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A CLASS OF LINEAR POSITIVE OPERATORS IN WEIGHTED SPACES

Ayşegül Erençin* — H. Gül İnce** — Ali Olgun***

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ABSTRACT. In this paper, we introduce a class of linear positive operators based on q-integers. For these operators we give some convergence properties in weighted spaces of continuous functions and present an application to differential equation related to q-derivatives. Furthermore, we give a Stancu-type remainder.

1. Introduction

The classical Meyer-Knig and Zeller (MKZ) [18] operators are defined by

$$M_n(f;x) = (1-x)^{n+1}\sum_{k=0}^{\infty} f\left(\frac{k}{n+k+1}\right)\binom{n+k}{k}x^k, \qquad x\in[0,1),\ \ n\in\mathbb{N}.$$

In order to give the monotonicity properties, Cheney and Sharma [7] introduced a slight modification of these operators as follows:

$$M_n^*(f;x) = (1-x)^{n+1}\sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k}x^k, \qquad x\in[0,1),\ \ n\in\mathbb{N}.$$

There are many works about these operators (see, for instance [1, 3, 6, 7, 8, 10, 16, 18]).

The study of approximation theory based on q-integers was firstly started by Phillips in [19]. The author proposed a generalization of Bernstein polynomials called q-Bernstein polynomials. After that many researchers studied in this area.

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Trif [21], introduced MKZ operators based on q-integers and studied the approximation and monotonicity properties of such operators. Since it was impossible to give an explicit expression for the second moment of these operators, in [9] Doğru and Duman presented another kind MKZ operators based on q-integers and investigated their statistical approximation properties. Later, Heping [14] gave an explicit formula for the second moment of the q-MKZ operators, defined by Trif, in terms of q-hypergeometric series and discussed further approximating properties of these operators. Recently, in [2] Özarslan and Duman have constructed a new generalization of MKZ operators based on q-integers and obtained a Korovkin type approximation theorem for them.

Before introducing the operators we recall some properties of the q-calculus. For any fixed real number $q > 0$ and nonnegative integer r, the q-integer $[r]_q$ and the q-factorial $[r]_q!$ are defined by

$$[r]_q = \begin{cases} \frac{1-q^r}{1-q}, & q \neq 1 \\ r, & q = 1 \end{cases}$$

and

$$[r]_q! = \begin{cases} [1]_q[2]_q \cdots [r]_q, & r \geq 1 \\ 1, & r = 0 \end{cases}$$

respectively. Also

$$\left(x,q\right)_n = \prod_{s=0}^{n-1} \left(1 - q^s x\right)$$

and q-binomial coefficients are defined by (see [13, p. 3, p. 6])

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q![n-r]_q!}, \qquad n \geq r \geq 0.$$

Throughout this work we shall assume that $q \in (0,1)$.

Rempulska and Skorupka in [20] defined the following operators

$$M_n\left(f, b_n; x\right) = \left(1 - \frac{x}{b_n}\right)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k} b_n\right) \binom{n+k}{k} \left(\frac{x}{b_n}\right)^k, \tag{1.1}$$

where $x \in \left[0, b_n\right)$, $n \in \mathbb{N}$, and introducing a generalization of these operators for differentiable functions in polynomial weighted spaces proved some convergence properties of that operators. In (1.1) b_n is a sequence of real numbers having the properties

$$1 \leq b_n < b_{n+1}, \quad \lim_{n\to\infty} b_n = \infty, \quad \lim_{n\to\infty} \frac{b_n}{n} = 0. \tag{1.2}$$

In this study, we present the q-generalization of the linear positive operators given by (1.1) as follows:

$$L_{n,q}(f;x) = P_{n,q}(x)\sum_{k=0}^{\infty} f\left(\frac{[k]_q}{[n+k]_q}b_n\right)\begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k, \tag{1.3}$$

where $x \in [0,b_n)$, $n \in \mathbb{N}$, $P_{n,q}(x) = \prod_{s=0}^{n}\left(1 - q^s\frac{x}{b_n}\right)$ and b_n is an increasing and unbounded sequence of positive numbers such that

$$\lim_{n\to\infty} b_n = \infty. \tag{1.4}$$

It is clear that for $x \in [0,b_n)$ and $q \in (0,1)$ these operators are linear and positive. Note that by the condition (1.4), the interval $[0,b_n)$ which is the domain of the operators $L_{n,q}$ expands infinity when $n \to \infty$. In this case, by means of the maximum norm we do not study the approximation properties of these operators. So we will give some convergence properties of the operators $L_{n,q}$ in weighted spaces of continuous functions on positive semi-axis with weighted norm.

