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Characterizations of the position vector of a surface curve in Euclidean 3-space

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Abstract

In this paper, we give some characterizations of position vector of a unit speed curve in a regular surface $M \subset \mathbb{E}^3$ which always lies in the planes spanned by $\{T, Z\}$, $\{T, Y\}$ and $\{Y, Z\}$, respectively, by using (curve-surface)-frame $\{T, Y, Z\}$ instead of Frenet frame $\{T, N, B\}$. We characterize such curves in terms of the geodesic curvature k_g, normal curvature k_n and geodesic torsion t_r. Furthermore, we give some characterization for the regular surface M by using the concept of transversality of surfaces in Euclidean 3-space.

1 Introduction

In the Euclidean space $\mathbb{E}^3$, it is well known that to each unit speed curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$, whose successive derivatives $\alpha'(s)$, $\alpha''(s)$ and $\alpha'''(s)$ are linearly independent vectors, one can associate the moving orthonormal Frenet frame $\{T, N, B\}$, consisting of the tangent, the principal normal and the binormal vector field respectively. Moreover, the planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are respectively known as the osculating, the rectifying and the normal plane. The rectifying curve in $\mathbb{E}^3$ is defined by *B. Y. Chen* in [2] as a curve for which position vector always lies in its rectifying plane. In particular, it is shown in [3] that there exist a simple relationship between the rectifying curves and the centrodes, which play some important roles in mechanics, kinematics as well as in differential geometry in defining the curves of constant precession. The rectifying curves are also studied in [3] as the extremal curves.

Key Words: Position vector, (curve-surface)-frame, geodesic curvature, normal curvature, geodesic torsion and transversal surfaces
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Also we know that the normal curves which position vector always lies in its normal plane are spherical curves. Necessary and sufficient conditions for a curve to be a spherical curve in Euclidean 3-space are given in [8] and [9]. The osculating curves which position vector always lies in its osculating plane are planar curves [2].

In this paper, we study the geometry of position vector of a unit speed curve in a regular surface $M \subset \mathbb{E}^3$ which always lies in the planes $\{T, Z\}$, $\{T, Y\}$ and $\{Y, Z\}$ respectively and characterize such curves in terms of the *geodesic curvature* k_g, *normal curvature* k_n and *geodesic torsion* t_r. Also the special cases for the surface curve, $k_g = 0$, $k_n = 0$ and $t_r = 0$ respectively are considered. Furthermore, we give some characterization for the regular surface M by using the concept of transversality of surfaces in Euclidean 3-space.

2 Preliminaries

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be arbitrary curve in the Euclidean space $\mathbb{E}^3$. Recall that the curve α is said to be of unit speed (or parameterized by arclength function s) if $< \alpha'(s), \alpha'(s) >= 1$, where $< \cdot, \cdot >$ is the standard scalar product of $\mathbb{E}^3$ given by

$$< X, Y >= x_1 y_1 + x_2 y_2 + x_3 y_3,$$

for each $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{E}^3$. In particular, the norm of a vector $X \in \mathbb{E}^3$ is given by $||X|| = \sqrt{< X, X >}$.

Let $\{T, N, B\}$ be the moving Frenet frame along the unit speed curve α, where T, N, B and denote respectively the tangent, the principal normal and the binormal vector fields. Then the Frenet formulas are given by (see [3]):

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{1}$$

The functions $k_1(s)$ and $k_2(s)$ are called respectively the first and the second curvature of the curve α.

Let M be regular surface in $\mathbb{E}^3$ and $\alpha : I \subset \mathbb{R} \to M$ is unit speed curve on M. Instead of the Frenet frame field on α, consider the frame field T, Y, Z where T is the unit tangent of α, Z is the surface unit normal restricted to α, and $Y = Z \times T$, thus we have

$$\begin{bmatrix} T' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & t_r \\ -k_n & -t_r & 0 \end{bmatrix} \begin{bmatrix} T \\ Y \\ Z \end{bmatrix}, \tag{2}$$

where *geodesic curvature* k_g is defined $k_g = k_1 \cos\gamma$, *normal curvature* k_n is defined by $k_n = k_1 \sin\gamma$ and *geodesic torsion* t_r is defined by $t_r = k_2 - \gamma'$, where γ is the measure of the angle between Z and B. This frame is called *(curve-surface)-frame.* It is well known that, if $k_g = 0$, then the curve is called geodesic, if $k_n = 0$, then the curve is called asymptotic and if $t_r = 0$, then the curve is called principal curve.

