Turkish Journal of Mathematics

Volume 40 | Number 3 Article 2

1-1-2016

Generalized Bertrand Curves with Spacelike $\left(1,3\right) $-Normal Plane in Minkowski Space-Time

ALİ UÇUM

OSMAN KEÇİLİOĞLU

KAZIM İLARSLAN

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2016) 40: 487 – 505

doi:10.3906/mat-1503-12

Generalized Bertrand Curves with Spacelike $(1,3)$-Normal Plane in Minkowski Space-Time

Ali UÇUM, Osman KEÇİLİOĞLU, Kazım İLARSLAN*
Department of Mathematics, Faculty of Sciences and Arts, Kırıkkale University, Kırıkkale, Turkey

Received: 03.03.2015 • **Accepted/Published Online:** 25.06.2015 • **Final Version:** 08.04.2016

Abstract: In this paper, we reconsider the $(1,3)$-Bertrand curves with respect to the casual characters of a $(1,3)$-normal plane that is a plane spanned by the principal normal and the second binormal vector fields of the given curve. Here, we restrict our investigation of $(1,3)$-Bertrand curves to the spacelike $(1,3)$-normal plane in Minkowski space-time. We obtain the necessary and sufficient conditions for the curves with spacelike $(1,3)$-normal plane to be $(1,3)$-Bertrand curves and we give the related examples for these curves.

Key words: Bertrand curve, Minkowski space-time, Frenet planes

1. Introduction

Much work has been done about the general theory of curves in a Euclidean space (or more generally in a Riemannian manifold). Now we have extensive knowledge on their local geometry as well as their global geometry. Characterization of a regular curve is one of the important and interesting problems in the theory of curves in Euclidean space. There are two ways widely used to solve these problems: figuring out the relationship between the Frenet vectors of the curves [15], and determining the shape and size of a regular curve by using its curvatures k_1 (or $\varkappa$) and k_2 (or τ).

In 1845, Saint Venant [21] proposed the question of whether the principal normal of a curve is the principal normal of another on the surface generated by the principal normal of the given one. Bertrand answered this question in [3], published in 1850. He proved that a necessary and sufficient condition for the existence of such a second curve is required; in fact, a linear relationship calculated with constant coefficients should exist between the first and second curvatures of the given original curve. In other words, if we denote the first and second curvatures of a given curve by k_1 and k_2 respectively, we have $\lambda k_1 + \mu k_2 = 1,\ \lambda, \mu \in R$. Since 1850, after the paper of Bertrand, the pairs of curves like this have been called conjugate Bertrand curves, or more commonly Bertrand curves [15].

There are many important papers on Bertrand curves in Euclidean space [4, 7, 20].

When we investigate the properties of Bertrand curves in Euclidean n-space, it is easy to see that either k_2 or k_3 is zero, which means that Bertrand curves in $\mathbb{E}^n$ $(n > 3)$ are degenerate curves [20]. This result was restated by Matsuda and Yorozu [17]. They proved that there were not any special Bertrand curves in $\mathbb{E}^n$ $(n > 3)$ and defined a new kind, which is called $(1,3)$-type Bertrand curves in 4-dimensional Euclidean

*Correspondence: kilarslan@yahoo.com
2010 *AMS Mathematics Subject Classification:* Primary 53C50; Secondary 53C40.

space. Bertrand curves and their characterizations were studied by many researchers in Minkowski 3-space and Minkowski space-time (see [1, 2, 8, 10, 12, 13, 22, 23] as well as in Euclidean space. In addition, there are some other studies about Bertrand curves such as [9, 14, 16, 19, 24].

Many researchers have dealt with $(1,3)$-type Bertrand curves in Minkowski space-time. However, they only considered the casual character of the curves. Therefore, there are some gaps in this approach. For example, they take no account of whether a Cartan null curve can have a nonnull Bertrand mate curve. In this paper, we reconsider $(1,3)$-type Bertrand curves in Minkowski space-time with respect to the casual character of the plane spanned by the principal normal and the second binormal of the curve. For now, we look into the spacelike case of the plane.

2. Preliminaries

The Minkowski space-time $\mathbb{E}_1^4$ is the Euclidean 4-space $\mathbb{E}^4$ equipped with an indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}_1^4$. Recall that a vector $v \in \mathbb{E}_1^4 \backslash \{0\}$ can be spacelike if $g(v,v) > 0$, timelike if $g(v,v) < 0$, and null (lightlike) if $g(v,v) = 0$. In particular, the vector $v = 0$ is said to be a spacelike. The norm of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$. Two vectors v and w are said to be orthogonal if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^4$ can locally be spacelike, timelike, or null (lightlike) if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike, or null [18].

A null curve α is parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$ [5]. On the other hand, a nonnull curve α is parametrized by the arc-length parameter s if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along a curve α in $\mathbb{E}_1^4$, consisting of the tangent, the principal normal, and the first binormal and the second binormal vector field respectively.

From [11], if α is a spacelike or a timelike curve whose Frenet frame $\{T, N, B_1, B_2\}$ contains only nonnull vector fields, the Frenet equations are given by

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2\kappa_1 & 0 & 0 \\ -\epsilon_1\kappa_1 & 0 & \epsilon_3\kappa_2 & 0 \\ 0 & -\epsilon_2\kappa_2 & 0 & -\epsilon_1\epsilon_2\epsilon_3\kappa_3 \\ 0 & 0 & -\epsilon_3\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.1}$$

where $g(T,T) = \epsilon_1$, $g(N,N) = \epsilon_2$, $g(B_1,B_1) = \epsilon_3$, $g(B_2,B_2) = \epsilon_4$, $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = -1$, $\epsilon_i \in \{-1,1\}$, $i \in \{1,2,3,4\}$. In particular, the following conditions hold:

$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(N,B_2) = g(B_1,B_2) = 0.$$

From [5, 6], if α is a null Cartan curve, the Cartan Frenet equations are given by

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ \kappa_2 & 0 & -\kappa_1 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ -\kappa_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.2}$$

where the first curvature $\kappa_1(s) = 0$ if $\alpha(s)$ is a null straight line or $\kappa_1(s) = 1$ in all other cases. In this case, the next conditions hold:

$$g(T,T) = g(B_1, B_1) = 0, \quad g(N,N) = g(B_2, B_2) = 1,$$

$$g(T,N) = g(T,B_2) = g(N,B_1) = g(N,B_2) = g(B_1,B_2) = 0, \quad g(T,B_1) = 1.$$

3. On $(1,3)$-Bertrand curves with spacelike plane $sp\{N, B_2\}$ in $\mathbb{E}_1^4$

In this section, we discuss $(1,3)$-Bertrand curves according to their $(1,3)$-normal planes, which are planes spanned by the principal normal vectors and second binormal vectors of the curves. Here we assume that the $(1,3)$-normal planes are spacelike. As a result, we obtain the necessary and sufficient conditions for the curves to be $(1,3)$-Bertrand curves with spacelike $(1,3)$-normal plane.