2. Convergence properties

In order to give some convergence properties of the operators $L_{n,q}$ we state the following lemma.

Lemma 1. *The operators* $L_{n,q}$ *defined by* (1.3) *verify*

$$L_{n,q}(1;x) = 1$$
$$L_{n,q}(t;x) = x$$
$$x^2 \le L_{n,q}(t^2;x) \le qx^2 + \frac{b_n}{[n]_q}x$$

for all $x \in [0,b_n)$.

Proof. By means of the following equation (see [4, p. 490])

$$\frac{1}{(x,q)_N} = \sum_{k=0}^{\infty}\begin{bmatrix} N+k-1 \\ k \end{bmatrix}_q x^k, \tag{2.1}$$

it is easy to verify that $L_{n,q}(1;q,x) = 1$ and $L_{n,q}(t;x) = x$.

A direct computation yields that

$$L_{n,q}\big(t^2;x\big) = P_{n,q}\big(x\big)\sum_{k=1}^{\infty}\frac{[k]_q}{[n+k]_q}b_n^2\frac{[n+k-1]_q!}{[n]_q![k-1]_q!}\left(\frac{x}{b_n}\right)^k. \tag{2.2}$$

Since $[k]_q = q[k-1]_q + 1$ for $k \geq 0$, we get

$$\begin{aligned} L_{n,q}\big(t^2;x\big) &= qx^2P_{n,q}\big(x\big)\sum_{k=2}^{\infty}\frac{1}{[n+k]_q}\frac{[n+k-1]_q!}{[n]_q![k-2]_q!}\left(\frac{x}{b_n}\right)^{k-2} \\ &\quad + b_nxP_{n,q}\big(x\big)\sum_{k=1}^{\infty}\frac{1}{[n+k]_q}\frac{[n+k-1]_q!}{[n]_q![k-1]_q!}\left(\frac{x}{b_n}\right)^{k-1} \\ &= qx^2P_{n,q}\big(x\big)\sum_{k=0}^{\infty}\frac{[n+k+1]_q}{[n+k+2]_q}\begin{bmatrix}n+k\\k\end{bmatrix}_q\left(\frac{x}{b_n}\right)^k \\ &\quad + b_nxP_{n,q}\big(x\big)\sum_{k=0}^{\infty}\frac{1}{[n+k+1]_q}\begin{bmatrix}n+k\\k\end{bmatrix}_q\left(\frac{x}{b_n}\right)^k. \end{aligned} \tag{2.3}$$

By using the inequalities

$$[n+k+1]_q < [n+k+2]_q, \qquad [n]_q < [n+k+1]_q$$

from (2.3) one has

$$L_{n,q}\big(t^2;x\big) \leq qx^2 + \frac{b_n}{[n]_q}x. \tag{2.4}$$

In a similar way that of (2.4) by using the facts that

$$[n+k+1]_q < [n+k+2]_q, \qquad [n+k+1]_q = \frac{[n+k+2]_q - 1}{q}$$

and taking into consideration $\frac{x}{b_n} \in \big[0,1\big)$ it can be proved that

$$x^2 \leq L_{n,q}\big(t^2;x\big). \tag{2.5}$$

Hence, combining (2.4) with (2.5) we arrive at the desired result. □

Lemma 2. *The operators $L_{n,q}$ defined by* (1.3) *verify*

$$\begin{aligned} L_{n,q_n}\Big(\big(t-x\big)^4;x\Big) &\leq \big(q_n^6 - 4q_n^3 + 6q_n - 3\big)x^4 \\ &\quad + \Big[\big(1+[2]_{q_n}+[3]_{q_n}\big)q_n^3 - 4\big(1+[2]_{q_n}\big)q_n + 6\Big]\frac{b_n}{[n]_{q_n}}x^3 \\ &\quad + \Big[\big(1+[2]_{q_n}+[2]_{q_n}^2\big)q_n - 4\Big]\frac{b_n^2}{[n]_{q_n}^2}x^2 + \frac{b_n^3}{[n]_{q_n}^3}x \end{aligned}$$

for all $x \in \big[0,b_n\big)$.

Since the proof is similar the proof of Lemma 1, we can omit it.

