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Research Article

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Osculating curves in 4-dimensional semi-Euclidean space with index 2

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Abstract: In this paper, we give the necessary and sufficient conditions for non-null curves with non-null normals in 4-dimensional Semi-Euclidian space with indeks 2 to be osculating curves. Also we give some examples of non-null osculating curves in $\mathbb{E}_2^4$.

Keywords: Semi-Euclidian space, Spacelike and timelike curves, Osculating curve, Curvature

MSC: 53C40, 53C50

1 Introduction

In the Euclidian space $\mathbb{E}^3$, it is well known that to each unit speed curve $\alpha : I \subset\rightarrow \mathbb{E}^3$,whose successive derivatives $\alpha^{'}(s), \alpha^{''}(s)$ and $\alpha^{'''}(s)$ are linearly independent vectors, one can associate the moving orthonormal Frenet frame $\{T, N, B\}$, consisting of the tangent, the principal normal and the binormal vector field, respectively. The planes spanned by $\{T, N\}, \{T, B\}$ and $\{N, B\}$ are respectively known as the osculating, rectifying and the normal plane. The rectifying curve in $\mathbb{E}^3$ is defined in [2] as a curve whose position vector (with respect to some chosen origin) always lies in its rectifying plane. It is shown in [1] that there exists a simple relationship between the rectifying curves and centrodes, which play some important roles in mechanics and kinematics. Some characterizations of rectifying curves in Minkowski space-time are given in [6].

It is well-known that the position vector of a curve in $\mathbb{E}^3$always lies in its osculating plane $B^{\perp} = Sp\{T, N\}$ if and only if its second curvature $k_2(s)$ is equal to zero for each s ([7]). The same property holds for timelike and spacelike curves (with non-null principal normal) in Minkowski 3-space. Osculating curves of first kind and second kind in Euclidian 4-space and Minkowski space time were studied by İlarslan and Nesovic in [4, 5].

In the light of the papers in [4, 5], in this paper we define the first kind and the second kind osculating curves in 4-dimensional semi-Euclidian space with index 2, by means of the orthogonal complements $B_2^{\perp}$ and $B_1^{\perp}$of binormal vector fields B_2 and B_1, respectively. We restrict our investigation of the first kind and the second kind osculating curves to timelike curves as well as to spacelike curves whose Frenet frame $\{T, N, B_1, B_2\}$ contains only non-null vector fields. We characterize such osculating curves in terms of their curvature functions and find the necessary and the sufficient conditions for such curves to be the osculating curve.

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2 Preliminaries

$\mathbb{E}_2^4$ is the Euclidean 4-space $\mathbb{E}^4$ equipped with indefinite flat metric given by

$$g = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}_2^4$. Recall that an arbitrary vector $v \in \mathbb{E}_2^4 \setminus \{0\}$ can be *spacelike, timelike or null(lightlike)*, if respectively holds $g(v, v) > 0$ or $g(v, v) < 0$ or $g(v, v) = 0$. In particular the vector $v = 0$ is a spacelike. The norm of a vector v is given by $||v|| = \sqrt{|g(v, v)|}$ and two vectors v and w are said to be orthogonal if $g(v, w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_2^4$, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null. A spacelike or timelike curve $\alpha(s)$ has unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. Recall that the pseudosphere, the pseudohyperbolic space and lightcone are hyperquadrics in $\mathbb{E}_2^4$, respectively defined by $S_2^3(m, r) = \{x \in \mathbb{E}_2^4 : g(x - m, x - m) = r^2\}$, $H_1^3 = \{x \in \mathbb{E}_2^4 : g(x - m, x - m) = -r^2\}$, $C^3(m) = \{x \in \mathbb{E}_2^4 : g(x - m, x - m) = 0\}$, where $r > 0$ is the radius and $m \in \mathbb{E}_2^4$ is the centre (or vertex) of hyperquadric ([8]).

Let $\{T, N, B_1, B_2\}$ be the non-null moving Frenet frame along a unit speed non-null curve α in $\mathbb{E}_2^4$, consisting of the tangent, principal normal, first binormal and second binormal vector field, respectively. If α is a non-null curve with non-null vector fields, then $\{T, N, B_1, B_2\}$ is an orthonormal frame. Accordingly, let us put

$$g(T, T) = \epsilon_0, \quad g(N, N) = \epsilon_1, \quad g(B_1, B_1) = \epsilon_2, \quad g(B_2, B_2) = \epsilon_3, \tag{1}$$

whereby $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, 1\}$. Then the Frenet equations read, ([3])

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -\epsilon_0\epsilon_1 k_1 & 0 & k_2 & 0 \\ 0 & -\epsilon_1\epsilon_2 k_2 & 0 & \kappa_3 \\ 0 & 0 & -\epsilon_2\epsilon_3 k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2}$$

where the following conditions are satisfied:

$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_2) = g(B_1, B_2) = 0, \tag{3}$$

The curve α lies fully in $\mathbb{E}_2^4$ if $k_3(s) \neq 0$ for each s.