Definition 1 *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ and $\beta^* : I^* \subset \mathbb{R} \to \mathbb{E}_1^4$ be C^∞-special Frenet curves in Minkowski space-time $\mathbb{E}_1^4$ and $f : I \to I^*$ a regular C^∞-map such that each point $\beta(s)$ of β corresponds to the point $\beta^*(s^*) = \beta^*(f(s))$ of β^* for all $s \in I$. Here s and s^* are arc-length parameters or pseudo-arc parameters of β and β^*, respectively. If the Frenet $(1,3)$-normal plane at each point $\beta(s)$ of β coincides with the Frenet $(1,3)$-normal plane at each point $\beta^*(s^*) = \beta^*(f(s))$ of β^* for all s, then β is called a $(1,3)$-Bertrand curve in Minkowski space-time $\mathbb{E}_1^4$ and β^* is called a $(1,3)$-Bertrand mate curve of β [10].*

Let $\beta : I \to \mathbb{E}_1^4$ be a $(1,3)$-Bertrand curve in $\mathbb{E}_1^4$ with the Frenet frame $\{T, N, B_1, B_2\}$ and the curvatures $\kappa_1, \kappa_2, \kappa_3$ and $\beta^* : I \to \mathbb{E}_1^4$ be a $(1,3)$-Bertrand mate curve of β with the Frenet frame $\{T^*, N^*, B_1^*, B_2^*\}$ and the curvatures $\kappa_1^*, \kappa_2^*, \kappa_3^*$. We assume that the $(1,3)$-normal plane spanned by $\{N, B_2\}$ is a spacelike plane. Since $sp\{N, B_2\} = sp\{N^*, B_2^*\}$ is a spacelike plane, we have the following four cases:

Case 1 *β is a spacelike or timelike curve with nonzero curvature functions $\kappa_1,\ \kappa_2,\ \kappa_3$ and spacelike vectors $N,\ B_2$, and β^* is also spacelike or timelike curve with nonzero curvature functions $\kappa_1^*,\ \kappa_2^*,\ \kappa_3^*$ and spacelike vectors $N^*,\ B_2^*$;*

Case 2 *β is a spacelike or timelike curve with nonzero curvature functions $\kappa_1,\ \kappa_2,\ \kappa_3$ and spacelike vectors $N,\ B_2$, and β^* is a Cartan null curve with curvature functions $\kappa_1^* = 1$, $\kappa_2^*,\ \kappa_3^* \neq 0$;*

Case 3 *β is a Cartan null curve with curvature functions $\kappa_1 = 1$, $\kappa_2,\ \kappa_3 \neq 0$, and β^* is a spacelike or timelike curve with nonzero curvature functions $\kappa_1^*,\ \kappa_2^*,\ \kappa_3^*$ and spacelike vectors $N^*,\ B_2^*$;*

Case 4 *β is a Cartan null curve with curvature functions $\kappa_1 = 1$, $\kappa_2,\ \kappa_3 \neq 0$ and β^* is also a Cartan null curve with curvature functions $\kappa_1^* = 1$, $\kappa_2^*,\ \kappa_3^* \neq 0$.*

In what follows, we consider these four cases separately.

Case 1. Let β be a spacelike or timelike curve with nonzero curvature functions $\kappa_1,\ \kappa_2,\ \kappa_3$ and spacelike vectors $N,\ B_2$, and β^* be also a spacelike or timelike curve with nonzero curvature functions $\kappa_1^*,\ \kappa_2^*,\ \kappa_3^*$ and spacelike vectors $N^*,\ B_2^*$. In this case, we have the following theorem.

Theorem 1 *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a spacelike or timelike curve parameterized by arc-length parameter s with the nonzero curvatures $\kappa_1, \kappa_2, \kappa_3$ and spacelike $(1,3)$-normal plane. Then the curve β is a $(1,3)$-Bertrand curve and its Bertrand mate curve is a spacelike or timelike curve with nonzero curvatures if and only if there exist constant real numbers a, b, $h \neq \mp 1$, μ satisfying*

$$a\kappa_2(s) - b\kappa_3(s) \neq 0, \tag{3.1}$$

$$1 = \epsilon_1 a\kappa_1(s) + \epsilon_3 h(a\kappa_2(s) - b\kappa_3(s)), \tag{3.2}$$

$$\mu\kappa_3(s) = h\kappa_1(s) - \kappa_2(s), \tag{3.3}$$

$$-\kappa_1(s)\kappa_2(s)(h^2+1) + h(\kappa_1^2(s) + \kappa_2^2(s) + \kappa_3^2(s)) \neq 0 \tag{3.4}$$

for all $s \in I$.

Proof We assume that $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ is a spacelike or timelike curve parameterized by arc-length parameter s with the nonzero curvatures $\kappa_1, \kappa_2, \kappa_3$ and spacelike $(1,3)$-normal plane, and the curve $\beta^* : I^* \subset \mathbb{R} \to \mathbb{E}_1^4$ is a spacelike or timelike $(1,3)$-Bertrand mate curve parameterized by arc-length parameter s^* with the nonzero curvatures $\kappa_1^*, \kappa_2^*, \kappa_3^*$ of the curve β. Then we can write the curve β^* as follows:

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + a(s)N(s) + b(s)B_2(s) \tag{3.5}$$

for all $s^* \in I^*$, $s \in I$ where $a(s)$ and $b(s)$ are C^∞-functions on I. Differentiating (3.5) with respect to s and using the Frenet formulae (2.1), we get

$$T^* f' = (1 - a\epsilon_1\kappa_1)T + a'N + \epsilon_3(a\kappa_2 - b\kappa_3)B_1 + b'B_2. \tag{3.6}$$

Multiplying equation (3.6) by N and B_2, respectively, we have

$$a' = 0 \text{ and } b' = 0. \tag{3.7}$$

Substituting (3.7) in (3.6), we find

$$T^* f' = (1 - a\epsilon_1\kappa_1)T + \epsilon_3(a\kappa_2 - b\kappa_3)B_1. \tag{3.8}$$

Multiplying equation (3.8) by itself, we obtain

$$\epsilon_1^*(f')^2 = \epsilon_1(1 - a\epsilon_1\kappa_1)^2 + \epsilon_3(a\kappa_2 - b\kappa_3)^2. \tag{3.9}$$

If we denote

$$\delta = \frac{1 - a\epsilon_1\kappa_1}{f'} \text{ and } \gamma = \frac{\epsilon_3(a\kappa_2 - b\kappa_3)}{f'}, \tag{3.10}$$

we get

$$T^* = \delta T + \gamma B_1. \tag{3.11}$$

Differentiating (3.11) with respect to s and using the Frenet formulae (2.1), we have

$$f'\kappa_1^* N^* = \delta' T + (\delta\kappa_1 - \gamma\kappa_2)N + \gamma' B_1 + \gamma\kappa_3 B_2. \tag{3.12}$$

Multiplying equation (3.12) by T and B_1, respectively, we get

$$\delta' = 0 \text{ and } \gamma' = 0. \tag{3.13}$$

From (3.10), we find

$$(1 - a\epsilon_1\kappa_1)\gamma = \epsilon_3(a\kappa_2 - b\kappa_3)\delta. \tag{3.14}$$

Assume that $\gamma = 0$. From (3.11), $T^* = \delta T$. Then

$$T^* = \pm T. \tag{3.15}$$

Differentiating (3.15) with respect to s and using the Frenet formulae (2.1), we find

$$f'\kappa_1^* N^* = \pm\kappa_1 N. \tag{3.16}$$

From (3.16), N is linearly dependent with N^*, which is a contradiction. Therefore, $\gamma \neq 0$. Since $\gamma \neq 0$, from (3.10), we find (3.1)

$$a\kappa_2 - b\kappa_3 \neq 0. \tag{3.17}$$

From (3.14), we have (3.2)

$$1 = a\epsilon_1\kappa_1 + h\epsilon_3(a\kappa_2 - b\kappa_3), \tag{3.18}$$

where $h \neq \mp 1$ from (3.14) and (3.9). Substituting (3.13) in (3.12), we get

$$f'\kappa_1^* N^* = (\delta\kappa_1 - \gamma\kappa_2)N + \gamma\kappa_3 B_2. \tag{3.19}$$

Multiplying equation (3.19) by itself, we obtain

$$(f')^2(\kappa_1^*)^2 = (\delta\kappa_1 - \gamma\kappa_2)^2 + \gamma^2\kappa_3^2. \tag{3.20}$$