For a fixed value of q with $0 < q < 1$ we have $\lim\limits_{n\to\infty} [n]_q = \frac{1}{1-q}$. So, to guarantee convergence properties of the operators $L_{n,q}$, we will replace q by a sequence q_n such that $\lim\limits_{n\to\infty} q_n = 1$.

Now we give approximation properties of the operators L_{n,q_n} in weighted spaces of continuous functions with the help of weighted Korovkin type theorem given by Gadjiev in [11, 12]. For this purpose, let us recall the notations and results of [11, 12].

$B_\rho\big[0,\infty\big)$: The space of all functions satisfying the condition

$$|f(x)| \le M_f \rho(x)$$

where $x \in \big[0,\infty\big)$, M_f is a positive constant depending only on f and $\rho(x) = 1 + x^2$.

$C_\rho\big[0,\infty\big)$: The space of all continuous functions in the space $B_\rho\big[0,\infty\big)$.

$C_\rho^0\big[0,\infty\big)$: The subspace of all functions $f \in C_\rho\big[0,\infty\big)$ for which

$$\lim_{x\to\infty} \frac{\big|f(x)\big|}{\rho(x)} < \infty.$$

The space $B_\rho\big[0,\infty\big)$ is a linear normed space with the following norm:

$$\big\|f\big\|_\rho = \sup_{x\in[0,\infty)} \frac{\big|f(x)\big|}{\rho(x)}.$$

Theorem A. ([11, 12]) *Let A_n be a sequence of positive linear operators acting from $C_\rho\big[0,\infty\big)$ to $B_\rho\big[0,\infty\big)$ satisfying the conditions*

$$\lim_{n\to\infty} \big\|A_n(t^\nu;x) - x^\nu\big\|_\rho = 0, \qquad \nu = 0,1,2.$$

Then for any function $f \in C_\rho^0\big[0,\infty\big)$,

$$\lim_{n\to\infty} \big\|A_n(f;x) - f(x)\big\|_\rho = 0$$

where $\rho(x) = 1 + x^2$.

We observe that a sequence of positive linear operators A_n acts from $C_\rho\big[0,\infty\big)$ to $B_\rho\big[0,\infty\big)$ if and only if

$$\big\|A_n(\rho;x)\big\|_\rho \le M_\rho$$

where ρ defined as in Theorem A and M_ρ is a positive constant. This fact is a simple corollary of the necessary and sufficient condition that $A_n(\rho;x) \le M\rho(x)$ given in [11, 12].

Lemma 3. *Let q_n be a sequence such that $\lim_{n\to\infty} q_n = 1$ for $0 < q_n < 1$. If $\lim_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$, then for all n the sequence of positive linear operators L_{n,q_n} acts from $C_\rho\big[0,\infty\big)$ to $B_\rho\big[0,\infty\big)$ where $\rho(x) = 1 + x^2$.*

Proof. By Lemma 1, we get

$$\sup_{x\in[0,b_n)} \frac{L_{n,q_n}\big(\rho;x\big)}{1+x^2} \le 1 + q_n + \frac{b_n}{[n]_{q_n}}.$$

Since, $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$ there is a positive constant M such that $q_n + \frac{b_n}{[n]_{q_n}} < M$ for each n. Thus we have

$$\sup_{x\in[0,b_n)} \frac{L_{n,q_n}\big(\rho;x\big)}{1+x^2} \le 1 + M$$

which completes the proof. □

We observe that for example, for $q_n = \frac{n}{n+1}$ and $b_n = \sqrt{n}$ the conditions of Lemma 3 are satisfied.

Theorem 1. *Let q_n be a sequence such that $\lim_{n\to\infty} q_n = 1$ for $0 < q_n < 1$. If $\lim_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$, then for each $f \in C_\rho^0\big[0,\infty\big)$,*

$$\lim_{n\to\infty} \sup_{x\in[0,b_n)} \frac{\big|L_{n,q_n}\big(f;x\big) - f\big(x\big)\big|}{1+x^2} = 0.$$