Let α be a non-null curve with non-null normals in $\mathbb{E}_2^4$. We define that α is *the first or the second kind osculating curve* in $\mathbb{E}_2^4$, if its position vector with respect to some chosen origin always lies in the orthogonal complement $B_2^{\perp}$ or $B_1^{\perp}$, respectively. The orthogonal complements $B_1^{\perp}$ and $B_2^{\perp}$ are non-degenerate hyperplanes of $\mathbb{E}_2^4$, spanned by $\{T, N, B_2\}$ and $\{T, N, B_1\}$, respectively.

Consequently, the position vector of the timelike and the spacelike first kind osculating curve (with non-null vector fields N and B_1), satisfies the equation

$$\alpha(s) = a(s)T(s) + b(s)N(s) + c(s)B_1(s), \tag{4}$$

and the position vector of the timelike and the spacelike second kind osculating curve (with non-null vector fields N and B_1), satisfies the equation

$$\alpha(s) = a(s)N(s) + b(s)B_1(s) + c(s)B_2(s), \tag{5}$$

for some differentiable functions $a(s)$, $b(s)$ and $c(s)$ in arclength function s.

3 Timelike and spacelike first kind osculating curves in $\mathbb{E}_2^4$

In this section we show that a non-null curve with non-null normals is the first kind osculating curve if and only if it lies fully in non-degenerate hyperplane of $\mathbb{E}_2^4$. In relation to that we give the following theorem.

Theorem 3.1. *Let α be a non-null curve in $\mathbb{E}_2^4$ with non-null vector fields N and B_1. Then α is congruent to the first kind osculating curve if and only if $k_3(s) = 0$ for each s.*

Proof. First assume that α is the first kind osculating curve in $\mathbb{E}_2^4$. Then its position vector satisfies relation (4). Differentiating relation (4) with respect to s and using Frenet equations (2), we easily find $k_3(s) = 0$.

Conversely, assume that the third curvature $k_3(s) = 0$ for each s. Let us decompose the position vector of α with respect to the orthonormal frame $\{T, N, B_1, B_2\}$ by

$$\alpha = \epsilon_0 g(\alpha, T)T + \epsilon_1 g(\alpha, N)N + \epsilon_2 g(\alpha, T)B_1 + \epsilon_3 g(\alpha, B_2)B_2, \tag{6}$$

Since $k_3(s) = 0$, relation (2) implies that B_2 is a constant vector and $g(\alpha, B_2) =$ constant. Substituting this in (6), we conclude that α is congruent to the first kind osculating curve. This completes the proof of the theorem. □

Corollary 3.2. *Every non-null curve with non-null vector fields N and B_1 lying fully in non-degenerate hyperplane in $\mathbb{E}_2^4$ is the first kind osculating curve.*

4 Timelike and spacelike second kind osculating curves in $\mathbb{E}_2^4$

In this section, we characterize non-null second kind osculating curves in $\mathbb{E}_2^4$ with non-null vector fields N and B_1 in terms of their curvatures. Let $\alpha = \alpha(s)$ be the unit speed non-null second kind osculating curve in $\mathbb{E}_2^4$ with non-null vector fields N and B_1 and non-zero curvatures $k_1(s), k_2(s)$ and $k_3(s)$. By definition, the position vector of the curve α satisfies the equation (5), for some differentiable functions $a(s)$, $b(s)$ and $c(s)$. Differentiating equation (5) with respect to s and using the Frenet equations (2), we obtain

$$T = \left(a' - \epsilon_0\epsilon_1 b k_1\right) T + \left(a k_1 + b'\right) N + \left(b k_2 - \epsilon_2\epsilon_3 c k_3\right) B_1 + c' B_2 \, .$$

