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FIXED POINT RESULTS FOR α-ADMISSIBLE MULTIVALUED F-CONTRACTIONS

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Abstract. In this study, we give some fixed point results for multivalued mappings using Pompeiu-Hausdorff distance on complete metric space. For this, we consider the α-admissibility of multivalued mappings. Our results are real generalizations of Mizoguchi-Takahashi fixed point theorem. We also provide an example showing this fact. Finally, we obtain some ordered fixed point results for multivalued mappings.

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1. Introduction and Preliminaries

Let (X,d) be a metric space. Denote by $P(X)$, the family of all nonempty subsets of X, $CB(X)$ the family of all nonempty, closed and bounded subsets of X and $K(X)$ the family of all nonempty compact subsets of X. It is well known that $H : CB(X) \times CB(X) \to \mathbb{R}$ defined by, for every $A, B \in CB(X)$,

$$H(A,B) = \max\left\{ \sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A) \right\}$$

is a metric on $CB(X)$, which is called the Pompeiu-Hausdorff metric induced by d, where $d(x,B) = \inf\{d(x,y) : y \in B\}$. We can find detailed information about the Pompeiu-Hausdorff metric in [6, 11]. An element $x \in X$ is said to be a fixed point of a multivalued mapping $T : X \to P(X)$ if $x \in Tx$. Let $T : X \to CB(X)$. Then, we say that T is called multivalued contraction if there exists $L \in [0,1)$ such that

$$H(Tx,Ty) \le Ld(x,y)$$

for all $x, y \in X$ (see [17]). In 1969, Nadler [17] proved that every multivalued contraction mappings on complete metric spaces has a fixed point.

Inspired by his result, various fixed point theorems concerning multivalued contractions appeared in the last decades. Concerning these, the following theorem was proved by Mizoguchi and Takahashi [15] that is, in fact, a partial answer to a question proposed by Reich [22]:

Theorem 1 ([15]). *Let (X,d) be a complete metric space and $T : X \to CB(X)$ is a mapping such that*

$$H(Tx,Ty) \leq k(d(x,y))d(x,y),$$

for all $x,y \in X$, $x \neq y$, where a function $k : (0,\infty) \to [0,1)$ satisfies

$$\limsup_{t \to s^+} k(t) < 1 \text{ for all } s \geq 0.$$

Then T has a fixed point in X.

We can find both a simple proof of Theorem 1 and an example showing that it is real generalization of Nadler's in [24]. We can also find a lot of generalizations of Mizoguchi-Takahashi's fixed point theorem in the literature [5, 7, 8].

In 2012, Samet et al [23] introduced the concept of α-ψ-contractive and α-admissible mapping and established various fixed point theorems for such mappings on complete metric spaces (See [1, 12, 16, 18]). Asl et al [4] also defined the notion of α-admissible and α_*-admissible for multivalued mappings as follows: Let (X,d) be a metric space, $T : X \to P(X)$ and $\alpha : X \times X \to [0,\infty)$ be a function. We say that T is an α-admissible mapping whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq 1$ implies $\alpha(y,z) \geq 1$ for all $z \in Ty$ and T is an α_*-admissible mapping whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq 1$ implies $\alpha_*(Tx,Ty) \geq 1$, where $\alpha_*(Tx,Ty) = \inf\{\alpha(a,b) : a \in Tx, b \in Ty\}$. It is clear that α_*-admissible mapping is also α-admissible, but the converse may not be true as shown in Example 15 of [13]. This situation also will be mentioned in Example 1. We say that α has (B) property whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$, then $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}$.

Consider the collection Ψ of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n th iterate of ψ. It is clear that for each $\psi \in \Psi$, we have $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$. Let $T : X \to CB(X)$ be a mapping. Then, we say that T is called multivalued α-ψ-contractive whenever

$$\alpha(x,y)H(Tx,Ty) \leq \psi(d((x,y))$$

for all $x,y \in X$ and T is called multivalued α_*-ψ-contractive whenever

$$\alpha_*(Tx,Ty)H(Tx,Ty) \leq \psi(d((x,y)).$$

The results for these type mappings are given by [4, 16] as follows:

Theorem 2 ([16]). *Let (X,d) be a complete metric space, $\alpha : X \times X \to [0,\infty)$ be a function, $\psi \in \Psi$ be a strictly increasing map and $T : X \to CB(X)$ be an α-admissible and α-ψ-contractive multifunction on X. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$. If T is continuous or α has (B) property, then T has a fixed point.*

Theorem 3 ([4]). *Let (X,d) be a complete metric space, $\alpha : X \times X \to [0,\infty)$ be a function, $\psi \in \Psi$ be a strictly increasing map and $T : X \to CB(X)$ be an α_*-admissible and α_*-ψ-contractive multifunction on X. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$. If α has (B) property, then T has a fixed point.*

Furthermore, some generalizations of Mizoguchi-Takashashi fixed point theorem using mappings of this type are given by Mınak and Altun [14] as follows:

Theorem 4 ([14]). *Let (X,d) be a metric space and $T : X \to CB(X)$ be an α-admissible multivalued mapping such that*

$$\alpha(x,y)H(Tx,Ty) \leq k(d(x,y))d(x,y)$$

for all $x,y \in X$, where $k : [0,\infty) \to [0,1)$ satisfies $\limsup_{t\to s^+} k(t) < 1$ for all $s \geq 0$. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$. If T is continuous or α has (B) property, then T has a fixed point in X.

Theorem 5 ([14]). *Let (X,d) be a metric space and $T : X \to CB(X)$ be an α_*-admissible multivalued mapping such that*

$$\alpha_*(Tx,Ty)H(Tx,Ty) \leq k(d(x,y))d(x,y)$$

for all $x,y \in X$, where $k : [0,\infty) \to [0,1)$ satisfies $\limsup_{t\to s^+} k(t) < 1$ for all $s \geq 0$. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$. If T is continuous or α has (B) property, then T has a fixed point in X.

In this paper, by considering the recent technique of Wardowski [25], we give some generalizations of Mizoguchi-Takahashi fixed point theorem. First, we recall the Wardowski's technique. Let $F : (0,\infty) \to \mathbb{R}$ be a function. For the sake of completeness, we will consider the following conditions:

(F1) F is strictly increasing, i.e., for all $\alpha,\beta \in (0,\infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$,

(F2) For each sequence $\{\alpha_n\}$ of positive numbers

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty,$$

(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$,

(F4) $F(\inf A) = \inf F(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$.

We denote by $\mathcal{F}$ and $\mathcal{F}_*$, the set of all functions F satisfying (F1)-(F3) and (F1)-(F4), respectively. It is clear that $\mathcal{F}_* \subset \mathcal{F}$. Some examples of the functions belonging $\mathcal{F}_*$ are $F_1(\alpha) = \ln\alpha$, $F_2(\alpha) = \alpha + \ln\alpha$, $F_3(\alpha) = -\frac{1}{\sqrt{\alpha}}$ and $F_4(\alpha) = \ln\left(\alpha^2 + \alpha\right)$. If we define $F_5(\alpha) = \ln\alpha$ for $\alpha \leq 1$ and $F_5(\alpha) = 2\alpha$ for $\alpha > 1$, then $F_5 \in \mathcal{F} \backslash \mathcal{F}_*$. If F satisfies (F1), then it satisfies (F4) if and only if it is right continuous.

By considering the class $\mathcal{F}$, Wardowski [25] introduced the concept of F-contraction, which is more general than ordinary contraction, as follows: Let (X,d) be a metric space and $T : X \to X$ be a map. If there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\tau + F(d(Tx,Ty)) \leq F(d(x,y)),$$

for all $x,y \in X$ with $d(Tx,Ty) > 0$, then T is called an F-contraction. As a real generalization of Banach contraction principle, Wardowski proved that every F-contraction on complete metric space has a unique fixed point. (See [25] for more detailed information about F-contractions).

By combining the ideas of Wardowski's and Nadler's, Altun et al [3] introduced the concept of multivalued F-contractions and obtained a fixed point result for these type mappings on complete metric spaces. Let (X,d) be a metric space and $T : X \to CB(X)$. Then T is said to be a multivalued F-contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\tau + F(H(Tx,Ty)) \leq F(d(x,y)), \tag{1.1}$$

for all $x,y \in X$ with $H(Tx,Ty) > 0$.

