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On the q analogue of Stancu-Beta operators

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ABSTRACT

In the present work we introduce the q analogue of well known Stancu-Beta operators. We estimate moments and establish direct results in terms of the modulus of continuity. We also present an asymptotic formula.

1. Introduction

Stancu [1] introduced Beta operators L_n of the second kind in order to approximate the Lebesgue integrable functions on the interval $(0, \infty)$ as

$$L_n(f; x) = \frac{1}{B(nx, n+1)} \int_0^\infty \frac{t^{nx-1}}{(1+t)^{nx+n+1}} f(t) \mathrm{d}t. \tag{1.1}$$

Abel and Gupta [2] and Gupta et al. [3] estimated the rate of convergence for functions and for functions with derivatives of bounded variation respectively, for the operators (1.1). As the q calculus has been one of the most interesting areas of research in the last decade, this motivated us to introduce the q analogue of the Stancu-Beta operators. First, we mention certain notation for q calculus as follows. For each nonnegative integer k, the q-integer $[k]_q$ is defined by

$$[k]_q := \begin{cases} (1-q^k)/(1-q), & q \neq 1 \\ k, & q = 1. \end{cases}$$

The q-factorial $[k]_q!$ is defined as

$$[k]_q! := \begin{cases} [k]_q [k-1]_q \cdots [1]_q, & k \geq 1 \\ 1, & k = 0. \end{cases}$$

For the integers $n, k,\ n \geq k \geq 0$, the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

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The q-improper integral is defined as (see [4])

$$\int_0^{\infty/A} f(x)\mathrm{d}_q x = (1-q)\sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A}, \quad A > 0.$$

We consider $(a+b)_q^n = \prod_{j=0}^{n-1}(a+q^j b)$. The q-Beta integral representations are as follows:

$$B_q(t,s) = K(A,t)\int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}}\mathrm{d}_q x,$$

where

$$K(A,t+1) = q^t K(A,t)$$

for $A > 0$ (see [5]).

The present work deals with the q analogue of the well known Stancu-Beta operators. Here we estimate moments, the recurrence relation, and some direct results in terms of the modulus of continuity of the q-Stancu-Beta operators.

2. q-Stancu-Beta operators and moments

Definition 1. For $0 < q < 1$, we propose the q analogue of Stancu-Beta operators as

$$L_n^q(f;x) = \frac{K\left(A,[n]_q x\right)}{B_q\left([n]_q x,[n]_q+1\right)}\int_0^{\infty/A}\frac{u^{[n]_q x-1}}{(1+u)_q^{[n]_q x+[n]_q+1}} f\left(q^{[n]_q x}u\right)\mathrm{d}_q u.$$

Lemma 1. *We have*

$$L_n^q(1;x) = 1, \qquad L_n^q(t;x) = x, \qquad L_n^q\left(t^2;x\right) = \frac{\left([n]_q x+1\right)x}{q\left([n]_q-1\right)}.$$

Proof. By the definition of q-Stancu-Beta operators, we have

$$\begin{aligned} L_n^q(1;x) &= \frac{K\left(A,[n]_q x\right)}{B_q\left([n]_q x,[n]_q+1\right)}\int_0^{\infty/A}\frac{u^{[n]_q x-1}}{(1+u)_q^{[n]_q x+[n]_q+1}}\mathrm{d}_q u \\ &= 1. \end{aligned}$$