It follows that

$$a' - \epsilon_0\epsilon_1 b k_1 = 1\,, \quad a k_1 + b' = 0\,, \quad b k_2 - \epsilon_2\epsilon_3 c k_3 = 0\,, \quad c' = 0, \tag{7}$$

and therefore

$$a(s) = -\frac{\epsilon_2\epsilon_3 c_0}{k_1}\left(\frac{k_3}{k_2}\right)', \quad b(s) = \epsilon_2\epsilon_3 c_0\left(\frac{k_3}{k_2}\right), \quad c(s) = c_0, \tag{8}$$

where $c_0 \in \mathbb{R}_0$. In this way functions $a(s)$, $b(s)$ and $c(s)$ are expressed in terms of curvature functions $k_1(s), k_2(s)$ and $k_3(s)$ of the curve α. Moreover, by using the first equation in (7) and relation (8), we easily find that the curvatures $k_1(s), k_2(s)$ and $k_3(s)$ satisfy the equation

$$\epsilon_2\epsilon_3\left(\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)'\right)' + \left(\frac{k_3}{k_2}\right)k_1 = -\frac{1}{c_0}, c_0 \in \mathbb{R}_0.$$

In this way, we obtain the following theorem.

Theorem 4.1. *Let $\alpha(s)$ be a unit speed non-null curve with non-null vector fields N, B_1 and B_2 with curvatures $k_1(s), k_2(s)$ and $k_3(s) \neq 0$ lying fully in $\mathbb{E}_2^4$. Then α is congruent to the second kind osculating curve if and only if there holds*

$$\epsilon_2\epsilon_3\left(\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)'\right)' + \left(\frac{k_3}{k_2}\right)k_1 = -\frac{1}{c_0} \tag{9}$$

where $\epsilon_2\epsilon_3 = \pm 1, c_0 \in_0$.

Proof. First assume that $\alpha(s)$ is congruent to the second kind osculating curve in $\mathbb{E}_2^4$. By using (8) and the first equation in relation (7), we easily find that relation (9) holds.

Conversely, assume that equation (9) is satisfied. Let us consider the vector $X \in \mathbb{E}_2^4$ given by

$$X(s) = \alpha(s) + \frac{\epsilon_2\epsilon_3 c_0}{k_1}\left(\frac{k_3}{k_2}\right)^{'} T(s) - \epsilon_2\epsilon_3 c_0 \left(\frac{k_3}{k_2}\right) N(s) - c_0 B_2(s)$$

By using relations (2) and (9) we easily find $X^{'}(s) = 0$, which means that X is a constant vector. Consequently, α is congruent to the second kind osculating curve. □

Recall that a unit speed non-null curve in $\mathbb{E}_2^4$ is called a W-curve, if it has constant curvature functions (see [9]). The following theorem gives the characterization of a non-null W-curves in $\mathbb{E}_2^4$ in terms of osculating curves.

Theorem 4.2. *Every non-null W-curve, with non-null vector fields N, B_1 and B_2 with curvatures $k_1(s), k_2(s)$ and $k_3(s) \neq 0$ lying fully in $\mathbb{E}_2^4$ is congruent to the second kind osculating curve.*

Proof. It is clear from Theorem 4.1. □

Example 4.3. *Let $\alpha(s)$ be a unit speed spacelike curve in $\mathbb{E}_2^4$ given by*

$$\alpha(s) = \frac{1}{15\sqrt{2}}\left(\sinh\left(3\sqrt{5}s\right), 9\cosh\left(\sqrt{5}s\right), 9\sinh\left(\sqrt{5}s\right), \cosh\left(3\sqrt{5}s\right)\right)$$

We easily obtain the Frenet vectors and curvatures as follows:

$$T(s) = \frac{1}{\sqrt{10}}\left(\cosh\left(3\sqrt{5}s\right), 3\sinh\left(\sqrt{5}s\right), 3\cosh\left(\sqrt{5}s\right), \sinh\left(3\sqrt{5}s\right)\right),$$

$$N(s) = \frac{\sqrt{2}}{2}\left(\sinh\left(3\sqrt{5}s\right), \cosh\left(\sqrt{5}s\right), \sinh\left(\sqrt{5}s\right), \cosh\left(3\sqrt{5}s\right)\right),$$

$$B_1(s) = \frac{\sqrt{10}}{10}\left(3\cosh\left(3\sqrt{5}s\right), \tfrac{1}{4}\sinh\left(\sqrt{5}s\right), \tfrac{1}{4}\cosh\left(\sqrt{5}s\right), 3\sinh\left(3\sqrt{5}s\right)\right),$$

$$B_2(s) = \frac{\sqrt{2}}{2}\left(\sinh\left(3\sqrt{5}s\right), -\tfrac{3}{4}\cosh\left(\sqrt{5}s\right), -\tfrac{3}{4}\sinh\left(\sqrt{5}s\right), \cosh\left(3\sqrt{5}s\right)\right),$$