Now, we give some information about transversality of maps and surfaces:

Definition 2.1 ([1]) Let M be a smooth manifold and N_1 and N_2 be smooth submanifold of M. Let f_1 and f_2 defined by

$$f_i : N_i \to M \text{ (i=1,2)}$$

be smooth maps. We will say that f_1 transversal to f_2 if $(p_1, p_2) \in N_1 \times N_2$ and $f_1(p_1) = q = f_2(p_2)$, then

$$(f_1)_*(T_{p_1}N_1) + (f_2)_*(T_{p_2}N_2) = T_qM.$$

From the above definition; we get the following results easily. Let M be 3-dimensional Euclidean space $\mathbb{E}^3$ and N_1, N_2 be surfaces in $\mathbb{E}^3$. In this case, we can define inclusion maps by $i_1 : N_1 \to \mathbb{E}^3$, $i_2 : N_2 \to \mathbb{E}^3$. Let $i_1(N_1) \cap i_2(N_2) = \alpha$ be a regular curve. If i_1 is not transversal to i_2 along α, then we get $(i_1)_*(T_pN_1) + (i_2)_*(T_pN_2) \neq T_pE^3$, where $p \in \alpha$. In this case, we say that the surfaces N_1 and N_2 are not transversal along α. Suppose that $(i_1)_*(T_pN_1)$ is not equal to $(i_2)_*(T_pN_2)$. Since dim $(i_1)_*(T_pN_1) = \dim(i_2)_*(T_pN_2) = 2$, we get

$$(i_1)_*(T_pN_1) + (i_2)_*(T_pN_2) = T_pE^3,$$

this is a contradiction. Thus we have

$$T_pN_1 = T_pN_2 \quad \text{and} \quad Z_1(p) = \pm Z_2(p),$$

where Z_1 and Z_2 are unit normal vectors of the surfaces N_1 and N_2 respectively. In addition we can easily see that $Y_1(p) = \pm Y_2(p)$, where $Y_1(p) = Z_1(p) \times T(p)$, and $Y_2(p) = Z_2(p) \times T(p)$. $\{T(p), Y_1(p), Z_1(p)\}$ and $\{T(p), Y_2(p), Z_2(p)\}$ are (curve-surface)-frames on N_1 and N_2, respectively, along the curve α

3 Characterizations of a surface curve which position vector always lies in the plane $sp\,\{T, Z\}$

Let $\alpha = \alpha(s)$ be a unit speed curve in a surface $M \subset \mathbb{E}^3$ with non-zero curvatures k_g, k_n and t_r. Firstly, assume that position vector of α always lies in the plane $sp\{T,Z\}$. By definition, position vector of the curve α satisfies the equation

$$\alpha(s)=\lambda(s)T+\mu(s)Z, \tag{3}$$

for some differentiable functions $\lambda(s)$ and $\mu(s)$.

Differentiating equation (3) with respect to s and using the (curve-surface)-frame given by equations (2), we obtain

$$T=(\lambda'-\mu k_n)\,T+(\lambda k_g-\mu t_r)\,Y+\left(\lambda k_n+\mu^{'}\right)Z.$$