Theorem 6 ([3])**.** *Let (X,d) be a complete metric space and $T : X \to K(X)$ be a multivalued F-contraction. Then, T has a fixed point in X.*

Note that Tx is compact for all $x \in X$ in Theorem 6. By adding the condition (F4) on $\mathcal{F}$, the compactness condition of Tx can be weakened. There are some detailed information about this situation in [2].

Theorem 7 ([3])**.** *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multivalued F-contraction with $F \in \mathcal{F}_*$, then T has a fixed point in X.*

On the other hand, taking τ as a function of $d(x,y)$ Olgun at al. [20] proved the following theorem, which is a generalization of Mizoguchi-Takahashi fixed point theorem for multivalued contractive mappings. These results are also nonlinear case of Theorem 7 (resp. Theorem 6).

Theorem 8 ([20])**.** *Let (X,d) be a complete metric space and $T : X \to CB(X)$ (resp. $K(X)$). If there exist $F \in \mathcal{F}_*$ (resp. $F \in \mathcal{F}$) and $\tau : (0,\infty) \to (0,\infty)$ such that*

$$\liminf_{t \to s^+} \tau(t) > 0, \text{ for all } s \geq 0, \tag{1.2}$$

satisfying

$$\tau(d(x,y)) + F(H(Tx,Ty)) \leq F(d(x,y))$$

for all $x,y \in X$ with $H(Tx,Ty) > 0$. Then, T has a fixed point in X.

The aim of this paper is to present some new fixed point results for multivalued F-contractions, by considering the α-admissibility and α_*-admissibility of a multivalued mappings on complete metric spaces.

2. THE RESULTS

Before we give our main results, we recall the following: Let X and Y be two metric spaces. Then, a multivalued mapping $T : X \to P(Y)$ is said to be upper semicontinuous (lower semicontinuous) if the inverse image of closed sets (open sets) is closed (open). A multivalued mapping is continuous if it is upper as well as lower semicontinuous. T is a closed multivalued mapping if the graph $GrT = \{(x,y) : x \in X, y \in Tx\}$ is a closed subset of $X \times Y$. If T is closed multivalued mapping, then it has closed values. If T is upper semicontinuous and closed values, then it is closed multivalued mapping (see Proposition 2.17 of [10]). A closed multivalued mapping may not be upper semicontionous. For example, let $T : [0,\infty) \to P([0,\infty))$ be defined by

$$Tx = \begin{cases} [0,x] \cup \{\frac{1}{x}\} & , \quad x > 0 \\ \{0\} & , \quad x = 0 \end{cases},$$

then T is closed multivalued mapping, but not upper semicontinuous since $T^{-1}(\mathbb{Z}^+) = \{\frac{1}{n} : n \in \mathbb{Z}^+\} \cup \mathbb{Z}^+$ is not closed, where $\mathbb{Z}^+$ is the set of positive integers. On the other hand, an upper semi continuous mapping may not be closed multivaled mapping unless it is closed values. For example, let $T : \mathbb{R} \to P(\mathbb{R})$ be defined by $Tx = [0,1)$, then T is upper semicontinuous, but not closed multivalued mapping. We can find more important properties of multivalued mappings (even when X and Y are two topological spaces) in [10, 11].

Let (X,d) be a metric space, $T : X \to CB(X)$ and $\alpha : X \times X \to [0,\infty)$ be two mappings. Define a set

$$T_\alpha = \{(x,y) : \alpha(x,y) \geq 1 \text{ and } H(Tx,Ty) > 0\} \subset X \times X.$$

Given $F \in \mathcal{F}$, we say that T is a multivalued (α,F)-contraction if there exists a function $\tau : (0,\infty) \to (0,\infty)$ such that

$$\tau(d(x,y)) + F(H(Tx,Ty)) \leq F(d(x,y)) \tag{2.1}$$

for all $(x,y) \in T_\alpha$. In this case, the function τ is called the contractive factor of T.

Theorem 9. *Let (X,d) be a complete metric space and $T : X \to K(X)$ be an α-admissible and multivalued (α,F)-contraction with contractive factor τ. Suppose that*

$$\liminf_{t \to s^+} \tau(t) > 0, \text{ for all } s \geq 0 \tag{2.2}$$

and there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$. If T is closed multivalued mapping or α has (B) property, then T has a fixed point.

