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# ON FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS OF FENG-LIU TYPE

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Abstract. In the present paper, considering the Jleli and Samet's technique we give many fixed point results for multivalued mappings on complete metric spaces without using the Pompeiu-Hausdorff metric. Our results are real generalization of some related fixed point theorems including the famous Feng and Liu's result in the literature. We also give some examples to both illustrate and show that our results are proper generalizations of the mentioned theorems.

## 1. Introduction and preliminaries

Let $X$ be any nonempty set. An element $x \in X$ is said to be a fixed point of a multivalued mapping $T : X \to P(X)$ if $x \in Tx$, where $P(X)$ denotes the family of all nonempty subsets of. Let $(X, d)$ be a metric space. We denote the family of all nonempty closed and bounded subsets of $X$ by $CB(X)$, the family of all nonempty closed subsets of $X$ by $C(X)$ and the family of all nonempty compact subsets of $X$ by $K(X)$. It is clear that $K(X) \subseteq CB(X) \subseteq C(X) \subseteq P(X)$. For $A, B \in C(X)$, let

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\},$$

where $d(x, B) = \inf \{d(x, y) : y \in B\}$. Then $H$ is called generalized Pompei-Hausdorff distance on $C(X)$. It is well known that $H$ is a metric on $CB(X)$, which is called Pompei-Hausdorff metric induced by $d$. We can find detailed information about the Pompeiu-Hausdorff metric in [4, 9].

Let $T : X \to CB(X)$. Then, $T$ is called multivalued contraction if there exists $L \in [0, 1)$ such that $H(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$ (see [15]). In 1969, Nadler [15] proved that every multivalued contraction on complete metric space has a fixed point. Then, the fixed point theory of multivalued contraction has been further developed in different directions by many authors, in particular, by Reich [17], Mizoguchi-Takahashi [14], Klim-Wardowski [12],

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Berinde-Berinde [3], Ćirić [5] and many others [6, 7, 11, 18]. Also, Feng and Liu [8] gave the following theorem without using generalized Pompei-Hausdorff distance. To state their result, we give the following notation for a multivalued mapping $T : X \to C(X)$: let $b \in (0,1)$ and $x \in X$ define

$$I_b^x = \{y \in Tx : bd(x,y) \leq d(x,Tx)\}.$$

**Theorem 1** ([8]). *Let $(X,d)$ be a complete metric space and $T : X \to C(X)$. If there exists a constant $c \in (0,1)$ such that there is $y \in I_b^x$ satisfying*

$$d(y,Ty) \leq cd(x,y)$$

*for all $x \in X$. Then $T$ has a fixed point in $X$ provided that $c < b$ and the function $x \to d(x,Tx)$ lower semi-continuous.*

As mentioned in Remark 1 of [8], we can see that Theorem 1 is a real generalization of Nadler's.

On the other hand, a new type of contractive mappings has been introduced by Jleli and Samet [10] for single-valued mappings. Throughout this study, we called it as $\theta$-contraction (see [13]).

Let $\Theta$ be the set of all functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

($\Theta_1$) $\theta$ is nondecreasing,

($\Theta_2$) for each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$,

($\Theta_3$) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

Let $(X,d)$ be a metric space, $T : X \to X$. Then, we say that $T$ is $\theta$-contraction if there exist $k \in (0,1)$ and $\theta \in \Theta$ such that

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k \tag{1.1}$$

for all $x,y \in X$ with $d(Tx,Ty) > 0$.

If we consider the different type of mapping $\theta$ in above, we obtain some of variety of contractions. For example, let $\theta : (0,\infty) \to (1,\infty)$ be given by $\theta(t) = e^{\sqrt{t}}$. It is clear that $\theta \in \Theta$. Then (1.1) turns to

$$d(Tx,Ty) \leq k^2 d(x,y) \tag{1.2}$$

for all $x,y \in X$ with $d(Tx,Ty) > 0$.