It follows that

$$\begin{aligned}\lambda'-\mu k_n &=1,\\ \lambda k_g-\mu t_r &=0,\\ \lambda k_n+\mu^{'} &=0,\end{aligned} \tag{4}$$

and therefore

$$\begin{aligned}\lambda(s)= &\ \tfrac{ct_r}{k_g}e^{-\int\frac{t_rk_n}{k_g}ds},\\ \mu(s)= &\ ce^{-\int\frac{t_rk_n}{k_g}ds},\end{aligned} \tag{5}$$

where $c\in\mathbb{R}_0$. In this way, the functions $\lambda(s)$ and $\mu(s)$ are expressed in terms of the curvature functions $k_g(s)$, $k_n(s)$ and $t_r(s)$ of the curve α. Moreover, by using the first equation in (4) and relation (5), we easily find that the curvatures $k_g(s)$, $k_n(s)$ and $t_r(s)$ satisfy equation

$$\left(\frac{t_r}{k_g}\right)^{'}-\left(\left(\frac{t_r}{k_g}\right)^2+1\right)k_n=\frac{1}{c}e^{\int\frac{t_rk_n}{k_g}ds}. \tag{6}$$

Conversely, assume that non-zero curvatures $k_g(s)$, $k_n(s)$ and $t_r(s)$ of an arbitrary unit speed curve α in $M\subset\mathbb{E}^3$, satisfy equation (6). Let us consider the vector $X\in M\subset\mathbb{E}^3$ given by

$$X(s)=\alpha(s)-\frac{ct_r}{k_g}e^{-\int\frac{t_rk_n}{k_g}ds}T(s)-ce^{-\int\frac{t_rk_n}{k_g}ds}Z(s),$$

by using the relations (2) and (6) we easily find $X'(s)=0$, which means that X is a constant vector. This implies that α is congruent to a curve which position vector always lies in the plane $sp\{T,Z\}$. In this way, the following theorem is proved.

Theorem 3.1. *Let $\alpha(s)$ be a unit speed curve in $M\subset\mathbb{E}^3$, with non-zero curvatures $k_g(s)$, $k_n(s)$ and $t_r(s)$. Then α is congruent to a curve which* *position vector always lies in the plane $sp\{T,Z\}$ if and only if*

$$\left(\frac{t_r}{k_g}\right)' - \left(\left(\frac{t_r}{k_g}\right)^2 + 1\right)k_n = \frac{1}{c}e^{\int \frac{t_r k_n}{k_g}ds}, \quad c \in \mathbb{R}_0.$$

In the next theorem, we give the necessary and the sufficient conditions for a surface curve which position vector always lies in the plane $sp\{T,Z\}$.

Theorem 3.2. *Let $\alpha(s)$ be a unit speed curve in $M \subset \mathbb{E}^3$, with non-zero curvatures $k_g(s)$, $k_n(s)$ and $t_r(s)$. Then position vector of α always lies in the plane $sp\{T,Z\}$ if and only if*

$$\langle \alpha, T\rangle = \frac{ct_r}{k_g}e^{-\int \frac{t_r k_n}{k_g}ds} \quad \text{and} \quad \langle \alpha, Z\rangle = ce^{-\int \frac{t_r k_n}{k_g}ds}, \qquad c \in \mathbb{R}_0. \tag{7}$$

Proof. If $\alpha(s)$ be a unit speed curve in $M \subset \mathbb{E}^3$, with non-zero curvatures $k_g(s)$, $k_n(s)$ and $t_r(s)$ and position vector of α always lies in the plane $sp\{T,Z\}$, then from equation (5) we get equation (7) easily. Conversely, assume that equation (7) holds. Then $\langle \alpha, Z\rangle = ce^{-\int \frac{t_r k_n}{k_g}ds}$. Differentiating the previous equation with respect to s and using (2), we find $\langle \alpha, Y\rangle = 0$, which implies that position vector of α always lies in the plane $sp\{T,Z\}$.

Now, we consider the following special cases when the curve is a geodesic (i.e. $k_g = 0$), asymptotic (i.e. $k_n = 0$), principal (i.e. $t_r = 0$) curve, respectively.