Proof. We have seen that L_{n,q_n} acts from $C_\rho\big[0,\infty\big)$ to $B_\rho\big[0,\infty\big)$ for all n. Therefore, to complete the proof applying Theorem A to the operators

$$A_n\big(f;x\big) = \begin{cases} L_{n,q_n}\big(f;x\big), & \text{if } x \in \big[0,b_n\big) \\ f(x), & \text{if } x \ge b_n \end{cases}$$

it is sufficient to verify that the conditions

$$\lim_{n\to\infty} \sup_{x\in[0,b_n)} \frac{\big|L_{n,q_n}\big(t^\nu;x\big) - x^\nu\big|}{1+x^2} = 0, \qquad \nu = 0,1,2,$$

are valid. From Lemma 1, we immediately see that this condition is satisfied for $\nu = 0,1$. Again by Lemma 1, one gets

$$\sup_{x\in[0,b_n)} \frac{\big|L_{n,q_n}\big(t^2;x\big) - x^2\big|}{1+x^2} \le 1 - q_n + \frac{b_n}{[n]_{q_n}}.$$

Since $\lim\limits_{n\to\infty} q_n = 1$ and $\lim\limits_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$, this implies that

$$\lim_{n\to\infty} \sup_{x\in[0,b_n)} \frac{\left|L_{n,q_n}\left(t^2;x\right) - x^2\right|}{1+x^2} = 0.$$

Thus the proof is completed. □

Now recall the definition of the weighted modulus of continuity $\Omega\big(f;\delta\big)$ which given in [15]. Let $f \in C_\rho^0\big[0,\infty\big)$. The weighted modulus of continuity of f is defined by

$$\Omega\big(f;\delta\big) = \sup_{|h|\le\delta,\, x\in[0,b_n)} \frac{\left|f\left(x+h\right) - f\left(x\right)\right|}{\left(1+h^2\right)\left(1+x^2\right)}. \tag{2.6}$$

In [15], it is proved that

$$\lim_{\delta\to 0} \Omega\big(f;\delta\big) = 0$$

and

$$\left|f\left(t\right) - f\left(x\right)\right| \le 2\left(1+\delta^2\right)\Omega\left(f;\delta\right)\left(1+x^2\right)\left(1+\frac{|t-x|}{\delta}\right)\left(1+(t-x)^2\right) \tag{2.7}$$

for each $f \in C_\rho^0\big[0,\infty\big)$ and $\delta > 0$.

Theorem 2. *Let q_n be a sequence such that $\lim\limits_{n\to\infty} q_n = 1$ for $0 < q_n < 1$. If $\lim\limits_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$, then for each $f \in C_\rho^0\big[0,\infty\big)$ the inequality*

$$\sup_{x\in[0,b_n)} \frac{\left|L_{n,q_n}\left(f;x\right) - f\left(x\right)\right|}{1+x^2} \le 2w\big(f;\delta_{n,q_n}\big),$$

where $w\big(f;\delta\big)$ is the usual modulus of continuity of the function f on the semi-axis $\big[0,\infty\big)$ and $\delta_{n,q_n} = \sqrt{1-q_n+\frac{b_n}{[n]_{q_n}}}$, is valid for sufficiently large n.

P r o o f. By the linearity and monotonicity of the operators L_{n,q_n} we have

$$\left|L_{n,q_n}\left(f;x\right) - f\left(x\right)\right| \le L_{n,q_n}\Big(\left|f\left(t\right) - f\left(x\right)\right|;x\Big). \tag{2.8}$$

On the other hand, by using the well known property of the modulus of continuity $w\big(f;\delta\big)$

$$\left|f\left(t\right) - f\left(x\right)\right| \le \left(1+\frac{|t-x|}{\delta}\right)\omega\left(f;\delta\right)$$

for every $\delta > 0$ and applying the Cauchy-Schwarz inequality, from (2.8) it follows that

$$\left|L_{n,q_n}\left(f;x\right) - f\left(x\right)\right| \le w\left(f;\delta\right)\left(1+\frac{1}{\delta}\sqrt{L_{n,q_n}\left((t-x)^2;x\right)}\right). \tag{2.9}$$

By means of Lemma 1 we have

$$\begin{aligned} L_{n,q_n}\big((t-x)^2;x\big) &= \Big[L_{n,q_n}\big(t^2;x\big)-x^2\Big]-2x\Big[L_{n,q_n}\big(t;x\big)-x\Big] \\ &= \big|L_{n,q_n}\big(t^2;x\big)-x^2\big| \\ &\le \big(1-q_n\big)x^2+\frac{b_n}{[n]_{q_n}}x \end{aligned}$$

which yields

$$\sup_{x\in[0,b_n)}\frac{L_{n,q_n}\big((t-x)^2;x\big)}{(1+x^2)^2}\le 1-q_n+\frac{b_n}{[n]_{q_n}}.$$