Substituting (3.10) in (3.20), we find

$$(f')^2(\kappa_1^*)^2 = \frac{(a\kappa_2 - b\kappa_3)^2}{(f')^2}[(h\kappa_1 - \kappa_2)^2 + \kappa_3^2]. \tag{3.21}$$

Substituting (3.18) in (3.9), we have

$$(f')^2 = \epsilon_1^*\epsilon_1(a\kappa_2 - b\kappa_3)^2[h^2 - 1] \tag{3.22}$$

where $h^2 \neq 1$. Substituting (3.22) in (3.21), we get

$$(f')^2(\kappa_1^*)^2 = \frac{\epsilon_1^*\epsilon_1}{h^2 - 1}[(h\kappa_1 - \kappa_2)^2 + \kappa_3^2]. \tag{3.23}$$

If we denote

$$\lambda_1 = \frac{(\delta\kappa_1 - \gamma\kappa_2)}{f'\kappa_1^*} = \frac{\epsilon_3(a\kappa_2 - b\kappa_3)}{(f')^2\kappa_1^*}[(h\kappa_1 - \kappa_2)] \tag{3.24}$$

$$\lambda_2 = \frac{\gamma\kappa_3}{f'\kappa_1^*} = \frac{\epsilon_3(a\kappa_2 - b\kappa_3)}{(f')^2\kappa_1^*}\kappa_3, \tag{3.25}$$

we get

$$N^* = \lambda_1 N + \lambda_2 B_2. \tag{3.26}$$

Differentiating (3.26) with respect to s and using the Frenet formulae (2.1), we find

$$-\epsilon_1^* f' \kappa_1^* T^* + \epsilon_3^* f' \kappa_2^* B_1^* = -\epsilon_1 \kappa_1 \lambda_1 T + \lambda_1' N + \epsilon_3(\lambda_1 \kappa_2 - \lambda_2 \kappa_3) B_1 + \lambda_2' B_2. \tag{3.27}$$

Multiplying equation (3.27) by N and B_2 respectively, we obtain

$$\lambda_1' = 0 \text{ and } \lambda_2' = 0. \tag{3.28}$$

From (3.24) and (3.25), since $\lambda_2 \neq 0$, we have (3.3)

$$\mu \kappa_3 = h\kappa_1 - \kappa_2 \tag{3.29}$$

where $\mu = \lambda_1/\lambda_2$. Substituting (3.28) in (3.27), we find

$$-\epsilon_1^* f' \kappa_1^* T^* + \epsilon_3^* f' \kappa_2^* B_1^* = -\epsilon_1 \kappa_1 \lambda_1 T + \epsilon_3(\lambda_1 \kappa_2 - \lambda_2 \kappa_3) B_1. \tag{3.30}$$

From (3.8) and (3.30), we obtain

$$\epsilon_3^* f' \kappa_2^* B_1^* = A(s) T + B(s) B_1, \tag{3.31}$$

where

$$A(s) = \frac{\epsilon_1 \epsilon_3 (a\kappa_2 - b\kappa_3)}{(f')^2 (h^2 - 1)\kappa_1^*} [-\kappa_1 \kappa_2 (h^2+1) + h(\kappa_1^2 + \kappa_2^2 + \kappa_3^2)] \tag{3.32}$$

and

$$B(s) = \frac{\epsilon_1 \epsilon_3 h (a\kappa_2 - b\kappa_3)}{(f')^2 (h^2 - 1)\kappa_1^*} [-\kappa_1 \kappa_2 (h^2+1) + h(\kappa_1^2 + \kappa_2^2 + \kappa_3^2)]. \tag{3.33}$$

Since $\epsilon_3^* f' \kappa_2^* B_1^* \neq 0$, we get (3.4)

$$-\kappa_1 \kappa_2 (h^2+1) + h(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 \neq 0. \tag{3.34}$$

Conversely, we assume that $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ is a spacelike or timelike curve parameterized by arc-length parameter s with the nonzero curvatures $\kappa_1, \kappa_2, \kappa_3$ and spacelike $(1,3)$-normal plane, and the relations (3.1),(3.2),(3.3), (3.4) hold for constant real numbers $a, b, h \neq \mp 1, \mu$. Then we can define a curve β^* as follows:

$$\beta^*(s^*) = \beta(s) + aN(s) + bB_2(s). \tag{3.35}$$

Differentiating (3.35) with respect to s and using the Frenet formulae (2.1), we find

$$\frac{d\beta^*}{ds} = (1 - a\epsilon_1 \kappa_1) T + \epsilon_3 (a\kappa_2 - b\kappa_3) B_1. \tag{3.36}$$

From (3.36) and (3.2), we get

$$\frac{d\beta^*}{ds} = \epsilon_3 (a\kappa_2 - b\kappa_3)[hT + B_1]. \tag{3.37}$$

From (3.37), we have

$$f' = \frac{ds^*}{ds} = \left\| \frac{d\beta^*}{ds} \right\| = m_1 (a\kappa_2 - b\kappa_3) \sqrt{\epsilon_1 m_2 \left(h^2 - 1\right)} > 0 \tag{3.38}$$

where $m_1 = \mp 1$ such that $m_1(a\kappa_2 - b\kappa_3) > 0$ and $m_2 = \mp 1$ such that $\epsilon_1 m_2 \left(h^2 - 1\right) > 0$. Now, by rewriting (3.37), we obtain

$$T^* f' = \epsilon_3(a\kappa_2 - b\kappa_3)[hT + B_1]. \tag{3.39}$$

Substituting (3.38) in (3.39), we find

$$T^* = \frac{\epsilon_3 m_1}{\sqrt{\epsilon_1 m_2 \left(h^2 - 1\right)}}[hT + B_1], \tag{3.40}$$

which implies that $g(T^*, T^*) = m_2 = \epsilon_1^*$. Differentiating (3.40) with respect to s and using the Frenet formulae (2.1), we find

$$\frac{dT^*}{ds^*} = \frac{\epsilon_3 m_1}{f'\sqrt{\epsilon_1 m_2 \left(h^2 - 1\right)}}[(h\kappa_1 - \kappa_2)N + \kappa_3 B_2]. \tag{3.41}$$

Using (3.41), we have

$$\kappa_1^* = \left\| \frac{dT^*}{ds^*} \right\| = \frac{\sqrt{(h\kappa_1 - \kappa_2)^2 + \kappa_3^2}}{f'\sqrt{\epsilon_1 m_2 \left(h^2 - 1\right)}} > 0. \tag{3.42}$$

From (3.41) and (3.42), we have

$$N^* = \frac{1}{\kappa_1^*}\frac{dT^*}{ds^*} = \frac{\epsilon_3 m_1}{\sqrt{(h\kappa_1 - \kappa_2)^2 + \kappa_3^2}}[(h\kappa_1 - \kappa_2)N + \kappa_3 B_2], \tag{3.43}$$

which leads to $g(N^*, N^*) = 1$. If we denote

$$\lambda_3 = \frac{\epsilon_3 m_1(h\kappa_1 - \kappa_2)}{\sqrt{(h\kappa_1 - \kappa_2)^2 + \kappa_3^2}} \text{ and } \lambda_4 = \frac{\epsilon_3 m_1 \kappa_3}{\sqrt{(h\kappa_1 - \kappa_2)^2 + \kappa_3^2}}, \tag{3.44}$$

we obtain

$$N^* = \lambda_3 N + \lambda_4 B_2. \tag{3.45}$$

Differentiating (3.45) with respect to s and using the Frenet formulae (2.1), we find

$$f'\frac{dN^*}{ds^*} = -\epsilon_1 \lambda_3 \kappa_1 T + \lambda_3' N + \epsilon_3(\kappa_2\lambda_3 - \kappa_3\lambda_4)B_1 + \lambda_4' B_2. \tag{3.46}$$