Proof. Suppose that T has no fixed point. Then for all $x \in X$, $d(x,Tx) > 0$. Let x_0 and x_1 be as mentioned in the hypothesis, then $H(Tx_0,Tx_1) > 0$ (otherwise

$d(x_1,Tx_1)=0$, this is a contradiction). Therefore $(x_0,x_1)\in T_\alpha$, thus we can use the condition (2.1) for x_0 and x_1. Then considering (F1) we have

$$F(d(x_1,Tx_1))\le F(H(Tx_0,Tx_1))\le F(d(x_1,x_0))-\tau(d(x_1,x_0)). \tag{2.3}$$

Since Tx_1 is compact, there exists $x_2\in Tx_1$ such that $d(x_1,x_2)=d(x_1,Tx_1)$. From (2.3),

$$F(d(x_1,x_2))\le F(H(Tx_0,Tx_1))\le F(d(x_1,x_0))-\tau(d(x_1,x_0)).$$

Also, since T is an α-admissible mapping $\alpha(x_1,x_2)\ge 1$. Again, since $x_2\in Tx_1$, then $H(Tx_1,Tx_2)>0$. Therefore, $(x_1,x_2)\in T_\alpha$, so we can use (2.1) for x_1 and x_2. Then

$$F(d(x_2,Tx_2))\le F(H(Tx_1,Tx_2))\le F(d(x_2,x_1))-\tau(d(x_2,x_1)).$$

Since Tx_2 is compact, there exists $x_3\in Tx_2$ such that $d(x_2,x_3)=d(x_2,Tx_2)$. Therefore, we have

$$F(d(x_2,x_3))\le F(H(Tx_1,Tx_2))\le F(d(x_2,x_1))-\tau(d(x_2,x_1)).$$

By induction, we can find a sequence $\{x_n\}$ in X such that $x_{n+1}\in Tx_n$, $(x_n,x_{n+1})\in T_\alpha$ and

$$F(d(x_n,x_{n+1}))\le F(d(x_n,x_{n-1}))-\tau(d(x_n,x_{n-1})) \tag{2.4}$$

for all $n\in\mathbb{N}$. Denote $a_n=d(x_n,x_{n+1})$ for $n\in\mathbb{N}_0$, then $a_n>0$ and from (2.4) $\{a_n\}$ is decreasing and hence convergent. We show that $\lim_{n\to\infty}a_n=0$. From (2.2) there exists $\gamma>0$ and $n_0\in\mathbb{N}$ such that $\tau(a_n)>\gamma$ for all $n>n_0$. Therefore, we obtain

$$\begin{aligned}
F(a_n)&\le F(a_{n-1})-\tau(a_{n-1})\\
&\le F(a_{n-2})-\tau(a_{n-1})-\tau(a_{n-2})\\
&\;\;\vdots\\
&\le F(a_0)-\tau(a_{n-1})-\tau(a_{n-2})-\cdots-\tau(a_0)\\
&\le F(a_0)-\tau(a_{n-1})-\tau(a_{n-2})-\cdots-\tau(a_{n_0})\\
&= F(a_0)-[\tau(a_{n-1})+\tau(a_{n-2})+\cdots+\tau(a_{n_0})]\\
&\le F(a_0)-(n-n_0)\gamma
\end{aligned} \tag{2.5}$$

for all $n>n_0$. Letting $n\to\infty$ in the above inequlity, we have $\lim_{n\to\infty}F(a_n)=-\infty$ and by (F2) $\lim_{n\to\infty}a_n=0$.

Now from (F3) there exists $k\in(0,1)$ such that

$$\lim_{n\to\infty}a_n^kF(a_n)=0. \tag{2.6}$$

By (2.5) we get for all $n>n_0$

$$\begin{aligned}
a_n^kF(a_n)-a_n^kF(a_0)&\le a_n^k[F(a_0)-(n-n_0)\gamma]-a_n^kF(a_0)\\
&=-a_n^k(n-n_0)\gamma\le 0.
\end{aligned}$$

Taking into account (2.6), we get from the above inequality

$$\lim_{n\to\infty} n a_n^k = 0.$$

Therefore, there exists $n_1 \in \mathbb{N}$ such that $n a_n^k \leq 1$ for all $n \geq n_1$. Consequently, we have

$$a_n \leq \frac{1}{n^{\frac{1}{k}}} \text{ for all } n \geq n_1.$$

Now, let $m,n \in \mathbb{N}$ be such that $m > n \geq n_1$. Then, we have

$$\begin{aligned} d(x_m,x_n) &\leq d(x_m,x_{m-1}) + d(x_{m-1},x_{m-2}) + \cdots + d(x_{n+1},x_n) \\ &< \sum_{i=n}^{\infty} d(x_{i+1},x_i) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Since the series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent, we have $\lim_{n\to\infty} d(x_m,x_n) = 0$ for all $m > n$. Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since (X,d) is a complete metric space, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

If T is closed multivalued mapping, then since $(x_n,x_{n+1}) \to (z,z)$, we have $z \in Tz$, which is a contradiction.