It is clear that for $x,y \in X$ such that $d(Tx,Ty) = 0$, the inequality (1.2) also holds. Therefore $T$ is an ordinary contraction. Similarly, let $\theta : (0,\infty) \to (1,\infty)$ be given by $\theta(t) = e^{\sqrt{te^t}}$. It is clear that $\theta \in \Theta$. Then (1.1) turns to

$$\frac{d(Tx,Ty)}{d(x,y)} e^{d(Tx,Ty)-d(x,y)} \leq k^2 \tag{1.3}$$

for all $x,y \in X$ with $d(Tx,Ty) > 0$.

In addition, we have concluded that every $\theta$-contraction $T$ is a contractive mapping, i.e., $d(Tx,Ty) < d(x,y)$ for all $x,y \in X$, $Tx \neq Ty$. Thus, every $\theta$-contraction is a continuous mapping. On the other side, Example in [10] shows that the mapping $T$ is not ordinary contraction but, it is a $\theta$-contraction with $\theta(t) = e^{\sqrt{te^t}}$. Thus, the following theorem, which was given as a corollary by Jleli and Samet, is a proper generalization of Banach Contraction Principle.

**Theorem 2** (Corollary 2.1 of [10])**.** *Let $(X,d)$ be a complete metric space and $T : X \to X$ be a $\theta$-contraction. Then $T$ has a unique fixed point in $X$.*

Later, the technique of Jleli and Samet was used by Mınak et al. [13] for multivalued mappings. They introduced the concept of multivalued $\theta$-contraction and obtained some fixed point results for these type of mappings on complete metric spaces.

Let $(X,d)$ be a metric space and $T : X \to CB(X)$. Then, we say that $T$ is a multivalued $\theta$-contraction if there exist $\theta \in \Theta$ and $k \in (0,1)$ such that

$$\theta(H(Tx,Ty)) \le [\theta(d(x,y))]^k$$

for all $x,y \in X$ with $H(Tx,Ty) > 0$. We can easily obtain that every multivalued contraction is also multivalued $\theta$-contraction with $\theta(t) = e^{\sqrt{t}}$.

**Theorem 3** ([13])**.** *Let $(X,d)$ be a complete metric space and $T : X \to K(X)$ be a multivalued $\theta$-contraction. Then $T$ has a fixed point in $X$.*

Example 1 in [13] shows that we cannot take $CB(X)$ instead of $K(X)$ in Theorem 3. However, we can take $CB(X)$ instead of $K(X)$ by adding the following weak condition on $\theta$:

$(\Theta_4)$ $\theta(\inf A) = \inf \theta(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$.

Note that, if $\theta$ satisfies $(\Theta_1)$, then it satisfies $(\Theta_4)$ if and only if it is right continuous. Let

$$\Xi = \{\theta \,|\, \theta : (0,\infty) \to (1,\infty) \text{ satisfies } (\Theta_1)\text{-}(\Theta_4)\}.$$

**Theorem 4** ([13])**.** *Let $(X,d)$ be a complete metric space and $T : X \to CB(X)$ be a multivalued $\theta$-contraction with $\theta \in \Xi$. Then $T$ has a fixed point in $X$.*

In the present paper, considering the Jleli and Samet's [10] technique, a new type contraction for multivalued mappings in metric spaces is introduced and the conditions guaranteeing the existence of a fixed point for such mappings in complete metric spaces are established. Our results generalize, improve and extend not only the results derived by Nadler [15] and Feng-Liu [8] but also various other related results in the literature.

## 2. Main results

Let $T : X \to P(X)$, $\theta \in \Theta$ and $s \in (0,1]$. Define the set $\theta_s^x \subseteq X$ as

$$\theta_s^x = \{y \in Tx : [\theta(d(x,y))]^s \le \theta(d(x,Tx))\}$$

for $x \in X$ with $d(x,Tx) > 0$. It is clear that if $s \le t$, then $\theta_t^x \subseteq \theta_s^x$ for fixed $x \in X$.

For this set, we need the consider the following cases:

Case 1. If $T : X \to K(X)$, then we have $\theta_s^x \neq \emptyset$ for all $s \in (0,1]$ and $x \in X$ with $d(x,Tx) > 0$. Indeed, since $Tx$ is compact, we have $y \in Tx$ such that $d(x,y) = d(x,Tx)$ for every $x \in X$ . Therefore, we have $\theta(d(x,y)) = \theta(d(x,Tx))$ for every $x \in X$ with $d(x,Tx) > 0$. Thus $y \in \theta_s^x$ for all $s \in (0,1]$.