Differentiating (3.3) with respect to s, we have

$$(h\kappa_1' - \kappa_2')\kappa_3 - (h\kappa_1 - \kappa_2)\kappa_3' = 0. \tag{3.47}$$

Differentiating (3.44) with respect to s and using (3.47), we get

$$\lambda_3' = 0 \text{ and } \lambda_4' = 0. \tag{3.48}$$

Substituting (3.44) and (3.48) in (3.46), we obtain

$$\frac{dN^*}{ds^*} = \frac{m_1\kappa_1(h\kappa_1 - \kappa_2)}{f'\sqrt{(h\kappa_1 - \kappa_2)^2 + \kappa_3^2}}T + \frac{m_1[\kappa_2(h\kappa_1 - \kappa_2) - \kappa_3^2]}{f'\sqrt{(h\kappa_1 - \kappa_2)^2 + \kappa_3^2}}B_1. \tag{3.49}$$

From (3.40) and (3.42), we find

$$\epsilon_1^*\kappa_1^*T^* = \frac{-m_1\sqrt{(h\kappa_1-\kappa_2)^2+\kappa_3^2}}{f'(h^2-1)}[hT+B_1]. \tag{3.50}$$

From (3.49) and (3.50), we get

$$\frac{dN^*}{ds^*}+\epsilon_1^*\kappa_1^*T^* = \frac{P(s)}{R(s)}\left[T+hB_1\right] \tag{3.51}$$

where

$$\begin{aligned} P(s) &= -m_1[-\kappa_1\kappa_2(h^2+1)+h(\kappa_1^2+\kappa_2^2+\kappa_3^2)] \neq 0, \\ R(s) &= f'(h^2-1)\sqrt{(h\kappa_1-\kappa_2)^2+\kappa_3^2} \neq 0. \end{aligned} \tag{3.52}$$

Using (3.51) and (3.52), we have κ_2^* as

$$\kappa_2^* = \left|\frac{P(s)}{R(s)}\right|\sqrt{\epsilon_3 m_3\left(h^2-1\right)} \tag{3.53}$$

where $m_3 = \pm 1$ such that $\epsilon_3 m_3\left(h^2-1\right) > 0$. Consider (3.51), (3.52), and (3.53) together, we obtain B_1^* as

$$B_1^* = \frac{\epsilon_3^*}{\kappa_2^*}[\frac{dN^*}{ds^*}+\epsilon_1^*\kappa_1^*T^*] = \frac{m_4\epsilon_3^*}{\sqrt{\epsilon_3 m_3\left(h^2-1\right)}}[T+hB_1] \tag{3.54}$$

where $m_4 = \left|\frac{P(s)}{R(s)}\right|/\frac{P(s)}{R(s)} = \pm 1$. From (3.54), we have $g(B_1^*, B_1^*) = m_3 = \epsilon_3^* = -\epsilon_1^*$. Besides, we can define a unit vector B_2^* as $B_2^* = -\lambda_4 N + \lambda_3 B_2$; that is,

$$B_2^* = \frac{m_1\epsilon_3}{\sqrt{(h\kappa_1-\kappa_2)^2+\kappa_3^2}}\left[-\kappa_3 N+(h\kappa_1-\kappa_2)B_2\right]. \tag{3.55}$$

Lastly, from (3.54) and (3.55), we get κ_3^* as

$$\kappa_3^* = g(\frac{dB_1^*}{ds^*}, B_2^*) = \frac{m_1 m_4\epsilon_3\epsilon_3^*\kappa_1\kappa_3(h^2-1)}{f'\sqrt{\epsilon_3 m_3\left(h^2-1\right)}\sqrt{(h\kappa_1-\kappa_2)^2+\kappa_3^2}} \neq 0.$$

Consequently, we find that β^* is a timelike or spacelike curve and a $(1,3)$-Bertrand mate curve of the curve β since span$\{N^*, B_2^*\}$ =span$\{N, B_2\}$. □

Case 2. Let β be a spacelike or timelike curve with nonzero curvature functions κ_1, κ_2, κ_3 and spacelike vectors N, B_2, and β^* be a Cartan null curve with curvature functions $\kappa_1^* = 1$, κ_2^*, $\kappa_3^* \neq 0$. In this case, we get the following theorem.

Theorem 2 *(i) Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a spacelike or timelike curve parameterized by arc-length parameter s with the nonzero curvatures $\kappa_1, \kappa_2, \kappa_3$ and spacelike $(1,3)$-normal plane. If the curve β is a $(1,3)$-Bertrand curve and its Bertrand mate curve is a Cartan null curve with nonzero third curvature then there exist constant real numbers a, b, $h = \mp 1$, μ satisfying*

$$a\kappa_2(s) - b\kappa_3(s) \neq 0, \tag{3.56}$$

$$1 = \epsilon_1 a\kappa_1(s) + \epsilon_3 h(a\kappa_2(s) - b\kappa_3(s)), \tag{3.57}$$

$$\mu\kappa_3(s) = h\kappa_1(s) - \kappa_2(s), \tag{3.58}$$

and

$$P_1^2(s) = P_2^2(s), \tag{3.59}$$

where

$$\begin{aligned} P_1(s) &= 2\mu^3\kappa_1(s)\kappa_3(s) + h\mu^2\left(\kappa_2^2(s) - \kappa_1^2(s)\right) + 2\mu\kappa_3(s)\left(\kappa_1(s) - h\kappa_2(s)\right) \\ &\quad + h\kappa_3^2(s), \end{aligned}$$

$$P_2(s) = 2\mu^3\kappa_1(s)\kappa_3(s) + \mu^2\left(\kappa_2^2(s) - \kappa_1^2(s) - 2\kappa_3^2(s)\right) - \kappa_3^2(s) \neq 0,$$

for all $s \in I$.

(*ii*) *Let* $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ *be a spacelike or timelike curve parameterized by arc-length parameter* s *with the nonzero constant curvatures* $\kappa_1, \kappa_2, \kappa_3$ *and spacelike* $(1,3)$*-normal plane. If the curve* β *satisfies the conditions* (3.56), (3.57), (3.58), (3.59), *and*

$$\kappa_1(s)P_1(s) - (\kappa_2(s) + \mu\kappa_3(s))P_2(s) \neq 0, \tag{3.60}$$

for all $s \in I$, *then the curve* β *is a* $(1,3)$*-Bertrand curve and its Bertrand mate curve is a Cartan null curve with nonzero third curvature.*

Proof The theorem can be proved by a similar technique to that in the first theorem. Therefore, we omit the proof of the theorem. □

Case 3. Let β be a Cartan null curve with curvature functions $\kappa_1 = 1$, κ_2, $\kappa_3 \neq 0$, and β^* be a spacelike or timelike curve with nonzero curvature functions κ_1^*, κ_2^*, κ_3^* and spacelike vectors N^*, B_2^*. Then we have the following theorem.

Theorem 3 *Let* $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ *be a Cartan null curve parameterized by pseudo-arc parameter* s *with the curvatures* $\kappa_1 = 1, \kappa_2, \kappa_3 \neq 0$. *Then the curve* β *is a* $(1,3)$*-Bertrand curve and its Bertrand mate curve is a spacelike or timelike curve with nonzero curvatures if and only if there exist constant real numbers* $a \neq 0$, b, h, μ *satisfying*

$$a\kappa_2(s) - b\kappa_3(s) = ah - 1, \tag{3.61}$$

$$\mu\kappa_3(s) = h + \kappa_2(s), \tag{3.62}$$

$$h^2 - \kappa_2^2(s) - \kappa_3^2(s) \neq 0, \tag{3.63}$$

for all $s \in I$.