Case 3.1. If α is a geodesic curve in $M \subset \mathbb{E}^3$ and position vector always lies in the plane $sp\{T,Z\}$. From equation (3), we get

$$\begin{aligned} \lambda' - \mu k_n &= 1, \\ \mu t_r &= 0, \\ \lambda k_n + \mu' &= 0. \end{aligned} \tag{8}$$

From $\mu t_r = 0$, we have $\mu = 0$ or $t_r = 0$. If $\mu = 0$, then from equation (8) we get $\lambda k_n = 0$ and $\lambda' = 1$. Thus $\lambda(s) \neq 0$ and $k_n = 0$. $k_g = 0$ and $k_n = 0$ implies that $k_1 = 0$ which means that α is a straight line. If $t_r = 0$ then $k_n = k_1$ and $k_2 = 0$. So we get the (curve-surface)- frame of the curve α as follows

$$\begin{aligned} T' &= k_1 Z, \\ Y' &= 0, \\ Z' &= -k_1 T. \end{aligned}$$

In this case, the relationship between the (curve-surface)- frame and Frenet frame of the curve α are $Z = N$ and $Y = B$. Thus we have the following corollary:

Corollary 3.1. *If $\alpha(s)$ is unit speed geodesic curve in $M \subset \mathbb{E}^3$, with curvatures $k_n(s)$, $t_r(s)$ and position vector of α always lies in the plane $sp\{T, Z\}$ then $\alpha(s)$ is a asymptotic curve or a principal curve in M.*

Case 3.2. If α is a principal curve in $M \subset\mathbb{E}^3$ and position vector lies in the plane $sp\{T, Z\}$. From equation (8), we get $k_n = k_1 \sin\gamma$ =constant. Since $t_r = k_2 - \gamma^{'} = 0$, we get $\gamma(s) = \int k_2 ds$, and $k_1 A \sin(\int k_2 ds) = 1$ where A is non-zero constant. If we take derivative two times of the last equation respect to s, we find $\frac{k_2}{k_1} + \left(\frac{1}{k_2}(\frac{1}{k_1})^{'}\right)^{'} = 0$. The previous equation is differential of equation $(\frac{1}{k_1})^2 + (\frac{1}{k_2}(\frac{1}{k_1})^{'})^2 =$ constant. Which means that the curve α is a spherical curves with respect to Frenet frame. Then we give the following corollary.

Corollary 3.2. *If $\alpha(s)$ is unit speed principal curve in $M \subset \mathbb{E}^3$, with curvatures $k_g(s)$, $k_n(s)$ and position vector of α always lies in the plane $sp\{T, Z\}$ then $\alpha(s)$ is a spherical curve.*

Case 3.3. If α is a asymptotic curve in $M \subset\mathbb{E}^3$ and position vector always lies in the plane $sp\{T, Z\}$. From equation (6), we get $\frac{t_r}{k_g} = as + b$, where a, b some constants with $a \neq 0$. Since $k_n = k_1 \sin\gamma = 0$, we find that $\gamma = k\pi (k = 0, 1)$ so by using the definition of t_r and k_g, we have $t_r = k_2$ and $k_g = \pm k_1$, thus we have $\frac{k_2}{k_1} = a_0 s + b_0$, where a_0, b_0 some constants with $a_0 \neq 0$. Which means that, according to the [1], α is a rectifying curves i.e. position vector of the curve α is always lies in its rectifying plane with respect to the Frenet frame.

Corollary 3.3. *If $\alpha(s)$ is unit speed asymptotic curve in $M \subset \mathbb{E}^3$, with curvatures k_g, t_r and position vector of α always lies in the plane $sp\{T, Z\}$ then $\alpha(s)$ is a rectifying curve with respect to the Frenet frame.*

4 Characterizations of a surface curve which position vector always lies in the plane $sp\{T, Y\}$

Let $\alpha = \alpha(s)$ be a unit speed curve in a regular surface $M \subset \mathbb{E}^3$ with non-zero curvatures k_g, k_n and t_r. We assume that position vector of α always lies in the plane $sp\{T, Y\}$. By definition, position vector of the curve α satisfies equation

$$\alpha(s) = \lambda(s)T + \mu(s)Y, \tag{9}$$

for some differentiable functions $\lambda(s)$ and $\mu(s)$. Differentiating equation (9) with respect to s and using the (curve-surface)-frame given by equations (2), we obtain

$$T = (\lambda' - \mu k_g)\, T + \left(\lambda k_g + \mu^{'}\right) Y + (\lambda k_n + \mu t_r)\, Z$$