Case 2. If $T : X \to C(X)$, then $\theta_s^x$ may be empty for some $x \in X$ and $s \in (0,1]$. For example, let $X = \{0\} \cup (1,2)$, $d(x,y) = |x-y|$ and $\theta(t) = e^{\sqrt{t}}$ for $t \leq 1$ and $\theta(t) = 9t$ for $t > 1$. Define $T : X \to C(X)$ by $T0 = (1,2)$ and $Tx = \{0\}$ for $x \in (1,2)$. Then, $d(0,T0) = 1 > 0$ and

$$\begin{aligned}\theta_{\frac{1}{2}}^0 &= \{y \in T0 : [\theta(d(0,y))]^{\frac{1}{2}} \leq \theta(d(0,T0))\} \\ &= \{y \in (1,2) : [\theta(y)]^{\frac{1}{2}} \leq \theta(1)\}\} \\ &= \{y \in (1,2) : 3\sqrt{y} \leq e\} \\ &= \emptyset.\end{aligned}$$

Case 3. If $T : X \to C(X)$ (even if $T : X \to P(X)$) and $\theta \in \Xi$, then we have $\theta_s^x \neq \emptyset$ for all $s \in (0,1)$ and $x \in X$ with $d(x,Tx) > 0$. To see this, let $x \in X$ with $d(x,Tx) > 0$ and

$$\alpha = \inf\{\theta(d(x,y)) : y \in Tx\}.$$

Then by $(\Theta_4)$, we have

$$\begin{aligned}\alpha &= \inf\{\theta(d(x,y)) : y \in Tx\} \\ &= \theta(\inf\{d(x,y) : y \in Tx\}) \\ &= \theta(d(x,Tx)) > 1.\end{aligned}$$

Now we claim that for all $s \in (0,1)$ there exists $y \in Tx$ such that

$$[\theta(d(x,y))]^s \leq \alpha.$$

Assume the contrary, that is, there exists $s \in (0,1)$ such that $[\theta(d(x,y))]^s > \alpha$ for all $y \in Tx$. Then

$$\alpha < [\theta(d(x,y))]^s < \theta(d(x,y)) \tag{2.1}$$

for all $y \in Tx$ and some $s \in (0,1)$. Taking infimum over $y \in Tx$ in (2.1), we get $\alpha = \alpha^s$ for some $s \in (0,1)$, which contradicts to $\alpha > 1$. Therefore, our claim is true, which implies $\theta_s^x \neq \emptyset$.

Considering the above facts, we give the following theorems:

**Theorem 5.** *Let $(X,d)$ be a complete metric space and $T : X \to K(X)$. If there exist $k \in (0,1)$ and $\theta \in \Theta$ such that there is $y \in \theta_s^x$ $(s \in (0,1])$ satisfying*

$$\theta(d(y,Ty)) \leq [\theta(d(x,y))]^k$$

*for all $x \in X$ with $d(x,Tx) > 0$. Then $T$ has a fixed point in $X$ provided that $k < s$ and the function $x \to d(x,Tx)$ is lower semi-continuous.*

*Proof.* Suppose that $T$ has no fixed point. Then, for all $x \in X$ we have $d(x, Tx) > 0$. Since $Tx \in K(X)$ for every $x \in X$, the set $\theta_s^x$ is nonempty for any $s \in (0,1]$. Let $x_0 \in X$ be any initial point, as $Tx_0 \in K(X)$, then there exists $x_1 \in \theta_s^{x_0}$ such that

$$\theta(d(x_1, Tx_1)) \leq [\theta(d(x_0, x_1))]^k$$

and for $x_1 \in X$, there exists $x_2 \in \theta_s^{x_1}$ satisfying

$$\theta(d(x_2, Tx_2)) \leq [\theta(d(x_1, x_2))]^k .$$

By the way, we can construct a sequence $\{x_n\}$ in $X$, where $x_{n+1} \in \theta_s^{x_n}$ and