Proof We assume that $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ is a Cartan null curve parameterized by pseudo-arc parameter s with the curvatures $\kappa_1 = 1, \kappa_2, \kappa_3 \neq 0$, and the curve $\beta^* : I^* \subset \mathbb{R} \to \mathbb{E}_1^4$ is a spacelike or timelike $(1,3)$-Bertrand mate curve parameterized by arc-length parameter s^* with nonzero curvatures $\kappa_1^*, \kappa_2^*, \kappa_3^*$ of the curve β. Then we can write the curve β^* as follows:

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + a(s)N(s) + b(s)B_2(s) \tag{3.64}$$

for all $s^* \in I^*$, $s \in I$ where $a(s)$ and $b(s)$ are C^∞-functions on I. Differentiating (3.64) with respect to s and using the Frenet formulae (2.2), we get

$$f'T^* = (1 + a\kappa_2 - b\kappa_3)T + a'N - aB_1 + b'B_2. \tag{3.65}$$

Multiplying equation (3.65) by N and B_2, respectively, we have

$$a' = 0 \text{ and } b' = 0. \tag{3.66}$$

Substituting (3.66) in (3.65), we find

$$f'T^* = (1 + a\kappa_2 - b\kappa_3)T - aB_1. \tag{3.67}$$

Multiplying equation (3.67) by itself, we obtain

$$\epsilon_1^*(f')^2 = -2a(1 + a\kappa_2 - b\kappa_3). \tag{3.68}$$

If we denote

$$\delta = \frac{(1 + a\kappa_2 - b\kappa_3)}{f'} \text{ and } \gamma = \frac{-a}{f'}, \tag{3.69}$$

we get

$$T^* = \delta T + \gamma B_1. \tag{3.70}$$

Differentiating (3.70) with respect to s and using the Frenet formulae (2.2), we have

$$f'\kappa_1^* N^* = \delta' T + (\delta - \gamma\kappa_2)N + \gamma' B_1 + \gamma\kappa_3 B_2. \tag{3.71}$$

Multiplying equation (3.71) by T and B_1 respectively, we get

$$\delta' = 0 \text{ and } \gamma' = 0. \tag{3.72}$$

Assume that $\gamma = 0$. From (3.11), $T^* = \delta T$. Then

$$T^* = \pm T. \tag{3.73}$$

Differentiating (3.73) with respect to s and using the Frenet formulae (2.2), we get that N is linearly dependent with N^*, which is a contradiction. Since $\gamma \neq 0$, from (3.10), we find (3.61)

$$a\kappa_2 - b\kappa_3 = ah - 1,$$

where $h = -\delta/\gamma$. Substituting (3.72) in (3.71), we get

$$f'\kappa_1^* N^* = (\delta - \gamma\kappa_2)N + \gamma\kappa_3 B_2. \tag{3.74}$$

Multiplying equation (3.74) by itself and using (3.68) and (3.69), we have

$$(f')^2 (\kappa_1^*)^2 = \frac{\epsilon_3^*[(h + \kappa_2)^2 + \kappa_3^2]}{2h}. \tag{3.75}$$

If we denote

$$\lambda_1 = \frac{\delta - \gamma\kappa_2}{f'\kappa_1^*} = \frac{a\left(h+\kappa_2\right)}{\left(f'\right)^2\kappa_1^*}, \tag{3.76}$$

$$\lambda_2 = \frac{\gamma\kappa_3}{f'\kappa_1^*} = \frac{-a\kappa_3}{\left(f'\right)^2\kappa_1^*}, \tag{3.77}$$

from (3.74), we obtain

$$N^* = \lambda_1 N + \lambda_2 B_2. \tag{3.78}$$

Differentiating (3.78) with respect to s and using the Frenet formulae (2.2), we find

$$-\epsilon_1^* f'\kappa_1^* T^* + \epsilon_3^* f'\kappa_2^* B_1^* = \left(\lambda_1\kappa_2 - \lambda_2\kappa_3\right)T + \lambda_1' N - \lambda_1 B_1 + \lambda_2' B_2. \tag{3.79}$$

Multiplying equation (3.79) by N and B_2 respectively, we obtain

$$\lambda_1' = 0 \text{ and } \lambda_2' = 0. \tag{3.80}$$

From (3.76) and (3.77), since $\lambda_2 \neq 0$, we have (3.3)

$$\mu\kappa_3 = h + \kappa_2$$

where $\mu = -\frac{\lambda_1}{\lambda_2}$. Substituting (3.80) in (3.79), we find

$$-\epsilon_1^* f'\kappa_1^* T^* + \epsilon_3^* f'\kappa_2^* B_1^* = \left(\lambda_1\kappa_2 - \lambda_2\kappa_3\right)T - \lambda_1 B_1. \tag{3.81}$$

From (3.67) and (3.81), we obtain

$$\epsilon_3^* f'\kappa_2^* B_1^* = A(s)T + B(s)B_1$$

where

$$A(s) = \frac{-a}{2\left(f'\right)^2\kappa_1^*}\left[h^2 - \kappa_2^2 - \kappa_3^2\right],$$

and

$$B(s) = \frac{-a}{2\left(f'\right)^2\kappa_1^* h}\left[h^2 - \kappa_2^2 - \kappa_3^2\right].$$

Since $\epsilon_3^* f'\kappa_2^* B_1^* \neq 0$, we get (3.4)

$$h^2 - \kappa_2^2 - \kappa_3^2 \neq 0.$$

Conversely, we assume that $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ is a Cartan null curve parameterized by pseudo-arc parameter s with the curvatures $\kappa_1, \kappa_2, \kappa_3$ and the relations (3.61), (3.62), and (3.63) hold for constant real numbers a, b, h, μ. Then we can define a curve β^* as follows:

$$\beta^*(s^*) = \beta(s) + aN(s) + bB_2(s). \tag{3.82}$$

Differentiating (3.82) with respect to s and using the Frenet formulae (2.2), we find

$$\frac{d\beta^*}{ds} = (1 + a\kappa_2 - b\kappa_3)T - aB_1. \tag{3.83}$$

From (3.83) and (3.61), we get

$$\frac{d\beta^*}{ds} = a[hT - B_1]. \tag{3.84}$$

From (3.84), we have

$$f' = \frac{ds^*}{ds} = \left\| \frac{d\beta^*}{ds} \right\| = \sqrt{2m_1 a^2 h} > 0 \tag{3.85}$$

where $m_1 = \pm 1$ such that $2m_1 a^2 h > 0$. Now, by rewriting (3.84), we obtain

$$T^* f' = a[hT - B_1]. \tag{3.86}$$

Substituting (3.85) in (3.86), we find

$$T^* = \frac{m_2}{\sqrt{2m_1 h}}[hT - B_1] \tag{3.87}$$

where $m_2 = a/|a| = \mp 1$. From (3.87), we get $g(T^*, T^*) = -m_1 = \epsilon_1^*$. Differentiating (3.87) with respect to s and using the Frenet formulae (2.2), we find

$$\frac{dT^*}{ds^*} = \frac{m_2}{f'\sqrt{2m_1 h}}[(h + \kappa_2)N - \kappa_3 B_2]. \tag{3.88}$$

Using (3.88), we have

$$\kappa_1^* = \left\| \frac{dT^*}{ds^*} \right\| = \frac{\sqrt{(h + \kappa_2)^2 + \kappa_3^2}}{f'\sqrt{2m_1 h}} > 0. \tag{3.89}$$