Thus if we choose $\delta=\delta_{n,q_n}$ from (2.9) we find the desired result. □

Theorem 3. *Let q_n be a sequence such that $\lim_{n\to\infty} q_n = 1$ for $0<q_n<1$. If $\lim_{n\to\infty}\frac{b_n}{[n]_{q_n}}=0$, then for each $f\in C^0_\rho\big[0,\infty\big)$ the inequality*

$$\sup_{x\in[0,b_n)}\frac{\big|L_{n,q_n}\big(f;x\big)-f\big(x\big)\big|}{\big(1+x^2\big)^3}\le M\Omega\big(f;K^{1\backslash 4}_{n,q_n}\big)$$

is valid for sufficiently large n. Here M is a positive constant independent of n, $\Omega\big(f;\delta\big)$ is the weighted modulus of continuity defined by (2.6) and

$$\begin{aligned} K_{n,q_n}=\max\Bigg\{ & q_n^6-4q_n^3+6q_n-3, \\ & \Big[\big(1+[2]_{q_n}+[3]_{q_n}\big)q_n^3-4\big(1+[2]_{q_n}\big)q_n+6\Big]\frac{b_n}{[n]_{q_n}}, \\ & \Big[\big(1+[2]_{q_n}+[2]^2_{q_n}\big)q_n-4\Big]\frac{b_n^2}{[n]^2_{q_n}},\frac{b_n^3}{[n]^3_{q_n}}\Bigg\}. \end{aligned}$$

P r o o f. By the linearity and monotonicity of the operators L_{n,q_n} from (2.7) one gets

$$\begin{aligned} &\big|L_{n,q_n}\big(f;x\big)-f\big(x\big)\big| \\ &\le 2\big(1+\delta^2\big)\Omega\big(f;\delta\big)(1+x^2)L_{n,q_n}\left(\left(1+\frac{|t-x|}{\delta}\right)\big(1+(t-x)^2\big);x\right). \end{aligned}$$

For all $x\in\big[0,b_n\big)$ and $t\in\big[0,\infty\big)$ using the inequality (see [15, p. 361])

$$\left(1+\frac{|t-x|}{\delta}\right)\big(1+(t-x)^2\big)\le 2\big(1+\delta^2\big)\left(1+\frac{(t-x)^4}{\delta^4}\right)$$

we have

$$\big|L_{n,q_n}\big(f;x\big)-f\big(x\big)\big|\le 16\Omega\big(f;\delta\big)\big(1+x^2\big)\left[1+\frac{1}{\delta^4}L_{n,q_n}\big((t-x)^4;x\big)\right]. \quad (2.10)$$

From Lemma 2 and (2.10) it follows that

$$\left|L_{n,q_n}\big(f;x\big)-f\big(x\big)\right|\leq 16\Omega\big(f;\delta\big)\big(1+x^2\big)\left[1+\frac{1}{\delta^4}K_{n,q_n}\big(x^4+x^3+x^2+x\big)\right].$$

Choosing $\delta=K_{n,q_n}^{1\backslash 4}$ then we have

$$\left|L_{n,q_n}\big(f;x\big)-f\big(x\big)\right|\leq 16\Omega\big(f;K_{n,q_n}^{1\backslash 4}\big)\big(1+x^2\big)\big(x^4+x^3+x^2+x+1\big)$$

which gives the desired result. □

Remark 1. In Theorem 2, we obtained the rate of convergence of the operators L_{n,q_n} in the space $C_\rho\big[0,\infty\big)$ by means of the usual modulus of continuity and in Theorem 3, we presented the rate of convergence for the operators L_{n,q_n} in the space $C_{\rho^3}\big[0,\infty\big)$ with the help of the weighted modulus of continuity.

3. Application to differential equations

Finally, we shall give a functional differential equation related the q-derivatives for the operators $L_{n,q}$ defined by (1.3). In [3, 8, 10, 17, 22] there exist equations without q-integers similar to the one in Theorem 4.

We now introduce the q-derivative.