$$\theta(d(x_{n+1}, Tx_{n+1})) \leq [\theta(d(x_n, x_{n+1}))]^k . \tag{2.2}$$

We will verify that $\{x_n\}$ is a Cauchy sequence. Since $x_{n+1} \in \theta_s^{x_n}$, we have

$$[\theta(d(x_n, x_{n+1}))]^s \leq \theta(d(x_n, Tx_n)). \tag{2.3}$$

From (2.2) and (2.3) we have

$$\theta(d(x_{n+1}, Tx_{n+1})) \leq [\theta(d(x_n, Tx_n))]^{\frac{k}{s}} \tag{2.4}$$

and

$$\theta(d(x_{n+1}, x_{n+2})) \leq [\theta(d(x_n, x_{n+1}))]^{\frac{k}{s}} . \tag{2.5}$$

By the way, we can obtain

$$1 < \theta(d(x_n, x_{n+1})) \leq [\theta(d(x_0, x_1))]^{(\frac{k}{s})^n} \tag{2.6}$$

and

$$1 < \theta(d(x_n, Tx_n)) \leq [\theta(d(x_0, Tx_0))]^{(\frac{k}{s})^n} . \tag{2.7}$$

From (2.6), we get $\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = 1$. Thus, from $(\Theta_2)$, we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0^+$$

and so from $(\Theta_3)$ there exists $r \in (0,1)$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} = l.$$

As in the proof of Theorem 2.1 of [10], we can show that $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is a complete metric space, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. On the other hand, from (2.7) and $(\Theta_2)$ we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0^+.$$

Since $x \to d(x, Tx)$ is lower semi-continuous, then

$$0 \leq d(z, Tz) \leq \liminf_{n\to\infty} d(x_n, Tx_n) = 0,$$

which contradicts that $d(z, Tz) > 0$. Hence $T$ has a fixed point. □

If we analyze Example 1 in [13], we cannot take $CB(X)$ instead of $K(X)$ in Theorem 5. In the following theorem, we replace $C(X)$ by $K(X)$, but we need to take $\theta \in \Xi$.

**Theorem 6.** *Let $(X, d)$ be a complete metric space and $T : X \to C(X)$. If there exist $k \in (0, 1)$ and $\theta \in \Xi$ such that there is $y \in \theta_s^x$ $(s \in (0, 1))$ satisfying*

$$\theta(d(y, Ty)) \leq [\theta(d(x, y))]^k$$

*for all $x \in X$ with $d(x, Tx) > 0$. Then $T$ has a fixed point in $X$ provided that $k < s$ and the function $x \to d(x, Tx)$ is lower semi-continuous.*

*Proof.* Suppose that $T$ has no fixed point. Then, for all $x \in X$ we have $d(x, Tx) > 0$. Since $\theta \in \Xi$, for any $x \in X$ the set $\theta_s^x$ is nonempty for any $s \in (0, 1)$. Let $x_0 \in X$ be any initial point, then there exists $x_1 \in \theta_s^{x_0}$ such that

$$\theta(d(x_1, Tx_1)) \leq [\theta(d(x_0, x_1))]^k . \tag{2.8}$$

From $(\Theta_4)$ we can write

$$\theta(d(x_1, Tx_1)) = \inf_{y \in Tx_1} \theta(d(x_1, y)),$$

and so from (2.8) we have

$$\inf_{y \in Tx_1} \theta(d(x_1, y)) \leq [\theta(d(x_0, x_1))]^k < [\theta(d(x_0, x_1))]^{\frac{k+1}{2}} . \tag{2.9}$$

Then, from (2.9) there exists $x_2 \in Tx_1$ such that

$$\theta(d(x_1, x_2)) \leq [\theta(d(x_0, x_1))]^{\frac{k+1}{2}} .$$

By the way, we can construct a sequence $\{x_n\}$ in $X$ such that $x_{n+1} \in Tx_n$ and

$$\theta(d(x_n, x_{n+1})) \leq [\theta(d(x_{n-1}, x_n))]^{\frac{k+1}{2}}$$

for all $n \in \mathbb{N}$. Finally, in order to obtain the result it is enough to argue as in the proof of Theorem 5 by considering the closedness of $Tz$. □

*Remark* 1. If we take $\theta(t) = e^{\sqrt{t}}$, $k = \sqrt{c}$ and $s = \sqrt{b}$ in Theorem 6, then we have Theorem 1.