From (3.88) and (3.89), we have

$$N^* = \frac{1}{\kappa_1^*}\frac{dT^*}{ds^*} = \frac{m_2}{\sqrt{(h + \kappa_2)^2 + \kappa_3^2}}[(h + \kappa_2)N - \kappa_3 B_2], \tag{3.90}$$

which implies that $g(N^*, N^*) = 1$. If we denote

$$\lambda_3 = \frac{m_2(h + \kappa_2)}{\sqrt{(h + \kappa_2)^2 + \kappa_3^2}} \text{ and } \lambda_4 = \frac{-m_2\kappa_3}{\sqrt{(h + \kappa_2)^2 + \kappa_3^2}}, \tag{3.91}$$

we obtain

$$N^* = \lambda_3 N + \lambda_4 B_2. \tag{3.92}$$

Differentiating (3.92) with respect to s and using the Frenet formulae (2.1), we find

$$f'\frac{dN^*}{ds^*} = (\kappa_2\lambda_3 - \kappa_3\lambda_4)T + \lambda_3' N - \lambda_3 B_1 + \lambda_4' B_2. \tag{3.93}$$

Differentiating (3.62) with respect to s, we have

$$\kappa_2'\kappa_3 - (h + \kappa_2)\kappa_3' = 0. \tag{3.94}$$

Differentiating (3.91) with respect to s and using (3.94), we get

$$\lambda_3' = 0 \text{ and } \lambda_4' = 0. \tag{3.95}$$

Substituting (3.91) and (3.95) in (3.93), we obtain

$$\frac{dN^*}{ds^*} = \frac{m_2\left(h\kappa_2+\kappa_2^2+\kappa_3^2\right)}{f'\sqrt{(h+\kappa_2)^2+\kappa_3^2}}T - \frac{m_2(h+\kappa_2)}{f'\sqrt{(h+\kappa_2)^2+\kappa_3^2}}B_1. \tag{3.96}$$

From (3.87) and (3.89), we find

$$\epsilon_1^*\kappa_1^*T^* = \frac{-m_2\sqrt{(h+\kappa_2)^2+\kappa_3^2}}{2hf'}[hT-B_1]. \tag{3.97}$$

From (3.96) and (3.97), we get

$$\frac{dN^*}{ds^*} + \epsilon_1^*\kappa_1^*T^* = \frac{m_2\left(\kappa_2^2+\kappa_3^2-h^2\right)}{2f'\sqrt{(h+\kappa_2)^2+\kappa_3^2}}[T+\frac{1}{h}B_1]. \tag{3.98}$$

Using (3.98), we have κ_2^* as

$$\kappa_2^* = \frac{\left|\kappa_2^2+\kappa_3^2-h^2\right|}{f\sqrt{2\left|h\right|}\sqrt{(h+\kappa_2)^2+\kappa_3^2}} > 0. \tag{3.99}$$

Considering (3.98) and (3.99) together, we find B_1^* as

$$B_1^* = \frac{\epsilon_3^*}{\kappa_2^*}[\frac{dN^*}{ds^*}+\epsilon_1^*\kappa_1^*T^*] = \frac{\epsilon_3^*m_2m_3\sqrt{2\left|h\right|}}{2}[T+\frac{1}{h}B_1], \tag{3.100}$$

where $m_3 = \left(\kappa_2^2+\kappa_3^2-h^2\right)/\left|\kappa_2^2+\kappa_3^2-h^2\right| = \pm 1$ and $\epsilon_3^* = \pm 1$. From (3.100), we have $g(B_1^*, B_1^*) = m_1 = \epsilon_3^* = -\epsilon_1^*$. We can define a unit vector B_2^* as $B_2^* = -\lambda_4 N + \lambda_3 B_2$; that is,

$$B_2^* = \frac{m_2\kappa_3}{\sqrt{(h+\kappa_2)^2+\kappa_3^2}}N + \frac{m_2(h+\kappa_2)}{\sqrt{(h+\kappa_2)^2+\kappa_3^2}}B_2, \tag{3.101}$$

which implies that $g(B_2^*, B_2^*) = 1$. Lastly, from (3.100) and (3.101), we obtain κ_3^* as

$$\kappa_3^* = g(\frac{dB_1^*}{ds^*}, B_2^*) = \frac{\epsilon_3^*m_3\sqrt{2\left|h\right|}\kappa_3}{f'\sqrt{(h+\kappa_2)^2+\kappa_3^2}} \neq 0.$$

Consequently, we find that β^* is a timelike or spacelike curve, and a $(1,3)$-Bertrand mate curve of the curve β since span$\{N^*, B_2^*\}$ =span$\{N, B_2\}$. □

Case 4. Let β be a Cartan null curve with curvature functions $\kappa_1 = 1,\ \kappa_2,\ \kappa_3 \neq 0$ and β^* be also a Cartan null curve with curvature functions $\kappa_1^* = 1,\ \kappa_2^*,\ \kappa_3^* \neq 0$. In this case, we get the following theorem.

Theorem 4 *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a Cartan null curve parameterized by pseudo-arc parameter s with the curvatures $\kappa_1, \kappa_2, \kappa_3$. Then curve β is a $(1,3)$-Bertrand curve and its Bertrand mate curve is also a Cartan null curve with nonzero third curvature if and only if there exist constant real numbers $\lambda \neq 0,\ \delta,\ \gamma,\ \mu \neq 0$ satisfying*

$$1 + \lambda\kappa_2(s) - \mu\kappa_3(s) = 0, \tag{3.102}$$

$$\kappa_2^2(s) + \kappa_3^2(s) = \frac{\lambda^2}{\delta^4}, \tag{3.103}$$

$$-\frac{\kappa_2(s)}{\kappa_3(s)} = \gamma, \tag{3.104}$$

for all $s \in I$.

Proof We omit the proof of the theorem since it can be seen in [13]. □

4. Some examples

Example 1 *Let us consider the spacelike curve with the equation*

$$\beta(s) = \frac{1}{\sqrt{6}}\left(\sinh\left(\sqrt{3}s\right), \cosh\left(\sqrt{3}s\right), 3\sin s, -3\cos s\right).$$

The Frenet frame of β *is given by*

$$T(s) = \frac{1}{\sqrt{2}}\left(\cosh\left(\sqrt{3}s\right), \sinh\left(\sqrt{3}s\right), \sqrt{3}\cos s, \sqrt{3}\sin s\right),$$

$$N(s) = \frac{1}{\sqrt{2}}\left(\sinh\left(\sqrt{3}s\right), \cosh\left(\sqrt{3}s\right), -\sin s, \cos s\right),$$

$$B_1(s) = \frac{1}{\sqrt{2}}\left(\sqrt{3}\cosh\left(\sqrt{3}s\right), \sqrt{3}\sinh\left(\sqrt{3}s\right), \cos s, \sin s\right),$$

$$B_2(s) = \frac{1}{\sqrt{2}}\left(\sinh\left(\sqrt{3}s\right), \cosh\left(\sqrt{3}s\right), \sin s, -\cos s\right).$$

The curvatures of β *are* $k_1(s) = \sqrt{3}$, $k_2(s) = -2$, $k_3(s) = 1$. *Let us take* $a = 0$, $b = \sqrt{3}$, $h = 1/\sqrt{3}$, *and* $\mu = 3$ *in Theorem 1. Then it is obvious that the relations (3.1), (3.2), (3.3), and (3.4) hold. Therefore, curve* β *is a* $(1,3)$*-Bertrand curve in* E_1^4 *and the* $(1,3)$*-Bertrand mate curve* β^* *of curve* β *is a timelike curve given as follows:*

$$\beta^*(s) = \left(\frac{2\sqrt{6}}{3}\sinh\left(\sqrt{3}s\right), \frac{2\sqrt{6}}{3}\cosh\left(\sqrt{3}s\right), \sqrt{6}\sin s, -\sqrt{6}\cos s\right).$$