Now assume that α has (B) property. Since $\lim_{n\to\infty} x_n = z$ and $d(z,Tz) > 0$, then there exists $n_0 \in \mathbb{N}$ such that $d(x_{n+1},Tz) > 0$ for all $n \geq n_0$. Therefore, for all $n \geq n_0$

$$H(Tx_n,Tz) > 0,$$

thus $(x_n,z) \in T_\alpha$ for all $n \geq n_0$. From (2.1) and (F1), we have

$$\begin{aligned} F(d(x_{n+1},Tz)) &\leq F(H(Tx_n,Tz)) \\ &\leq F(d(x_n,z)) - \tau(d(x_n,z)) \end{aligned}$$

and so

$$d(x_{n+1},Tz) \leq d(x_n,z)$$

for all $n \geq n_0$. Passing to limit $n \to \infty$, we obtain $d(z,Tz) = 0$, which is a contradiction.

Therefore, T has a fixed point in X. □

Remark 1. Example 1 in [2] shows that we can not take $CB(X)$ instead of $K(X)$ in Theorem 9. However, we can take $CB(X)$ instead of $K(X)$ by adding the condition (F4) on F.

Theorem 10. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be an α-admissible and multivalued (α,F)-contraction with contractive factor τ. Suppose that $F \in \mathcal{F}_*$,*

$$\liminf_{t\to s^+} \tau(t) > 0, \textit{ for all } s \geq 0$$

and there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is closed multi-valued mapping or α has (B) property, then T has a fixed point.

Proof. We begin as in the proof of Theorem 9. Considering the condition (F4), we can write

$$F(d(x_1, Tx_1)) = \inf_{y \in Tx_1} F(d(x_1, y)).$$

Thus from

$$\begin{aligned} F(d(x_1, Tx_1)) &\leq F(H(Tx_0, Tx_1)) \\ &\leq F(d(x_1, x_0)) - \tau(d(x_1, x_0)) \end{aligned}$$

we have

$$\begin{aligned} \inf_{y \in Tx_1} F(d(x_1, y)) &\leq F(d(x_1, x_0)) - \tau(d(x_1, x_0)) \\ &< F(d(x_1, x_0)) - \frac{\tau(d(x_1, x_0))}{2}. \end{aligned}$$

Therefore, there exists $x_2 \in Tx_1$ such that

$$F(d(x_1, x_2)) \leq F(d(x_1, x_0)) - \frac{\tau(d(x_1, x_0))}{2}.$$

The rest of the proof can be completed as in the proof of Theorem 9. □

Remark 2. If we take $\alpha(x, y) = 1$ in Theorem 10, we obtain Theorem 8.

Remark 3. By taking $\alpha(x, y) = 1$ and $F(\alpha) = \ln \alpha$ in Theorem 10, we obtain the famous Mizoguchi-Takahashi's fixed point theorem with $k(t) = \exp(-\tau(t))$.

Now, we give an example showing that T is α-admissible and multivalued (α, F)-contraction with contractive factor τ, but not multivalued F-contraction. Therefore, Theorem 9 (resp. Theorem 10) can be applied to this example, but Theorem 8 can not. Also, We show that Theorems 1, 2, 3, 4 and 5 can not be applied to this example.

Example 1. Consider the complete metric space (X, d) where $X = \{0, 1, 2, \cdots\}$ and $d : X \times X \to [0, \infty)$ is given by

$$d(x, y) = \begin{cases} 0 & , \quad x = y \\ x + y & , \quad x \neq y \end{cases}.$$

Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \{x\} & , \quad x \in \{0, 1\} \\ \{0, x - 1\} & , \quad x \geq 2 \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 3 & , \quad \text{otherwise} \\ 0 & , \quad (x, y) \in \{(0, 1), (1, 0)\} \end{cases}.$$

Then it is clear that T is an α-admissible.