It follows that

$$\begin{aligned} \lambda' - \mu k_g &= 1, \\ \lambda k_g + \mu' &= 0, \\ \lambda k_n + \mu t_r &= 0, \end{aligned} \tag{10}$$

and therefore

$$\begin{aligned} \lambda(s) = & \ \tfrac{-ct_r}{k_n} e^{\int \frac{t_r k_g}{k_n} ds}, \\ \mu(s) = & \ c e^{\int \frac{t_r k_g}{k_n} ds}, \end{aligned} \tag{11}$$

where $c \in \mathbb{R}_0$. In this way, the functions $\lambda(s)$ and $\mu(s)$ are expressed in terms of the curvature functions $k_g(s)$, $k_n(s)$ and $t_r(s)$ of the curve α. Moreover, by using the first equation in (10) and relation (11), we easily find that the curvatures $k_g(s)$, $k_n(s)$ and $t_r(s)$ satisfy equation

$$\left(\frac{t_r}{k_n}\right)' + \left(\left(\frac{t_r}{k_n}\right)^2 + 1\right) k_g = \frac{-1}{c} e^{-\int \frac{t_r k_g}{k_n} ds}. \tag{12}$$

Conversely, assume that non-zero curvatures $k_g(s)$, $k_n(s)$ and $t_r(s)$, of an arbitrary unit speed curve α in $M \subset \mathbb{E}^3$, satisfy equation (12). Let us consider the vector $X \in M \subset \mathbb{E}^3$ given by

$$X(s) = \alpha(s) + \frac{ct_r}{k_n} e^{\int \frac{t_r k_g}{k_n} ds} T(s) - c e^{\int \frac{t_r k_g}{k_n} ds} Y(s).$$

By using the relations (2) and (12), we easily find $X'(s) = 0$, which means that X is a constant vector. This implies that α is congruent to a curve which position vector always lies in the plane $sp\{T, Y\}$. In this way, the following theorem is proved.

Theorem 4.1. *Let $\alpha(s)$ be a unit speed curve in $M \subset \mathbb{E}^3$, with non-zero curvatures $k_g(s)$, $k_n(s)$ and $t_r(s)$. Then α is congruent to a curve which position vector always lies in the plane $sp\{T, Y\}$ if and only if*

$$\left(\frac{t_r}{k_n}\right)' + \left(\left(\frac{t_r}{k_n}\right)^2 + 1\right) k_g = \frac{-1}{c} e^{-\int \frac{t_r k_g}{k_n} ds}, \quad c \in \mathbb{R}_0.$$

In the next theorem, we give the necessary and the sufficient conditions for a surface curve which position vector always lies in the plane $sp\{T, Y\}$.

Theorem 4.2. *Let $\alpha(s)$ be a unit speed curve in $M \subset \mathbb{E}^3$, with non-zero curvatures* $k_g(s)$, $k_n(s)$ and $t_r(s)$. *Then position vector of α always lies in the plane $sp\{T, Y\}$ if and only if*

$$\langle \alpha, T\rangle = \frac{-ct_r}{k_n} e^{\int \frac{t_r k_g}{k_n} ds} \quad and \quad \langle \alpha, Y\rangle = ce^{\int \frac{t_r k_g}{k_n} ds}, \quad c \in \mathbb{R}_0. \tag{13}$$

Proof. If $\alpha(s)$ be a unit speed curve in $M \subset \mathbb{E}^3$, with non-zero curvatures $k_g(s)$, $k_n(s)$, $t_r(s)$ and position vector of α always lies in the plane $sp\{T, Y\}$, then from equation (11) we get equation (13) easily. Conversely, assume that equation (13) holds. Then $\langle \alpha, Y\rangle = ce^{\int \frac{t_r k_g}{k_n} ds}$. Differentiating the previous equation with respect to s and using (2), we find $\langle \alpha, Z\rangle = 0$, which implies that position vector of α always lies in the plane $sp\{T, Y\}$.