where T and B_1 are spacelike vectors, N and B_2 are timelike vectors, $k_1(s) = 3, k_2(s) = 4$ and $k_3(s) = 5$. Since $g(\alpha, B_2) = 0$, α is congruent to second kind osculating curve. Also from Theorem 4.1. we find $c_0 = -\frac{4}{15}$. Thus we can write

$$\alpha(s) = \frac{1}{3}N(s) - \frac{4}{15}B_2(s)$$

Example 4.4. *Let $\alpha(s)$ be a unit speed timelike curve in $\mathbb{E}_2^4$ with the equation*

$$\alpha(s) = \frac{1}{\sqrt{15}}\left(\sinh\left(2\sqrt{5}s\right), \cosh\left(\sqrt{5}s\right), \sinh\left(\sqrt{5}s\right), \cosh\left(2\sqrt{5}s\right)\right)$$

We easily obtain the Frenet vectors and curvatures as follows:

$$T(s) = \frac{1}{\sqrt{3}}\left(2\cosh\left(2\sqrt{5}s\right), \sinh\left(\sqrt{5}s\right), \cosh\left(\sqrt{5}s\right), 2\sinh\left(2\sqrt{5}s\right)\right),$$

$$N(s) = \frac{1}{\sqrt{15}}\left(4\sinh\left(2\sqrt{5}s\right), \cosh\left(\sqrt{5}s\right), \sinh\left(\sqrt{5}s\right), 4\cosh\left(2\sqrt{5}s\right)\right),$$

$$B_1(s) = \frac{-1}{\sqrt{3}}\left(\cosh\left(2\sqrt{5}s\right), 2\sinh\left(\sqrt{5}s\right), 2\cosh\left(\sqrt{5}s\right), \sinh\left(2\sqrt{5}s\right)\right),$$

$$B_2(s) = \frac{-1}{\sqrt{15}}\left(\sinh\left(2\sqrt{5}s\right), 4\cosh\left(\sqrt{5}s\right), 4\sinh\left(\sqrt{5}s\right), \cosh\left(2\sqrt{5}s\right)\right),$$

where N and B_1 are spacelike vectors, T and B_2 are timelike vectors, $k_1(s) = 5, k_2(s) = 2$ and $k_3(s) = 2$. It can be easily verified that $g(\alpha, B_1) = 0$, which means that α is congruent to the second kind osculating curve. Also from Theorem 4.1. we find $c_0 = -\frac{1}{15}$. Thus we can write

$$\alpha(s) = -\frac{1}{5}N(s) + \frac{1}{5}B_2(s)$$

Remark. *The curve given in Example 4.3 lies fully in the pseudohyperbolic space H_1^3 with the equation $-x_1^2-x_2^2+x_3^2+x_4^2=\frac{-8}{45}$. The curve given in Example 4.4 lies fully in the light cone C^3 with the equation $-x_1^2-x_2^2+x_3^2+x_4^2=0$.*

Theorem 4.5. *Let $\alpha(s)$ be a unit speed non-null curve with non-null vector fields N, B_1 and B_2 with curvatures $k_1(s), k_2(s)$ and $k_3(s)\neq 0$ lying fully in $\mathbb{E}_2^4$. If α is the second kind osculating curve, then the following statements hold:*

i) The tangential and the principal normal component of the position vector α are respectively given by

$$\langle\alpha(s),T(s)\rangle=\frac{\epsilon_0\epsilon_2\epsilon_3c_0}{k_1}\left(\frac{k_3}{k_2}\right)', \quad \langle\alpha(s),N(s)\rangle=\epsilon_1\epsilon_2\epsilon_3c_0\left(\frac{k_3}{k_2}\right), \ c_0\in\mathbb{R}_0. \tag{10}$$

ii) The second binornmal component of the position vector α is a non-zero constant, i.e.

$$\langle\alpha(s),B_2(s)\rangle=c_0\epsilon_3, c_0\in\mathbb{R}_0 \tag{11}$$

Conversely, if $\alpha(s)$is a unit speed non-null curve with non-null vector fields N, B_1 and B_2 , lying fully in $\mathbb{E}_2^4$ and one of the statements (i) or (ii) hold, then α is congruent to the second kind osculating curve.

Proof. First assume that α is congruent to the second kind osculating curve in $\mathbb{E}_2^4$. By using relation (4) and (8), the position vector of α can be written as

$$\alpha(s)=-\frac{\epsilon_2\epsilon_3c_0}{k_1}\left(\frac{k_3}{k_2}\right)'T(s)+\epsilon_2\epsilon_3c_0\left(\frac{k_3}{k_2}\right)N(s)+c_0B_2(s) \tag{12}$$

Relation (12) easily implies that relations (10) and (11) are satisfied, which proves statements (i) and (ii).