*Remark* 2. We can present a simple proof of Theorem 3 by Theorem 5. In fact, suppose all conditions of Theorem 3 are satisfied. Since every multivalued $\theta$-contractions are multivalued nonexpansive and every multivalued nonexpansive mappings are upper semi-continuous, then $T$ is upper semi-continuous. Therefore, the function $x \to d(x, Tx)$ is lower semi-continuous (see Proposition 4.2.6 of [1]). On the other hand, since $T$ is multivalued $\theta$-contraction, for each $x \in X$ with $d(x, Tx) > 0$ and $y \in \theta_s^x$ $(k < s)$ we have

$$\theta(d(y, Ty)) \leq \theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^k .$$

Hence the hypotheses of Theorem 5 hold and so the existence of a fixed point has been proved. Similarly, we can prove Theorem 4 by considering Theorem 6.

The following example shows that Theorem 5 (resp. Theorem 6) is a proper generalization of Theorem 3 (resp. Theorem 4).

**Example 1.** Consider the complete metric space $(X,d)$ with $X=\{\frac{1}{3^{n-1}}:n\in\mathbb{N}\}\cup\{0\}$ and $d(x,y)=|x-y|$. Define a mapping $T:X\to C(X)$ as

$$Tx=\begin{cases}\{\frac{1}{3^n},1\}, & x=\frac{1}{3^{n-1}},\ n\geq 1\\ \{0,\frac{1}{3}\}, & x=0.\end{cases}$$

It is easy to see that

$$d(x,Tx)=\begin{cases}\frac{2}{3^n}, & x=\frac{1}{3^{n-1}},\ n>1\\ 0, & x\in\{0,1\},\end{cases}$$

and so $x\to d(x,Tx)$ is lower semi-continuous. Now, let $\theta(t)=e^{\sqrt{t}}$. If $d(x,Tx)>0$, then $x=\frac{1}{3^{n-1}}$, $n>1$. Thus for $y=\frac{1}{3^n}\in Tx$ we have

$$[\theta(d(x,y))]^s\leq\theta(d(x,y))=\theta(d(x,Tx))$$

for all $s\in(0,1]$ and

$$d(y,Ty)=\frac{2}{3^{n+1}}=\frac{1}{3}\frac{2}{3^n}=\frac{1}{3}d(x,y).$$

Therefore, $y\in\theta_s^x$ and

$$\theta(d(y,Ty))\leq[\theta(d(x,y))]^k$$

for $\frac{1}{\sqrt{3}}\leq k<s<1$. Hence, all conditions of Theorem 5 and Theorem 6 are satisfied and so $T$ has a fixed point.

On the other hand, since $H(T\frac{1}{3},T0)=\frac{2}{3}$ and $d(\frac{1}{3},0)=\frac{1}{3}$, then we have

$$\theta(H(T\frac{1}{3},T0))>\theta(d(\frac{1}{3},0))>\left[\theta(d(\frac{1}{3},0))\right]^k$$

for all $\theta\in\Theta$ and $k\in(0,1)$. Thus, $T$ is not multivalued $\theta$-contraction. Therefore, Theorem 3 and Theorem 4 can not be applied to this example.

In the following theorem, we replace $P(X)$ by $C(X)$, but we need to add an extra condition.