Now, we consider the following special cases when the curve is a geodesic (i.e. $k_g = 0$), asymptotic (i.e. $k_n = 0$), principal (i.e. $t_r = 0$) curve, respectively.

Case 4.1. If α is a asymptotic curve in $M \subset \mathbb{E}^3$ and position vector always lies in the plane $sp\{T, Y\}$. From equation (10), we get

$$\begin{aligned} \lambda' - \mu k_g &= 1, \\ \lambda k_g + \mu' &= 0, \\ \mu t_r &= 0. \end{aligned} \tag{14}$$

From $\mu t_r = 0$, we have $\mu = 0$ or $t_r = 0$. If $\mu = 0$, then from equation (6) we get $\lambda k_n = 0$ and $\lambda' = 1$. Thus $\lambda(s) \neq 0$ and $k_g = 0$. $k_g = 0$ and $k_n = 0$ implies that $k_1 = 0$ which means that α is a straight line. If $t_r = 0$ then $k_g = \pm k_1$ and $k_2 = 0$. So we get the (curve-surface)- frame of the curve α as follows

$$\begin{aligned} T' &= \pm k_1 Y, \\ Y' &= \pm(-k_1 T), \\ Z' &= 0. \end{aligned}$$

In this case, the relationship between the (curve-surface)- frame and Frenet frame of the curve α are $Z = \pm B$ and $Y = \mp N$. Thus we have the following corollaries:

Corollary 4.1. *If $\alpha(s)$ is unit speed asymptotic curve in $M \subset \mathbb{E}^3$, with curvatures $k_g(s)$, $t_r(s)$ and position vector of α always lies in the plane $sp\{T, Y\}$ then $\alpha(s)$ is a geodesic curve or a principal curve in M.*

Corollary 4.2. *Let $\alpha(s)$ be a unit speed asymptotic curve in $M \subset \mathbb{E}^3$, with curvatures $k_g(s)$, $t_r(s)$ and position vector of α always lies in the plane $sp\{T, Y\}$ if and only if $\alpha(s)$ is a planar curve.*

Also, by using definition 2.1., we have the following corollary:

Corollary 4.3. Let E be a plane and M be a regular surface in $\mathbb{E}^3$. Let α be a unit speed asymptotic curve with $k_g(s)$ *and* $t_r(s)$ in M. In this case, position vector of α lies in the plane $sp\{T, Y\}$ if and only if M and E are nowhere transversal along α, where $M \cap E = \alpha$.

Example. Let E be a plane in $\mathbb{E}^3$ and $\alpha : I \subset \mathbb{R} \to E$ be a unit speed curve (i.e. α is a planer curve). If we define the isometric immersion from $I \times \mathbb{R}$ to $\mathbb{E}^3$ by

$$x(s,t) = \gamma(s) + tN(s) + t^2 B$$

and $M = x(I \times \mathbb{R})$, then we can see that M and E are nowhere transversal along α. Thus position vector of α always lies in the plane $\mathrm{sp}\{T, Y\}$.

Case 4.2. If α is a principal curve in $M \subset \mathbb{E}^3$ and its position vector always lies in the plane $sp\{T, Y\}$. From equation (12), we get $k_g = k_1 \cos\gamma$ =constant. Since $t_r = k_2 - \gamma' = 0$, we get $\gamma(s) = \int k_2 ds$, and $k_1 A \cos(\int k_2 ds) = 1$ where A is non-zero constant. If we take derivative two times of the last equation respect to s , we find $\frac{k_2}{k_1} + \left(\frac{1}{k_2}(\frac{1}{k_1})'\right)' = 0$. The previous equation is differential of equation $(\frac{1}{k_1})^2 + (\frac{1}{k_2}(\frac{1}{k_1})')^2 =$ constant. Which means that the curve α is a spherical curves with respect to Frenet frame. Then we give the following corollary.