Next, we have

$$\begin{aligned} L_n^q(t;x) &= \frac{K\left(A,[n]_q x\right)q^{[n]_q x}}{B_q\left([n]_q x,[n]_q+1\right)}\int_0^{\infty/A}\frac{u^{[n]_q x}}{(1+u)_q^{[n]_q x+[n]_q+1}}\mathrm{d}_q u \\ &= \frac{K\left(A,[n]_q x\right)q^{[n]_q x}}{B_q\left([n]_q x,[n]_q+1\right)}\frac{B_q\left([n]_q x+1,[n]_q\right)}{K\left(A,[n]_q x+1\right)} \\ &= \frac{K\left(A,[n]_q x\right)q^{[n]_q x}}{B_q\left([n]_q x,[n]_q+1\right)}\frac{\Gamma_q\left([n]_q x+1\right)\Gamma_q\left([n]_q\right)}{K\left(A,[n]_q x+1\right)\Gamma_q\left([n]_q x+[n]_q+1\right)} \\ &= K\left(A,[n]_q x\right)q^{[n]_q x}\frac{\Gamma_q\left([n]_q x+[n]_q+1\right)}{\Gamma_q\left([n]_q x\right)\Gamma_q\left([n]_q+1\right)}\frac{1}{q^{[n]_q x}K\left(A,[n]_q x\right)}\frac{\Gamma_q\left([n]_q x+1\right)\Gamma_q\left([n]_q\right)}{\Gamma_q\left([n]_q x+[n]_q+1\right)} \\ &= \frac{\Gamma_q\left([n]_q x+1\right)\Gamma_q\left([n]_q\right)}{\Gamma_q\left([n]_q x\right)\Gamma_q\left([n]_q+1\right)} \\ &= \frac{[n]_q x\Gamma_q\left([n]_q x\right)\Gamma_q\left([n]_q\right)}{\Gamma_q\left([n]_q x\right)[n]_q\Gamma_q\left([n]_q\right)} = x. \end{aligned}$$

Finally, by using the q-Beta integral, we have

$$\begin{aligned} L_n^q\left(t^2;x\right) &= \frac{K\left(A,[n]_q x\right)q^{2[n]_q x}}{B_q\left([n]_q x,[n]_q+1\right)}\int_0^{\infty/A}\frac{u^{[n]_q x+1}}{(1+u)_q^{[n]_q x+[n]_q+1}}\mathrm{d}_q u \\ &= \frac{K\left(A,[n]_q x\right)q^{2[n]_q x}}{B_q\left([n]_q x,[n]_q+1\right)}\frac{B_q\left([n]_q x+2,[n]_q-1\right)}{K\left(A,[n]_q x+2\right)} \end{aligned}$$

$$= q^{2[n]_q x} K\left(A, [n]_q x\right) \frac{\Gamma_q\left([n]_q x + [n]_q + 1\right)}{\Gamma_q\left([n]_q x\right) \Gamma_q\left([n]_q + 1\right)}$$

$$\times \frac{1}{q^{\left([n]_q x + 1\right)} q^{[n]_q x} K\left(A, [n]_q x\right)} \frac{\Gamma_q\left([n]_q x + 2\right) \Gamma_q\left([n]_q - 1\right)}{\Gamma_q\left([n]_q x + [n]_q + 1\right)}$$

$$= \frac{\Gamma_q\left([n]_q x + 2\right) \Gamma_q\left([n]_q - 1\right)}{\Gamma_q\left([n]_q x\right) \Gamma_q\left([n]_q + 1\right)} \frac{1}{q}$$

$$= \frac{\left([n]_q x + 1\right) [n]_q x \Gamma_q\left([n]_q x\right) \Gamma_q\left([n]_q - 1\right)}{\Gamma_q\left([n]_q x\right) [n]_q \left([n]_q - 1\right) \Gamma_q\left([n]_q - 1\right)} \frac{1}{q}$$

$$= \frac{\left([n]_q x + 1\right) x}{q\left([n]_q - 1\right)}. \quad \square$$

Remark 1. Suppose that $q \in (0, 1)$; then for $x \in [0, \infty)$, we have

$$L_n^q(t - x; x) = 0$$

$$L_n^q\left((t - x)^2; x\right) = \frac{([n]_q - q[n]_q + q)x^2 + x}{q([n]_q - 1)}.$$

Remark 2. Suppose that $q \in (0, 1)$; then for $x \in [0, \infty)$, and proceeding along the lines of the proof of Lemma 1, we have the following formula for the mth-order moment:

$$L_n^q\left(t^m; x\right) = \frac{\Gamma_q\left([n]_q x + m\right) \Gamma_q\left([n]_q - m + 1\right)}{\Gamma_q\left([n]_q x\right) \Gamma_q\left([n]_q + 1\right) q^{m(m-1)/2}}.$$