Conversely, assume that the statement (i) holds. By taking the derivative of the equations $\langle\alpha(s),N(s)\rangle=\epsilon_1\epsilon_2\epsilon_3c_0\left(\frac{k_3}{k_2}\right)$,with respect to s and using (2) we get $\langle\alpha,B_1\rangle=0$, which means that α is congruent to the second kind osculating curve.

If the statement (ii) holds, in a similar way we conclude that α is congruent to the second kind osculating curve. □

Theorem 4.6. *Let $\alpha(s)$ be a non-null unit speed curve with non-null Frenet vectors and with curvatures $k_1(s), k_2(s)$ and $k_3(s)\neq 0$ lying fully in $\mathbb{E}_2^4$. If α is congruent to the second kind osculating curve then the following statements hold:*

i) If N and B_1 have the opposite casual characters then

$$\begin{aligned}\frac{k_3(s)}{k_2(s)}&=-\frac{1}{2c_0}\left(\int e^{-\int k_1(s)ds}ds+c_1\right)e^{\int k_1(s)ds}\\&+\frac{1}{2c_0}\left(\int e^{\int k_1(s)ds}ds+c_2\right)e^{-\int k_1(s)ds}\end{aligned} \tag{13}$$

where $c_0\in\mathbb{R}_0$ and $c_1,c_2\in\mathbb{R}$ ii) If N and B_1 have the same casual characters then

$$\frac{k_3(s)}{k_2(s)}=\frac{1}{c_0}\left(\int\sin\phi(s)ds+c_1\right)\cos ffi(s)-\frac{1}{c_0}\left(\int\cos\phi(s)ds+c_2\right)\sin ffi(s) \tag{14}$$

where $\phi(s)=\int k_1(s)ds$, $c_0\in\mathbb{R}_0$ and $c_1,c_2\in\mathbb{R}$. Conversely, if $\alpha(s)$ is a unit speed non-null curve with non-null vector fields N, B_1, lying fully in $\mathbb{E}_2^4$ and one of the statements (i) or (ii) holds, then α is congruent to the second kind osculating curve.

Proof. Let us first assume that $\alpha(s)$ is the unit speed non-null second kind osculating curve with non-null Frenet vector fields. From Theorem 4.1, the curvature functions of α satisfy the equation

$$\epsilon_2\epsilon_3\left(\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)'\right)'+\left(\frac{k_3}{k_2}\right)k_1=-\frac{1}{c_0}$$

Up to casual character of N and B_1 we have the following cases:

i) N and B_1 have the opposite casual character. Putting $\epsilon_2\epsilon_3 = -1$, $y(s) = \frac{k_3(s)}{k_2(s)}$ and $p(s) = \frac{1}{k_1(s)}$ in equation (9), we have,

$$-\frac{d}{ds}\left(p(s)\frac{dy}{ds}\right) + \frac{y(s)}{p(s)} = -\frac{1}{c_0}\ , c_0 \in \mathbb{R}_0$$

Next, by changing the variables in the above equation by $t(s) = \int \frac{1}{p(s)} ds$, we find

$$-\frac{d^2y}{dt^2} + y = -\frac{p(t)}{c_0}, c_0 \in \mathbb{R}_0$$

The solution of the previous differential equation is given by

$$y = -\frac{1}{2c_0}\left(\int p(t)e^{-t}dt + c_1\right)e^t + \frac{1}{2c_0}\left(\int p(t)e^{t}dt + c_2\right)e^{-t}$$

where $c_0 \in \mathbb{R}_0, c_1, c_2 \in \mathbb{R}$. Finally, since

$$y(s) = \frac{k_3(s)}{k_2(s)} \quad \text{and} \quad t(s) = \int \frac{1}{p(s)}ds,$$

we obtain

$$\begin{aligned}\frac{k_3(s)}{k_2(s)} = &-\frac{1}{2c_0}\left(\int e^{-\int k_1(s)ds}ds + c_1\right)e^{\int k_1(s)ds} \\ &+\frac{1}{2c_0}\left(\int e^{\int k_1(s)ds}ds + c_2\right)e^{-\int k_1(s)ds}.\end{aligned}$$

Conversely, if relation (13) holds, by taking the derivative of relation (13) two times with respect to s, we obtain that relation (9) is satisfied. Hence Theorem 4.1 implies that α is congruent to the first kind osculating curve.

In a similar way the statement (ii) can be proved. □

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