Now, we claim that T is a multivalued (α,F)-contraction with contractive factor $\tau(t)=1$ and $F(\alpha)=\alpha+\ln\alpha$. To see this have to show that

$$1+F(H(Tx,Ty))\leq F(d(x,y))$$

for all $(x,y)\in T_\alpha$ or equivalently

$$\frac{H(Tx,Ty)}{d(x,y)}e^{H(Tx,Ty)-d(x,y)}\leq e^{-1} \tag{2.7}$$

for all $(x,y)\in T_\alpha$. Note that

$$\begin{aligned}T_\alpha &= \{(x,y)\in X\times X : \alpha(x,y)\geq 1 \text{ and } H(Tx,Ty)>1\}\\ &= \{(x,y)\in X\times X : (x,y)\notin\{(0,1),(1,0)\} \text{ and } x\neq y\}.\end{aligned}$$

Thus, without loss of generality, we may assume $x>y$ for all $(x,y)\in T_\alpha$ in the following cases:

Case 1. Let $y=0$ and $x\geq 2$. Then $H(Tx,Ty)=x-1$ and $d(x,y)=x$, and so we have

$$\frac{H(Tx,Ty)}{d(x,y)}e^{H(Tx,Ty)-d(x,y)}\leq\frac{x-1}{x}e^{-1}\leq e^{-1}.$$

Case 2. Let $y=1$ and $x=2$. Then $H(Tx,Ty)=1$ and $d(x,y)=3$, and so we have

$$\frac{H(Tx,Ty)}{d(x,y)}e^{H(Tx,Ty)-d(x,y)}\leq\frac{1}{3}e^{-2}\leq e^{-1}.$$

Case 3. Let $y=1$ and $x>2$. Then $H(Tx,Ty)=x$ and $d(x,y)=x+1$, and so we have

$$\frac{H(Tx,Ty)}{d(x,y)}e^{H(Tx,Ty)-d(x,y)}\leq\frac{x}{x+1}e^{-1}\leq e^{-1}.$$

Case 4. Let $x>y\geq 2$ Then $H(Tx,Ty)=x+y-2$ and $d(x,y)=x+y$, and so we have

$$\frac{H(Tx,Ty)}{d(x,y)}e^{H(Tx,Ty)-d(x,y)}=\frac{x+y-2}{x+y}e^{-2}\leq e^{-2}\leq e^{-1}.$$

This shows that T is an multivalued (α,F)-contraction with contractive factor τ.
For $x_0=1$ and $x_1\in Tx_0=\{1\}$, we have $\alpha(x_0,x_1)=\alpha(1,1)=3\geq 1$.
Finally, since τ_d is discrete topology, T is upper semi continuous and hence closed multivalued mapping. By Theorem 9 (or Theorem 10), T has a fixed point in X.

On the other hand, since $H(T0,T1)=1=d(0,1)$, then for all $F\in\mathcal{F}$ and $\tau:(0,\infty)\to(0,\infty)$ satisfying inequality (1.2), we have

$$\tau(d(0,1))+F(H(T0,T1))>F(d(0,1)).$$

Therefore, Theorem 8 can not be applied to this example. Accordingly, T is not multivalued F-contraction and multivalued contraction.

Note that α has not (B) property. Indeed, considering the sequence $\{x_n\} = \{1,2,3,0,0,0,\cdots\}$ in X, then $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$ and $x_n \to 0$, but $\alpha(x_1, 0) = \alpha(1,0) = 0 \ngeq 1$.

Despite $\alpha(1,2) \geq 1$, but $\alpha_*(T1, T2) = 0$, then T is not an α_*-admissible. Thus, Theorems 3 and 5 can not be applied to this example.

Also, since $H(T0, T2) = 1$, $d(0,2) = 2$ and $\alpha(0,2) = 3$, then for all $\psi \in \Psi$ and $k : [0,\infty) \to [0,1)$ satisfies $\limsup_{t \to s^+} k(t) < 1$ for all $s \geq 0$, we have

$$\alpha(0,2)H(T0,T1) = 3 > 2k(d(0,2)) = k(d(0,2))d(0,2),$$

and

$$3 = \alpha(0,2)H(T0,T1) \nleq \psi(d(0,2)) < 2.$$

Thus, Theorems 2 and 4 can not be applied to this example.