The Frenet frame of β^* *is given by*

$$T^*(s) = \left(2\cosh\left(\sqrt{3}s\right), 2\sinh\left(\sqrt{3}s\right), \sqrt{3}\cos s, \sqrt{3}\sin s\right),$$

$$N^*(s) = \frac{1}{\sqrt{5}}\left(2\sinh\left(\sqrt{3}s\right), 2\cosh\left(\sqrt{3}s\right), -\sin s, \cos s\right),$$

$$B_1^*(s) = \left(-\sqrt{3}\cosh\left(\sqrt{3}s\right), -\sqrt{3}\sinh\left(\sqrt{3}s\right), -2\cos s, -2\sin s\right),$$

$$B_2^*(s) = \frac{1}{\sqrt{5}}\left(\sinh\left(\sqrt{3}s\right), \cosh\left(\sqrt{3}s\right), 2\sin s, -2\cos s\right).$$

The curvatures of β^* *are* $k_1^*(s) = \sqrt{30}/2$, $k_2^*(s) = 4\sqrt{10}/5$, $k_3^*(s) = 1/\sqrt{10}$ *and also the following equalities hold:*

$$\begin{aligned} N^*(s) &= \frac{3}{\sqrt{10}}N(s) + \frac{1}{\sqrt{10}}B_2(s), \\ B_2^*(s) &= \frac{-1}{\sqrt{10}}N(s) + \frac{3}{\sqrt{10}}B_2(s). \end{aligned}$$

Example 2 *Let us consider the timelike curve with the equation*

$$\beta(s) = \left(\sqrt{2}\sinh s, \sqrt{2}\cosh s, \sin s, \cos s\right).$$

The Frenet frame of β *is given by*

$$T(s) = \left(\sqrt{2}\cosh s, \sqrt{2}\sinh s, \cos s, -\sin s\right),$$

$$N(s) = \frac{\sqrt{3}}{3}\left(\sqrt{2}\sinh s, \sqrt{2}\cosh s, -\sin s, -\cos s\right),$$

$$B_1(s) = \left(-\cosh s, -\sinh s, -\sqrt{2}\cos s, \sqrt{2}\sin s\right),$$

$$B_2(s) = \frac{\sqrt{3}}{3}\left(\sinh s, \cosh s, \sqrt{2}\sin s, \sqrt{2}\cos s\right).$$

The curvatures of β *are* $k_1(s) = \sqrt{3}$, $k_2(s) = 2\sqrt{6}/3$, $k_3(s) = 1/\sqrt{3}$. *Let us take* $a = 0$, $b = \sqrt{6}$, $h = -1/\sqrt{2}$, *and* $\mu = -7/\sqrt{2}$ *in Theorem 1. Then it is obvious that the relations (3.1), (3.2), (3.3), and (3.4) hold. Therefore, curve* β *is a* $(1,3)$*-Bertrand curve in* E_1^4 *and the* $(1,3)$*-Bertrand mate curve* β^* *of curve* β *is a spacelike curve with timelike first normal vector given as follows:*

$$\beta^*(s) = \left(2\sqrt{2}\sinh s, 2\sqrt{2}\cosh s, 3\sin s, 3\cos s\right).$$

The Frenet frame of β^* *is given by*

$$T^*(s) = \left(2\sqrt{2}\cosh s, 2\sqrt{2}\sinh s, 3\cos s, -3\sin s\right),$$

$$N^*(s) = \frac{1}{\sqrt{17}}\left(2\sqrt{2}\sinh s, 2\sqrt{2}\cosh s, -3\sin s, -3\cos s\right),$$

$$B_1^*(s) = \left(-3\cosh s, -3\sinh s, -2\sqrt{2}\cos s, 2\sqrt{2}\sin s\right),$$

$$B_2^*(s) = \frac{1}{\sqrt{17}}\left(3\sinh s, 3\cosh s, 2\sqrt{2}\sin s, 2\sqrt{2}\cos s\right).$$

The curvatures of β^* *are* $k_1^*(s) = \sqrt{17}$, $k_2^*(s) = 12\sqrt{34}/17$, $k_3^*(s) = -1/\sqrt{17}$ *and also the following equalities hold:*

$$\begin{aligned} N^*(s) &= \frac{7}{\sqrt{51}}N(s) - \frac{\sqrt{2}}{\sqrt{51}}B_2(s), \\ B_2^*(s) &= \frac{\sqrt{2}}{\sqrt{51}}N(s) + \frac{7}{\sqrt{51}}B_2(s). \end{aligned}$$

Example 3 *For the same spacelike curve in Example 1, let us take* $a = 1$, $b = -1-\sqrt{3}$, $h = 1$, *and* $\mu = 2+\sqrt{3}$ *in (ii) of Theorem 2. Then it is obvious that the relations* (3.56), (3.57), (3.58), (3.59), *and* (3.60) *hold. Therefore, curve* β *is a* $(1,3)$*-Bertrand curve in* E_1^4 *and the* $(1,3)$*-Bertrand mate curve* β^* *of curve* β *is a Cartan null given as follows:*

$$\beta^*(s) = \left(-\frac{\sqrt{6}}{3}\sinh\left(\sqrt{3}s\right), -\frac{\sqrt{6}}{3}\cosh\left(\sqrt{3}s\right), -\sqrt{2}\sin s, \sqrt{2}\cos s\right).$$

The Frenet frame of β^* *is given by*

$$T(s) = -\frac{1}{2^{\frac{1}{4}}}\left(\cosh\left(\sqrt{3}s\right), \sinh\left(\sqrt{3}s\right), \cos s, \sin s\right),$$

$$N(s) = -\frac{1}{2}\left(\sqrt{3}\sinh\left(\sqrt{3}s\right), \sqrt{3}\cosh\left(\sqrt{3}s\right), -\sin s, \cos s\right),$$

$$B_1(s) = \frac{1}{2^{\frac{3}{4}}}\left(\cosh\left(\sqrt{3}s\right), \sinh\left(\sqrt{3}s\right), -\cos s, -\sin s\right),$$

$$B_2(s) = \frac{1}{2}\left(\sinh\left(\sqrt{3}s\right), \cosh\left(\sqrt{3}s\right), \sqrt{3}\sin s, -\sqrt{3}\cos s\right).$$

The curvatures of β^* *are* $k_1^*(s) = 1$, $k_2^*(s) = \sqrt{2}/4$, $k_3^*(s) = \sqrt{6}/4$ *and also the following equalities hold:*

$$N^*(s) = -\left(\frac{\sqrt{2+\sqrt{3}}}{2}\right)N(s) - \left(\frac{1}{2\sqrt{2+\sqrt{3}}}\right)B_2(s),$$

$$B_2^*(s) = -\left(\frac{1}{2\sqrt{2+\sqrt{3}}}\right)N(s) + \left(\frac{\sqrt{2+\sqrt{3}}}{2}\right)B_2(s).$$

Example 4 *(The null curve equation given in [13]) Let us consider the null curve with the equation*

$$\beta(s) = \frac{1}{\sqrt{2}}(\sinh s, \cosh s, \sin s, \cos s).$$

The Frenet frame of β *is given by*

$$T(s) = \frac{1}{\sqrt{2}}(\cosh s, \sinh s, \cos s, -\sin s),$$

$$N(s) = \frac{1}{\sqrt{2}}(\sinh s, \cosh s, -\sin s, -\cos s),$$

$$B_1(s) = \frac{1}{\sqrt{2}}(-\cosh s, -\sinh s, \cos s, -\sin s),$$

$$B_2(s) = \frac{1}{\sqrt{2}}(\sinh s, \cosh s, \sin s, \cos s).$$

The curvatures of β *are* $k_1(s) = 1$, $k_2(s) = 0$, $k_3(s) = -1$. *Let us take* $a = b = 1$, $h = 2$, *and* $\mu = -2$ *in Theorem 3. Then it is obvious that the relations (3.61), (3.62), and (3.63) hold. Therefore, curve* β *is a*