For $0<q<1$ the q-derivative $D_qf(x)$ of f denoted by

$$D_qf(x)=\frac{f(x)-f(qx)}{(1-q)x},\quad x\neq 0,\qquad D_qf(0)=\lim_{x\to 0}D_qf(x).$$

Theorem 4. *Let* $g_n(t)=\frac{q^nt}{b_n-q^nt}$ *for* $0<q<1$ *and also let* $f\in C_\rho^0\big[0,\infty\big)$ *and* $x\in\big[0,b_n\big)$. *Then the operators* $L_{n,q}$ *defined by* (1.3) *satisfies the functional differential equation*

$$\frac{q^n}{[n]_q}x\left(1-\frac{x}{b_n}\right)D_qL_{n,q}\big(f;x\big)$$
$$-\left(1-q^{n+1}\frac{x}{b_n}\right)L_{n,q}\big(fg_n;x\big)+\frac{q^n[n+1]_q}{[n]_qb_n}xL_{n,q}\big(f;x\big)=0$$

for each n.

P r o o f. From the definition of the q-derivative, one gets

$$\begin{aligned}&\frac{q^n}{[n]_q}x\left(1-\frac{x}{b_n}\right)D_qL_{n,q}(f;x)\\&=\frac{q^n}{[n]_q}\left(1-\frac{x}{b_n}\right)\left[\frac{L_{n,q}(f;x)-L_{n,q}(f;qx)}{1-q}\right]\\&=\frac{q^n}{(1-q)[n]_q}\left(1-\frac{x}{b_n}\right)L_{n,q}(f;x)-\frac{q^n}{(1-q)[n]_q}\left(1-\frac{x}{b_n}\right)\prod_{s=0}^{n}\left(1-q^{s+1}\frac{x}{b_n}\right)\\&\quad\times\sum_{k=0}^{\infty}f\left(\frac{[k]_q}{[n+k]_q}b_n\right)\begin{bmatrix}n+k\\k\end{bmatrix}_q q^k\left(\frac{x}{b_n}\right)^k.\end{aligned}$$

Since

$$\left(1-\frac{x}{b_n}\right)\prod_{s=0}^{n}\left(1-q^{s+1}\frac{x}{b_n}\right)=\left(1-q^{n+1}\frac{x}{b_n}\right)P_{n,q}(x)$$

we can write

$$\begin{aligned}&\frac{q^n}{[n]_q}x\left(1-\frac{x}{b_n}\right)D_qL_{n,q}(f;x)\\&=\frac{q^n}{(1-q)[n]_q}\left(1-\frac{x}{b_n}\right)L_{n,q}(f;x)\\&\quad-\frac{q^n}{(1-q)[n]_q}\left(1-q^{n+1}\frac{x}{b_n}\right)P_{n,q}(x)\sum_{k=0}^{\infty}f\left(\frac{[k]_q}{[n+k]_q}b_n\right)\begin{bmatrix}n+k\\k\end{bmatrix}_q q^k\left(\frac{x}{b_n}\right)^k.\end{aligned}\tag{3.1}$$

By the definition of q-integers, it is easily seen that

$$q^k=(q-1)[k]+1.$$

So, substitution of this into (3.1) gives

$$\begin{aligned}&\frac{q^n}{[n]_q}x\left(1-\frac{x}{b_n}\right)D_qL_{n,q}(f;x)\\&=\frac{q^n}{(1-q)[n]_q}\left[\left(1-\frac{x}{b_n}\right)-\left(1-q^{n+1}\frac{x}{b_n}\right)\right]L_{n,q}(f;x)\\&\quad+\left(1-q^{n+1}\frac{x}{b_n}\right)P_{n,q}(x)\sum_{k=1}^{\infty}f\left(\frac{[k]_q}{[n+k]_q}b_n\right)\frac{q^n[k]_q}{[n]_q}\begin{bmatrix}n+k\\k\end{bmatrix}_q\left(\frac{x}{b_n}\right)^k\end{aligned}$$

$$= -\frac{q^n[n+1]_q}{[n]_q b_n} x L_{n,q}(f;x) + \left(1 - q^{n+1}\frac{x}{b_n}\right) P_{n,q}(x)$$
$$\times \sum_{k=1}^{\infty} f\left(\frac{[k]_q}{[n+k]_q} b_n\right) \frac{q^n[k]_q}{[n]_q} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k.$$

Since

$$g_n\left(\frac{[k]_q}{[n+k]_q} b_n\right) = \frac{q^n[k]_q}{[n]_q}$$

the proof is completed. □

4. A Stancu-type remainder of $L_{n,q}$

In this section, we give Stancu-type remainder for the operators $L_{n,q}$ by means of the divided differences.

