\\
\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_q [n+k]_q &= \left[\begin{array}{c} n+k \\ k \end{array}\right]_q [n]_q
\end{aligned}$$

we have

$$\begin{aligned}
\mathscr{D}_q\left(\frac{B_{n,q}(f;x)}{x}\right) =\ &\sum_{k=1}^{\infty}\left[\begin{array}{c} n+k \\ k \end{array}\right]_q q^{\frac{k(k-1)}{2}}x^{k-1}(-x,q)_{n+k+1}^{-1}\\
&\left(f\left(\frac{[k+1]_q}{q^k[n]_q}\right)\frac{q^k[n]_q}{[k+1]_q}-f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\frac{q^{k-1}[n]_q}{[k]_q}\right)[k]_q\\
&-\frac{f(0)}{qx^2}(-x,q)_n^{-1}-[n]_q\frac{f(0)}{x}(-x,q)_{n+1}^{-1}.
\end{aligned}$$

Since $f(x)\geq 0$ and $\frac{f(x)}{x}$ is non-increasing for $x\in(0,\infty)$,

$$\mathscr{D}_q\left(\frac{B_{n,q}(f;x)}{x}\right)\leq 0$$

for all $q\in(0,\infty)$ and $x\in(0,\infty)$. $\square$

5. Monotonicity property

Now we give the following relation between two consecutive terms of the sequence $B_{n,q}(f)$. Note that similar result for classical Baskakov operators was given in [20].

Theorem 6. *If $f \in C(\mathbb{R}^+)$, then the following formula is valid*

$$\begin{aligned} & B_{n+1,q}(f,x) - B_{n,q}(f,x) \\ &= -\frac{q^n}{[n]_q[n+1]_q} \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}-2k} x^{k+1} (-x,q)_{n+k+1}^{-1} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \\ & \times \frac{[n+k+1]_q}{[n+1]_q} f\left[\frac{[k]_q}{q^{k-1}[n+1]_q}, \frac{[k+1]_q}{q^k[n+1]_q}, \frac{[k+1]_q}{q^k[n]_q}\right] \end{aligned}$$

P r o o f. Using the equality

$$1 = 1 + q^{n+k}x - q^{n+k}x,$$

from (2.1) we can write

$$\begin{aligned} & B_{n+1,q}(f;x) \\ &= \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^k (-x,q)_{n+k+1}^{-1} \\ &= \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^k (-x,q)_{n+k}^{-1} \\ & \quad - \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} q^{n+k} x^{k+1} (-x,q)_{n+k+1}^{-1} \\ &= f(0)(-x,q)_n^{-1} + \sum_{k=1}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^k (-x,q)_{n+k}^{-1} \\ & \quad - \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} q^{n+k} x^{k+1} (-x,q)_{n+k+1}^{-1} \end{aligned}$$

Thus, we have

$$\begin{aligned} & B_{n+1,q}(f;x) \\ &= f(0)(-x,q)_n^{-1} + \sum_{k=0}^{\infty} f\left(\frac{[k+1]_q}{q^k[n+1]_q}\right) \begin{bmatrix} n+k+1 \\ k+1 \end{bmatrix}_q q^{\frac{k(k-1)}{2}} q^k x^{k+1} (-x,q)_{n+k+1}^{-1} \\ & \quad - \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} q^{n+k} x^{k+1} (-x,q)_{n+k+1}^{-1}. \end{aligned}$$

Since

$$
\begin{aligned}
&B_{n,q}(f;x)\\
&= f(0)(-x,q)_n^{-1} + \sum_{k=1}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_q q^{\frac{k(k-1)}{2}} x^k (-x,q)_{n+k}^{-1}\\
&= f(0)(-x,q)_n^{-1} + \sum_{k=0}^{\infty} f\left(\frac{[k+1]_q}{q^{k}[n]_q}\right)\left[\begin{array}{c} n+k \\ k+1 \end{array}\right]_q q^{\frac{k(k-1)}{2}} q^k x^{k+1} (-x,q)_{n+k+1}^{-1},
\end{aligned}
$$

we have

$$
\begin{aligned}
&B_{n+1,q}(f,x) - B_{n,q}(f,x)\\
= &\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} q^k x^{k+1} (-x,q)_{n+k+1}^{-1} \left(f\left(\frac{[k+1]_q}{q^k[n+1]_q}\right)\left[\begin{array}{c} n+k+1 \\ k+1 \end{array}\right]_q \right.\\
&\left. -q^n f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right)\left[\begin{array}{c} n+k \\ k \end{array}\right]_q - f\left(\frac{[k+1]_q}{q^k[n]_q}\right)\left[\begin{array}{c} n+k \\ k+1 \end{array}\right]_q \right)
\end{aligned}
$$

Using the equalities

$$
\left[\begin{array}{c} n+k+1 \\ k+1 \end{array}\right]_q = \frac{[n+k+1]_q}{[k+1]_q}\left[\begin{array}{c} n+k \\ k \end{array}\right]_q
$$

and

$$
\left[\begin{array}{c} n+k \\ k+1 \end{array}\right]_q = \frac{[n]_q}{[k+1]_q}\left[\begin{array}{c} n+k \\ k \end{array}\right]_q,
$$

we can write

$$
\begin{aligned}
&B_{n+1,q}(f,x) - B_{n,q}(f,x)\\
&= -\sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} x^{k+1} (-x,q)_{n+k+1}^{-1} \left[\begin{array}{c} n+k \\ k \end{array}\right]_q\\
&\left(q^n f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right) - \frac{[n+k+1]_q}{[k+1]_q} f\left(\frac{[k+1]_q}{q^k[n+1]_q}\right) + \frac{[n]_q}{[k+1]_q} f\left(\frac{[k+1]_q}{q^k[n]_q}\right)\right).
\end{aligned}
$$

Using the inequalities

$$
\begin{aligned}
\frac{[k+1]_q}{q^k[n]_q} - \frac{[k]_q}{q^{k-1}[n+1]_q} &= \frac{[n+k+1]_q}{q^k[n]_q[n+1]_q},\\
\frac{[k+1]_q}{q^k[n+1]_q} - \frac{[k]_q}{q^{k-1}[n+1]_q} &= \frac{1}{q^k[n+1]_q}
\end{aligned}
$$

and

$$
\frac{[k+1]_q}{q^k[n]_q} - \frac{[k+1]_q}{q^k[n+1]_q} = \frac{q^n[k+1]_q}{q^k[n+1]_q[n]_q},
$$

we can easily seen that

$$f\left[\frac{[k]_q}{q^{k-1}[n+1]_q},\frac{[k+1]_q}{q^k[n+1]_q},\frac{[k+1]_q}{q^k[n]_q}\right]$$
$$=\frac{q^{2k}\,[n]_q\,[n+1]_q^2}{q^n\,[n+k+1]_q}\left(q^n f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right)\right.$$
$$\left.-\frac{[n+k+1]_q}{[k+1]_q}f\left(\frac{[k+1]_q}{q^k[n+1]_q}\right)+\frac{[n]_q}{[k+1]_q}f\left(\frac{[k+1]_q}{q^k[n]_q}\right)\right).$$

This proves the theorem. □

We know that a function f is convex if and only if all second order divided differences of f are nonnegative. Using this property and Theorem 6, we have following result:

Corollary 3. *If $f(x)$ is a convex function defined on $\mathbb{R}^+$, then the q-Baskakov operator $B_{n,q}(f,x)$ defined by (2.1) is strictly monotonically non-decreasing in n, unless f is the linear function (in which case $B_{n,q}(f,x)=B_{n+1,q}(f,x)$ for all n).*

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