**Theorem 7.** *Let $(X,d)$ be a complete metric space and $T:X\to P(X)$. Suppose there exist $k\in(0,1)$ and $\theta\in\Xi$ such that there is $y\in\theta_s^x$ satisfying $d(y,Ty)>0$ and*

$$\theta(d(y,Ty))\leq[\theta(d(x,y))]^k$$

*for all $x\in X$ with $d(x,Tx)>0$. If there exists $x_0\in X$ with $d(x_0,Tx_0)>0$ such that for all convergent sequence $\{x_n\}$ with $x_n\in Tx_{n-1}$, we have $T(\lim x_n)$ is closed, then $T$ has a fixed point in $X$ provided that $k<s$ and the function $x\to d(x,Tx)$ is lower semi-continuous.*

*Proof.* Since $d(x_0, Tx_0) > 0$, then there exists $x_1 \in \theta_s^{x_0}$ such that $d(x_1, Tx_1) > 0$ and

$$\theta(d(x_1, Tx_1)) \leq [\theta(d(x_0, x_1))]^k .$$

Also, since $d(x_1, Tx_1) > 0$, there exists $x_2 \in \theta_s^{x_1}$ satisfying $d(x_2, Tx_2) > 0$ and

$$\theta(d(x_2, Tx_2)) \leq [\theta(d(x_1, x_2))]^k .$$

Continuing this process, we get an iterative sequence $\{x_n\}$ as in the proof of Theorem 5 such that $x_n \in Tx_{n-1}$ and $\{x_n\}$ is a Cauchy sequence. Since $X$ is complete, $\{x_n\}$ converges to a point of $X$, say $z$. By the hypotheses, we have $Tz$ is closed. On the other hand from (2.7) and $(\Theta_2)$ we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0^+.$$

Since $x \to d(x, Tx)$ is lower semi-continuous, then

$$0 \leq d(z, Tz) \leq \liminf_{n\to\infty} d(x_n, Tx_n) = 0$$

and so $z \in Tz$. Hence $T$ has a fixed point. □

If we take $\theta(t) = e^{\sqrt{t}}$, $k = \sqrt{c}$ and $s = \sqrt{b}$ in Theorem 7, then we obtain the following corollary:

**Corollary 1.** *Let $(X, d)$ be a complete metric space and $T : X \to P(X)$. Suppose there exists $c \in (0, 1)$ such that there is $y \in I_b^x$ satisfying*

$$0 < d(y, Ty) \leq cd(x, y) \tag{2.10}$$

*for each $x \in X$ with $d(x, Tx) > 0$. If there exists $x_0 \in X$ with $d(x_0, Tx_0) > 0$ such that for all convergent sequence $\{x_n\}$ with $x_n \in Tx_{n-1}$, we have $T(\lim x_n)$ is closed, then $T$ has a fixed point in $X$ provided that $c < b$ and the function $x \to d(x, Tx)$ is lower semi-continuous.*

**Example 2.** Consider the complete metric space $(X, d)$ with $X = [0, 1]$ and $d(x, y) = |x - y|$ . Define $T : X \to P(X)$ as

$$Tx = \begin{cases} (0, \frac{x}{4}], & x \in (0, 1] \\ \{0\}, & x = 0. \end{cases}$$

Since $Tx$ is not closed for some $x \in X$, both Nadler's and Feng-Liu's results can not be applied to this example. On the other hand if we take $\frac{1}{4} \leq c < b$ and $x_0 \in (0, 1]$, then all conditions of Corollary 1 are satisfied. Indeed, if $d(x, Tx) > 0$, then $x \in (0, 1]$ and so, for $y = \frac{x}{4} \in Tx$, we have

$$bd(x, y) = bd(x, \frac{x}{4}) = b\frac{3x}{4} < \frac{3x}{4} = d(x, Tx)$$

and

$$0 < d(y, Ty) = d(\frac{x}{4}, T\frac{x}{4}) = \frac{3x}{16} = \frac{1}{4}\frac{3x}{4} \leq c\frac{3x}{4} = cd(x, y).$$

That is, $y \in I_b^x$ for any $x \in X$ with $d(x, Tx) > 0$ and (2.10) is satisfied. Now, let $x_0 \in (0, 1]$, then we have $0 < x_n \leq \frac{x_0}{4^n}$ for the sequence $\{x_n\}$ with

$x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ converges to 0 and $T0$ is closed. Finally, the function $f(x) = d(x, Tx) = \frac{3x}{4}$ is lower semi-continuous. Therefore all conditions of Corollary 1 are satisfied and so $T$ has a fixed point.

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