Since α_*-admissible mapping is also α-admissible, we can obtain following corollary.

Corollary 1. *Let (X,d) be a complete metric space and $T : X \to K(X)$ be an α_*-admissible and multivalued (α, F)-contraction with contractive factor τ. Suppose that*

$$\liminf_{t \to s^+} \tau(t) > 0, \text{ for all } s \geq 0$$

and there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is closed multivalued mapping or α has (B) property, then T has a fixed point.

Recently, there have been so many interesting developments in fixed point theory in metric spaces endowed with a partial order. The first result in this direction for single valued maps was given by Ran and Reurings [21], where they extended the Banach contraction principle in partially ordered sets with some application to a matrix equation. Later, many important results have been obtained for both single and multivalued mappings on metric spaces endowed with a partial order (see for example [14, 19]). By [12], we know that the fixed point results for α-admissible mappings are closely related to fixed point theory on partially ordered metric spaces. Following, we will present a fixed point result for multivalued mappings on metric spaces endowed with a partial order. In 2004, Feng and Liu [9] defined relations between two sets. Let X be a nonempty set and $\preceq$ be a partial order on X. Let A, B be two nonempty subsets of X, the relations between A and B are defined as follows:

(a) $A \prec_1 B$: if for every $a \in A$, there exists $b \in B$ such that $a \preceq b$,

(b) $A \prec_2 B$: if for every $b \in B$, there exists $a \in A$ such that $a \preceq b$,

(c) $A \prec B$: if $A \prec_1 B$ and $A \prec_2 B$.

$\prec_1$ and $\prec_2$ are different relations between A and B. For example, let $X = \mathbb{R}$, $A = [\frac{1}{2}, 1]$, $B = [0,1]$, $\preceq$ be usual order on X, then $A \prec_1 B$ but $A \nprec_2 B$; if $A = [0,1]$, $B = [0, \frac{1}{2}]$, then $A \prec_2 B$ while $A \nprec_1 B$. $\prec_1$, $\prec_2$and $\prec$ are reflexive and transitive, but are not antisymmetric. For instance, let $X = \mathbb{R}$, $A = [0,3]$, $B = [0,1] \cup [2,3]$, $\preceq$

be usual order on X, then $A \prec B$ and $B \prec A$, but $A \neq B$. Hence, they are not partial orders. Note that if A is a nonempty subset of X with $A \prec_1 A$, then A is singleton. (see [9]).

Corollary 2. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exist a metric d in X such that (X,d) is complete metric space. Let $T : X \to CB(X)$(resp. $K(X)$) be a closed multivalued mapping such that*

$$\tau(d(x,y)) + F(H(Tx,Ty)) \leq F(d(x,y))$$

for all $(x,y) \in T_{\preceq}$, where $\tau : (0,\infty) \to (0,\infty)$ be a function satisfying

$$\liminf_{t \to s^+} \tau(t) > 0, \text{ for all } s \geq 0$$

and $T_{\preceq} = \{(x,y) \in X \times X : x \preceq y$ and $H(Tx,Ty) > 0\}$. Assume that for each $x \in X$ and $y \in Tx$ with $x \preceq y$, we have $y \preceq z$ for all $z \in Ty$ and there exist $x_0 \in X$, $x_1 \in Tx_0$ such that $\{x_0\} \prec_1 Tx_0$, then T has a fixed point.

Proof. Define a mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & , \quad x \preceq y \\ \\ 0 & , \quad \text{otherwise} \end{cases}.$$

Then $T_{\preceq} = T_{\alpha}$. That is, T is (α,F)-contraction with contractive factor τ. Also, since $\{x_0\} \prec_1 Tx_0$, then there exists $x_1 \in Tx_0$ such that $x_0 \preceq x_1$ and so $\alpha(x_0,x_1) \geq 1$. Now let $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq 1$, then $x \preceq y$ and so by the hypotheses we have $y \preceq z$ for all $z \in Ty$. Therefore, $\alpha(y,z) \geq 1$ for all $z \in Ty$. This shows that T is α-admissible. Therefore, from Theorem 10 (resp. Theorem 9), T has a fixed point in X. □

Remark 4. We can give similar result using $\prec_2$ instead of $\prec_1$.

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