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Some fixed point theorems for contractive mapping in ordered vector metric spaces

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Abstract

In this paper, considering an order relation on a vector metric space which is introduced by Çevik and Altun in 2009, we present some fundamental fixed point results. Then, we provide some nontrivial examples show that the investigation of this work is significant.

Keywords: Fixed point, Riesz space, vector metric space.
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1. Introduction

Recent works about contraction mapping on metric spaces, show us there is a tendency to weaken the conditions of this mapping by considering especially (partially) ordered metric spaces. This began with Ran and Reurings in 2004 [9]. They were inspired by classical Banach fixed point theorem and showed the existence of fixed point for contraction mapping on ordered metric spaces. In addition to that they applied the results of their works to matrix equations. Many mathematicians have worked on ordered metric spaces since that time (see for example [4, 5]). Nieto and Rodríguez-López [7, 8] presented some new results for contractions in ordered metric spaces. They extended the work of Ran and Reuring by considering convergent sequences whose successive terms are comparable (monotone or non-monotone) instead of the continuity of contraction mapping [7, 8]. On the other hand, Çevik and Altun [3] expanded the concepts of metric spaces to vector metric spaces by using Riesz space valued metric (called vector metric). They obtained Banach contraction principle and some fixed point results for these spaces [3]. Also in [2], vectorially continuous function between vector metric spaces was defined. By this concept, the continuity can be given for more general spaces. In fact, the known description of continuity could

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be extended to all vector metric spaces in topological sense. However, this extension would not yield fruitful results. Although sequential continuity is equivalent to continuity in metric spaces specially, and first countable topological spaces in general, an appropriate extension could be attained by a version of sequentially continuity that does not fit this concept. It is provided by vectorial convergence and named as vectorial continuity.

In this work, we consider ordered vector metric spaces for the first time. Thus, the conditions of the contraction mappings on ordered metric space are even more weakened by this work. Since the real numbers equipped with the usual order is a Riesz space, the works in [7] and [8] become a special case of ours.

2. Preliminaries

Let X be a nonempty set equipped with an order relation $\leqslant$. A function f from X to itself is said to be order-preserving, if $f(x) \leqslant f(y)$ and is said to be order-reversing, if $f(y) \leqslant f(x)$ whenever $x, y \in X$ with $x \leqslant y$. A lattice is an ordered set that every two elements a and b have a supremum $a \vee b$ and an infimum $a \wedge b$. An ordered vector space is a (real) vector space equipped with an order relation $\leqslant$ which is compatible with the vector space operations. An ordered set $E = (E, \leqslant)$ is called a Riesz space or vector lattice, if it is an ordered vector space and a lattice. If $a \in E$, then we define $|a| = a \vee (-a)$. The element $|a|$ is called module of a. The notation $a_n \downarrow a$ means that (a_n) is an order-reversing sequence and a is the infimum for the set $\{a_n : n \in \mathbb{N}\}$. A Riesz space E is said to be Archimedean, if $\frac{1}{n}a \downarrow 0$ holds for every $a \in E^+$. A sequence (b_n) is said to be order-convergent (or o-convergent) to b in E, if there is a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $|b_n - b| \leqslant a_n$ for all n, and written $b_n \xrightarrow{o} b$. Furthermore (b_n) is said to be order-Cauchy (or o-Cauchy), if there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $|b_n - b_{n+p}| \leqslant a_n$ holds for all n and p. The Riesz space E is o-Cauchy complete, if every o-Cauchy sequence is o-convergent. The Riesz space E is called Dedekind (σ-)complete, if every nonempty (countable) subset of E that is bounded from above has a supremum. For notations and other facts regarding Riesz space we refer to [1] and [6].

We recall now some useful definitions for our main results and the concept of vector metric space introduced in [3].

Definition 2.1. Let X be a non-empty set and let E be a Riesz space. The mapping $d : X \times X \to E$ is said to be a vector metric (or E-metric), if it satisfies the following properties:

(vm1) $d(x, y) = 0$, if and only if $x = y$;

(vm2) $d(x, y) \leqslant d(x, z) + d(y, z)$, for all $x, y, z \in X$.

Also the triple (X, d, E) is said to be vector metric space.

By (vm1) and (vm2), the conditions of nonnegative and symmetry hold in a vector metric space.

Definition 2.2.

(a) A sequence (x_n) in a vector metric space (X, d, E) vectorial converges (or is E-converges) to some $x \in E$, written $x_n \xrightarrow{d,E} x$, if there is a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $d(x_n, x) \leqslant a_n$ for all n.

(b) A sequence (x_n) is called an E-Cauchy sequence whenever there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leqslant a_n$ holds for all n and p.

(c) A vector metric space (X, d, E) is called E-complete if each E-Cauchy sequence in X E-converges to a limit in X.

We give the following result without proof.