3. Direct theorems

We denote by $C_B[0, \infty)$ the space of real valued continuous bounded functions f on the interval $[0, \infty)$; the norm $\|.\|$ on the space $C_B[0, \infty)$ is given by

$$\|f\| = \sup_{0 \le x < \infty} |f(x)|.$$

Peetre's K-functional is defined by

$$K_2(f, \delta) = \inf[\{\|f - g\| + \delta \|g''\| : g \in W^2\}],$$

where $W^2 = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$. By [6], there exists a positive constant $C > 0$ such that $K_2(f, \delta) \le C\omega_2(f, \delta^{1/2}),\ \delta > 0$ where the second-order modulus of smoothness is given by

$$\omega_2(f, \sqrt{\delta}) = \sup_{0 < h \le \sqrt{\delta}} \sup_{0 \le x < \infty} |f(x + 2h) - 2f(x + h) + f(x)|.$$

Also for $f \in C_B[0, \infty)$ the usual modulus of continuity is given by

$$\omega(f, \delta) = \sup_{0 < h \le \delta} \sup_{0 \le x < \infty} |f(x + h) - f(x)|.$$

Theorem 1. *Suppose that $f \in C_B[0, \infty)$ and $0 < q < 1$. Then for all $x \in [0, \infty)$ and $n \in N$, there exists an absolute constant $C > 0$ such that*

$$|L_n^q(f; x) - f(x)| \le C\omega_2\left(f, \delta_n(x)\right),$$

where $\delta_n^2(x) = \frac{([n]_q - q[n]_q + q)x^2 + x}{q([n]_q - 1)}$.

Proof. Suppose that $g \in W^2$. From Taylor's expansion

$$g(t) = g(x) + g'(x)(t - x) + \int_x^t (t - u)g''(u)\mathrm{d}u, \quad t \in [0, \infty)$$

and Lemma 1, we get

$$L_n^q(g;x) = g(x) + L_n^q\left(\int_x^t (t-u)g''(u)\mathrm{d}u, x\right).$$

Hence

$$\begin{aligned}|L_n^q(g;x) - g(x)| &\le \left|L_n^q\left(\int_x^t (t-u)g''(u)\mathrm{d}u, x\right)\right| \le L_n^q\left(\left|\int_x^t |t-u||g''(u)|\mathrm{d}u\right|, x\right)\\ &\le L_n^q\left((t-x)^2, x\right)\|g''\|.\end{aligned}$$

Using Remark 1, we obtain

$$|L_n^q(g;x) - g(x)| \le \frac{([n]_q - q[n]_q + q)x^2 + x}{q([n]_q - 1)}\|g''\|.$$

On the other hand, by the definition of $L_n^q(f;x)$, we have

$$|L_n^q(f;x)| \le \|f\|.$$

Next,

$$\begin{aligned}|L_n^q(f;x) - f(x)| &\le |L_n^q(f-g;x) - (f-g)(x)| + |L_n^q(g;x) - g(x)|\\ &\le \|f-g\| + \frac{([n]_q - q[n]_q + q)x^2 + x}{q([n]_q - 1)}\|g''\|.\end{aligned}$$

Hence taking the infimum on the right hand side over all $g \in W^2$, we get

$$|L_n^q(f;x) - f(x)| \le CK_2\left(f, \delta_n^2(x)\right).$$

In view of the property of the K-functional for every $q \in (0,1)$, we get

$$|L_n^q(f,x) - f(x)| \le C\omega_2\left(f, \delta_n(x)\right).$$

This completes the proof of the theorem. □

Let $B_{x^2}[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)| \le M_f\left(1+x^2\right)$, where M_f is a constant depending only on f. We denote by $C_{x^2}[0,\infty)$ the subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$. Also, let $C_{x^2}^*[0,\infty)$ be the subspace of all functions $f \in C_{x^2}[0,\infty)$ for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on $C_{x^2}^*[0,\infty)$ is $\|f\|_{x^2} = \sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}$.