Corollary 4.2. *If $\alpha(s)$ is unit speed principal curve in $M \subset \mathbb{E}^3$, with curvatures $k_n(s)$, $k_n(s)$ and position vector of α always lies in the plane $sp\{T, Y\}$ then $\alpha(s)$ is a spherical curve with respect to the Frenet frame.*

Case 4.3. If α is a geodesic curve in $M \subset \mathbb{E}^3$ and position vector always lies in the plane $sp\{T, Y\}$. From equation (12), we get $\frac{t_r}{k_n} = as + b$, where a, b some constants with $a \neq 0$. Since $k_g = k_1 \cos\gamma = 0$, we find that $\gamma = \frac{\pi}{2}$ so by using the definition of t_r and k_n, we have $t_r = k_2$ and $k_n = k_1$, thus we have $\frac{k_2}{k_1} = a_0 s + b_0$, where a_0, b_0 some constants with $a_0 \neq 0$.Which means that, according to the [1], α is a rectifying curves i.e. position vector of the curve α always lies its rectifying plane with respect to the Frenet frame.

Corollary 4.3. *If $\alpha(s)$ is unit speed geodesic curve in $M \subset \mathbb{E}^3$, with curvatures $k_n(s)$, $t_r(s)$ and position vector of α always lies in the plane $sp\{T, Y\}$ then $\alpha(s)$ is a rectifying curve with respect to the Frenet frame.*

5 Characterizations of a surface curve which position vector always lies in the plane $sp\{Y, Z\}$

Let $\alpha = \alpha(s)$ be a unit speed curve in a surface $M \subset \mathbb{E}^3$ with non-zero curvatures k_g, k_n and t_r. Now, assume that position vectors of α always lies in the plane $sp\{Y, Z\}$. By definition, position vector of the curve α satisfies equation

$$\alpha(s) = \lambda(s)Y + \mu(s)Z, \tag{15}$$

for some differentiable functions $\lambda(s)$ and $\mu(s)$. Differentiating equation (15) with respect to s and using the (curve-surface)-frame given by equations (2), we obtain

$$T = -\left(\lambda k_g + \mu k_n\right) T + \left(\lambda' - \mu^{'} t_r\right) Y + \left(\mu^{'} + \lambda t_r\right) Z$$

It follows that

$$\begin{aligned} \lambda k_g + \mu k_n &= -1, \\ \lambda' - \mu^{'} t_r &= 0, \\ \mu^{'} + \lambda t_r &= 0. \end{aligned} \tag{16}$$

From second and third equations in (16), we easily find that $\lambda^2 + \mu^2 =$ constant. Since
$< \alpha, \alpha >= \lambda^2 + \mu^2$, we find that $< \alpha, \alpha >=$ constant, which means that α is a spherical curve with respect to Frenet frame. We give the following theorem.

Theorem 5.1. *Let $\alpha(s)$ be a unit speed curve in $M \subset \mathbb{E}^3$, with non-zero curvatures $k_g(s)$, $k_n(s)$ and $t_r(s)$. Then position vector of α always lies in the plane $sp\{Y, Z\}$ if and only if α is a spherical curve.*

Now the question is that the curves whose position vectors lie on the plane $sp\{Y, Z\}$ would take place only on the sphere.

The answers of the above question is given by the following corollary by using definition 2.1.:

Corollary 5.1. Let M be a regular surface and $S^2(m, r)$ be a $2-$sphere with radius r and origin $m \in \mathbb{E}^3$ in $\mathbb{E}^3$. Let α be a unit speed curve with non-zero curvatures $k_g(s)$, $k_n(s)$ *and* $t_r(s)$.in M. In this case, position vector of α lies in the plane $sp\{Y, Z\}$ if and only if M and $S^2(m, r)$ are nowhere transversal along α, where $M \cap S^2(m, r) = \alpha$. **Acknowledgment.** The authors are very grateful to the referee for his/her useful comments and suggestions which improved the first version of the paper.

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