Let $x_0, x_1, \ldots, x_n$ be distinct points in the domain of f. Then the divided differences of a function f is given by

$$f[x_0, x_1, \ldots, x_n] = \sum_{r=0}^{n} \frac{f(x_r)}{\prod\limits_{j\neq r}^{n} (x_r - x_j)}$$

where r remains fixed and j takes all values from 0 to n, except r (see [5]).

Theorem 5. *If* $x \in [0, b_n) \setminus \left\{\frac{[k]_q}{[n+k]_q} b_n \;:\; k = 0, 1, 2, \ldots\right\}$ *then the following formula is valid*

$$L_{n,q}(f;x) - f(x) = b_n x P_{n,q}(x) \sum_{k=0}^{\infty} \frac{q^k}{[n+k+1]_q} f\left[x, \frac{[k]_q}{[n+k]_q} b_n, \frac{[k+1]_q}{[n+k+1]_q} b_n\right]$$
$$\times \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k.$$

P r o o f. By the linearity of the operators $L_{n,q}$, we can write

$$L_{n,q}(f;x) - f(x) = -P_{n,q}(x) \sum_{k=0}^{\infty} \left\{ f(x) - f\left(\frac{[k]_q}{[n+k]_q} b_n\right) \right\} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k. \tag{4.1}$$

Since,

$$f(x) - f\left(\frac{[k]_q}{[n+k]_q} b_n\right) = \left(x - \frac{[k]_q}{[n+k]_q} b_n\right) f\left[x, \frac{[k]_q}{[n+k]_q} b_n\right]$$

and

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q \frac{[k]_q}{[n+k]_q} b_n = \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_q b_n$$

from (4.1) it follows that

$$\begin{aligned} &L_{n,q}(f;x) - f(x) \\ &= b_n P_{n,q}(x) \sum_{k=1}^{\infty} f\left[x, \frac{[k]_q}{[n+k]_q} b_n\right] \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \\ &\quad - x P_{n,q}(x) \sum_{k=0}^{\infty} f\left[x, \frac{[k]_q}{[n+k]_q} b_n\right] \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \\ &= x P_{n,q}(x) \sum_{k=0}^{\infty} \left\{ f\left[x, \frac{[k+1]_q}{[n+k+1]_q} b_n\right] - f\left[x, \frac{[k]_q}{[n+k]_q} b_n\right] \right\} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k . \end{aligned} \tag{4.2}$$

By substituting

$$\begin{aligned} &f\left[x, \frac{[k+1]_q}{[n+k+1]_q} b_n\right] - f\left[x, \frac{[k]_q}{[n+k]_q} b_n\right] \\ &= f\left[x, \frac{[k]_q}{[n+k]_q} b_n, \frac{[k+1]_q}{[n+k+1]_q} b_n\right] \left(\frac{[k+1]_q}{[n+k+1]_q} - \frac{[k]_q}{[n+k]_q} \right) b_n \end{aligned}$$

into (4.2) and by using the equalities

$$\frac{[k+1]_q}{[n+k+1]_q} - \frac{[k]_q}{[n+k]_q} = \frac{q^k [n]_q}{[n+k]_q [n+k+1]_q}$$

and

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q \frac{q^k [n]_q}{[n+k]_q [n+k+1]_q} = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \frac{q^k}{[n+k+1]_q}$$

we arrive at the desired result. □

We recall that a function f is convex on $I \subset \mathbb{R}$ if and only if its divided differences is nonnegative. So we now state the following remark.

Remark 2. If f is convex on $[0, b_n)$, then $L_{n,q}(f;x) - f(x) \geq 0$ for all $n \in \mathbb{N}$, $x \in [0, b_n) \setminus \left\{ \frac{[k]_q}{[n+k]_q} b_n : \ k = 0, 1, 2, \dots \right\}$ and $0 < q < 1$.

Remark 3. When $q = 1$ all of the results of this paper is valid for the operators $M_n\big(f, b_n; x\big)$ defined by Rempulska and Skorupka [20].

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* *Abant İzzet Baysal University*
Faculty of Arts and Sciences
Department of Mathematics
14280, Bolu
TURKEY
E-mail: erencina@hotmail.com

** *Gazi University*
Faculty of Arts and Sciences
Department of Mathematics
06500, Teknikokullar Ankara
TURKEY
E-mail: ince@gazi.edu.tr

*** *Kirikkale University*
Faculty of Arts and Sciences
Department of Mathematics
71450, Yahşihan
Kirikkale
TURKEY
E-mail: aolgun@kku.edu.tr