Theorem 2. *Let $f \in C_{x^2}^*[0,\infty)$ be such that $f', f'' \in C_{x^2}^*[0,\infty)$ and $q = q_n \in (0,1)$ such that $q_n \to 1$ as $n \to \infty$; then the following equality holds:*

$$\lim_{n\to\infty}[n]_{q_n}\left(L_n^{q_n}(f;x) - f(x)\right) = \frac{x(1+x)}{2}f''(x)$$

uniformly on $[0,A]$, $A > 0$.

Proof. By Taylor's formula we may write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + r(t,x)(t-x)^2, \tag{3.1}$$

where $r(t,x)$ is the remainder term and $\lim_{t\to x} r(t,x) = 0$. Applying $L_n^{q_n}(f;x)$ to (3.1), we obtain

$$\begin{aligned}[n]_{q_n}\left(L_n^{q_n}(f;x) - f(x)\right) &= [n]_{q_n}L_n^{q_n}(t-x;x)f'(x) + [n]_{q_n}L_n^{q_n}\left((t-x)^2;x\right)\frac{f''(x)}{2}\\ &\quad + [n]_{q_n}L_n^{q_n}\left(r(t,x)(t-x)^2;x\right).\end{aligned}$$

By the Cauchy–Schwartz inequality, we have

$$L_n^{q_n}\left(r(t,x)(t-x)^2;x\right) \le \sqrt{L_n^{q_n}\left(r^2(t,x);x\right)}\sqrt{L_n^{q_n}\left((t-x)^4;x\right)}. \tag{3.2}$$

Observe that $r^2(x,x) = 0$ and $r^2(.,x) \in C_{x^2}^*[0,\infty)$. Then it follows from Theorem 1 that

$$\lim_{n\to\infty} L_n^{q_n}\left(r^2(t,x);x\right) = r^2(x,x) = 0 \tag{3.3}$$

uniformly with respect to $x \in [0, A]$. Now from (3.2), (3.3) and Remark 2, we get immediately

$$\lim_{n\to\infty} [n]_{q_n} L_n^{q_n} \left(r\,(t,x)\,(t-x)^2\,; x\right) = 0.$$

Finally using Remark 1, we get the following:

$$\begin{aligned}
&\lim_{n\to\infty} [n]_{q_n} \left(L_n^{q_n}(f; x) - f(x)\right) \\
&\quad = \lim_{n\to\infty} [n]_{q_n} \left(f'(x) L_n^{q_n}\left((t-x)\,; x\right) + \frac{1}{2} f''(x) L_n^{q_n}\left((t-x)^2, x\right) + L_n^{q_n}\left(r\,(t,x)\,(t-x)^2\,; x\right)\right) \\
&\quad = \frac{x(1+x)}{2} f''(x). \quad \square
\end{aligned}$$

4. Weighted approximation

In this section we shall discuss the weighted approximation theorem.

Theorem 3. *Suppose that $q = q_n$ satisfies $0 < q_n < 1$ and suppose that $q_n \to 1$ as $n \to \infty$. For each $f \in C_{x^2}^*[0, \infty)$, we have*

$$\lim_{n\to\infty} \left\| L_n^{q_n}(f) - f \right\|_{x^2} = 0.$$

Proof. Using the theorem in [7] we see that it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty} \left\| L_n^{q_n}(t^\nu; x) - x^\nu \right\|_{x^2} = 0, \quad \nu = 0, 1, 2. \tag{4.1}$$

Since $L_n^{q_n}(1, x) = 1$ and $L_n^{q_n}(t, x) = x$, the first and second conditions of (4.1) are fulfilled for $\nu = 0$ and $\nu = 1$.

We can write

$$\left\| L_n^{q_n}\left(t^2, x\right) - x^2 \right\|_{x^2} \le \sup_{x\in[0,\infty)} \frac{\left|[n]_{q_n} - q_n\,[n]_{q_n} - q_n\right|}{q_n\left([n]_{q_n} - 1\right)} \frac{x^2}{1+x^2} + \sup_{x\in[0,\infty)} \frac{1}{q_n\left([n]_{q_n} - 1\right)} \frac{x}{1+x^2}$$

which implies that

$$\lim_{n\to\infty} \left\| L_n^{q_n}\left(t^2, x\right) - x^2 \right\|_{x^2} = 0.$$

Thus the proof is completed. $\square$

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