$(1,3)$-Bertrand curve in E_1^4 and the $(1,3)$-Bertrand mate curve β^* of curve β is a timelike curve given as follows:

$$\beta^*(s)=\left(\frac{3}{2}\sqrt{2}\sinh s,\frac{3}{2}\sqrt{2}\cosh s,\frac{1}{2}\sqrt{2}\sin s,\frac{1}{2}\sqrt{2}\cos s\right).$$

The Frenet frame of β^* is given by

$$T^*(s)=\left(\frac{3}{4}\sqrt{2}\cosh s,\frac{3}{4}\sqrt{2}\sinh s,\frac{1}{4}\sqrt{2}\cos s,-\frac{1}{4}\sqrt{2}\sin s\right),$$

$$N^*(s)=\left(\frac{3}{10}\sqrt{10}\sinh s,\frac{3}{10}\sqrt{10}\cosh s,-\frac{1}{10}\sqrt{10}\sin s,-\frac{1}{10}\sqrt{10}\cos s\right),$$

$$B_1^*(s)=\left(-\frac{1}{4}\sqrt{2}\cosh s,-\frac{1}{4}\sqrt{2}\sinh s,-\frac{3}{4}\sqrt{2}\cos s,\frac{3}{4}\sqrt{2}\sin s\right),$$

$$B_2^*(s)=\left(\frac{1}{10}\sqrt{10}\sinh s,\frac{1}{10}\sqrt{10}\cosh s,\frac{3}{10}\sqrt{10}\sin s,\frac{3}{10}\sqrt{10}\cos s\right).$$

The curvatures of β^* are $k_1^*(s)=\sqrt{5}/4$, $k_2^*(s)=3\sqrt{5}/20$, $k_3^*(s)=1/\sqrt{5}$ and also the following equalities hold:

$$N^*(s)=\frac{2}{\sqrt{5}}N(s)+\frac{1}{\sqrt{5}}B_2(s),$$

$$B_2^*(s)=\frac{-1}{\sqrt{5}}N(s)+\frac{2}{\sqrt{5}}B_2(s).$$

Example 5 *For the same null curve in Example 3, let us take $a=1$, $b=-3$, $h=-2$, and $\mu=2$ in Theorem 3. Then it is obvious that the relations (3.61), (3.62), and (3.63) hold. Therefore, curve β is a $(1,3)$-Bertrand curve in E_1^4 and the $(1,3)$-Bertrand mate curve β^* of curve β is a spacelike curve with timelike first binormal vector given as follows:*

$$\beta^*(s)=\frac{-1}{\sqrt{2}}(\sinh s,\cosh s,3\sin s,3\cos s).$$

The Frenet frame of β^* is given by

$$T^*(s)=\frac{-1}{2\sqrt{2}}(\cosh s,\sinh s,3\cos s,-3\sin s),$$

$$N^*(s)=\frac{-1}{\sqrt{10}}(\sinh s,\cosh s,-3\sin s,-3\cos s),$$

$$B_1^*(s)=\frac{1}{2\sqrt{2}}(3\cosh s,3\sinh s,\cos s,-\sin s),$$

$$B_2^*(s)=\frac{-1}{\sqrt{10}}(3\sinh s,3\cosh s,\sin s,\cos s).$$

The curvatures of β^* are $k_1^*(s)=\sqrt{5}/4$, $k_2^*(s)=3\sqrt{5}/20$, $k_3^*(s)=-1/\sqrt{5}$ and also the following equalities *hold:*

$$N^{*}(s) = -\frac{2}{\sqrt{5}}N(s)+\frac{1}{\sqrt{5}}B_2(s),$$
$$B_2^{*}(s) = -\frac{1}{\sqrt{5}}N(s)-\frac{2}{\sqrt{5}}B_2(s).$$

Acknowledgment

The authors express thanks to the referees for their valuable suggestions. The first author would like to thank TUBITAK (The Scientific and Technological Research Council of Turkey) for their financial supports during his PhD studies.

References

[1] Balgetir H, Bektaş M, Ergüt M. Bertrand curves for non-null curves in 3-dimensional Lorentzian space. Hadronic J 2004; 27: 229–236.

[2] Balgetir H, Bektaş M, Inoguchi J. Null Bertrand curves in Minkowski 3-space and their characterizations. Note Mat 2004; 23: 7–13.

[3] Bertrand JM. Mémoire sur la théorie des courbes á double courbure. Comptes Rendus 1850; 15: 332–350 (in French).

[4] Bioche C. Sur les courbes de M. Bertrand. Bull Soc Math France 1889; 17: 109–112 (in French).

[5] Bonnor WB. Null curves in a Minkowski space-time. Tensor 1969; 20: 229–242.

[6] Bonnor WB. Curves with null normals in Minkowski space-time. In: Dadhich NJ, editor. A Random Walk in Relativity and Cosmology. New York, NY, USA: Wiley, 1985, pp. 33–47.

[7] Burke JF. Bertrand curves associated with a pair of curves. Mathematics Magazine 1960; 34: 60–62.

[8] Ekmekci N, İlarslan K. On Bertrand curves and their characterization. Differ Geom Dyn Syst 2001; 3: 17–24.

[9] Ersoy S, İnalcık A. Generalized spacelike Bertrand curves in Minkowski 5-space. Quaestiones Mathematicae 2014; 37: 19–29.

[10] Gök İ, Nurkan SK, İlarslan K. On pseudo null Bertrand curves in Minkowski space-time. Kyungpook Math J 2014; 54: 685–697.

[11] İlarslan K, Nesovic E. Spacelike and timelike normal curves in Minkowski space-time. Publications de L'Institut Math 2009; 85: 111–118.

[12] Jin DH. Null Bertrand curves in a Lorentz manifold. J Korea Soc Math Educ Ser B Pure Appl Math 2008; 15: 209–215.

[13] Kahraman F, Gök İ, İlarslan K. Generalized null Bertrand curves in Minkowski space-time. Scientic Annals of "Al.I. Cuza" University of Iasi Mathematica 2014; 60: 489–502.

[14] Kim CY, Park JH, Yorozu S. Curves on the unit 3-sphere $S^3(1)$ in Euclidean 4-space $\mathbb{R}^4$. Bull Korean Math Soc 2013; 50: 1599–1622.

[15] Kuhnel W. Differential Geometry: Curves-Surfaces-Manifolds. Wiesbaden, Germany: Braunschweig, 1999.

[16] Lucas P, Ortega-Yagües JA. Bertrand curves in the three-dimensional sphere. J Geom Phys 2012; 62: 1903–1914.

[17] Matsuda H, Yorozu S. Notes on Bertrand curves. Yokohama Math J 2003; 50: 41–58.

[18] O'Neill B. Semi-Riemannian Geometry with Applications to Relativity. New York, NY, USA: Academic Press, 1983.

[19] Oztekin HB. Weakened Bertrand curves in the Galilean space G^3. Journal of Advanced Mathematical Studies 2009; 2: 69–76.

[20] Pears LR. Bertrand curves in Riemannian space. J London Math Soc 1935; 1–10: 180–183.

[21] Saint Venant B. Mémoire sur les lignes courbes non planes. Journal de l'Ecole Polytechnique 1845; 18: 1–76 (in French).

[22] Uçum A, Keçilioğlu O, İlarslan K. Generalized Bertrand curves with timelike (1,3)-normal plane in Minkowski space-time. Kuwait J Sci 2015; 42: 10–27.

[23] Whittemore JK. Bertrand curves and helices. Duke Math J 1940; 6: 235–245.

[24] Yilmaz MY, Bektaş M. General properties of Bertrand curves in Riemann-Otsuki space. Nonlinear Anal-Theor 2008; 69: 3225–3231.