Lemma 2.3. *Let (X, d, E) be a vector metric space and (x_n) be a sequence in X. Then, $x_n \overset{d,E}{\to} x$, if and only if $d(x_n, x) \overset{o}{\to} 0$. Also, the sequence (x_n) is E-Cauchy, if and only if $d(x_n, x_{n+p}) \overset{o}{\to} 0$ for all n and p.*

Let us recall the vectorial continuity defined in [2]. This concept will be used throughout this paper.

Definition 2.4. Let (X, d, E) and (Y, ρ, F) be vector metric spaces, and let $x \in X$. A function $f : X \to Y$ is said to be vectorially continuous at x, if $x_n \xrightarrow{d,E} x$ in X implies $f(x_n) \xrightarrow{\rho,F} f(x)$ in Y. The function f is said to be vectorially continuous, if it is vectorially continuous at each point of X.

3. Main results

Let (X, d) be a metric space equipped with an order relation $\leqslant$ and let f be a contraction mapping from X to itself. In [7–9], the contraction condition on f is applied only to comparable elements, thus that is weakened with the order relation $\leqslant$ on X. Ran and Reurings [9] showed the existence of fixed point for the continuous mapping f in a metric space equipped with an order relation $\leqslant$. Later, in [7] and [8], Nieto and Rodríguez-López removed the continuity of f, and they put one of the following conditions:

(i) if an order-preserving sequence (x_n) converges to x, then $x_n \leqslant x$ for all n;

(ii) if an order-reversing sequence (y_n) converges to y, then $y \leqslant y_n$ for all n.

Note that every metric space is a vector metric space since $\mathbb{R}$ with usual order is a Riesz space. Here, we extend the results given for metric spaces in [7] and [8] to vector metric spaces. This is the result of the fact that vectorial convergence is used instead of usual convergence.

We use the notation $\lesseqgtr$ for comparable elements according to the order relation $\leqslant$ on X. That is, $x \lesseqgtr y$ if and only if $x \leqslant y$ or $y \leqslant x$. Throughout this section, the contraction condition on any function f from X to itself is the following.

$$\begin{gathered}\text{If there exists a constant } k \in [0, 1) \text{ such that}\\ d(f(x), f(y)) \leqslant kd(x, y) \text{ for all } x, y \in X \text{ with } x \lesseqgtr y.\end{gathered} \tag{3.1}$$

Theorem 3.1. *Let (X, d, E) be an E-complete vector metric space equipped with an order relation $\leqslant$ and E is Archimedean. Let f be an order-preserving mapping (according to $\leqslant$) from X to itself, and one of the followings is satisfied:*

(a) *The mapping f is vectorially continuous; or*

(b) *$x_n \leqslant x$ for all n whenever (x_n) order-preserving sequence and $x_n \overset{d,E}{\to} x$.*

If there exists an element u in X such that $u \leqslant f(u)$ and if f satisfies the condition (3.1), then f has a fixed point in X.

Proof. Let $u = x_0$. Define the sequence (x_n) by $x_n = f(x_{n-1})$ for all n. We have

$$x_{n+1} = f(x_n) = f^2(x_{n-1}) = \ldots = f^n(x_1) = f^{n+1}(x_0),$$

for all n. Since the mapping f is order-preserving, then for arbitrary n

$$x_0 \leqslant f(x_0) = x_1 \leqslant \cdots \leqslant f(x_{n-1}) = f^n(x_0) = x_n,$$

holds. Since the sequence (x_n) is order-preserving, we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f(x_{n-1}), f(x_n)) \\ &\leqslant kd(x_{n-1}, x_n) = kd(f(x_{n-2}), f(x_{n-1})) \end{aligned}$$

$$
\begin{aligned}
&\leqslant k^2 d(x_{n-2}, x_{n-1}) = k^2 d(f(x_{n-3}), f(x_{n-2})) \\
&\vdots \\
&\leqslant k^n d(x_0, x_1).
\end{aligned}
$$

Thus, for all n and p

$$
\begin{aligned}
d(x_n, x_{n+p}) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\
&\leqslant \left(k^n + k^{n+1} + \cdots + k^{n+p-1}\right) d(x_0, x_1) \\
&\leqslant \frac{k^n}{1-k} d(x_0, x_1).
\end{aligned}
$$

Then the sequence (x_n) is E-Cauchy since E is Archimedean. By the E-completeness of X, there exists x in X such that $x_n \overset{d,E}{\to} x$. Hence there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x) \leqslant a_n$ for all n.
Case 1. Due to the vectorial continuity of f there exists another sequence (b_n) in E such that $b_n \downarrow 0$ and $d(f(x_n), f(x)) \leqslant b_n$ for all n. Since

$$
\begin{aligned}
d(f(x), x) &\leqslant d(f(x), f(x_n)) + d(f(x_n), x) \\
&\leqslant b_n + d(x_{n+1}, x) \\
&\leqslant b_n + a_{n+1} \leqslant b_n + a_n,
\end{aligned}
$$

for all n and $b_n + a_n \downarrow 0$, then $d(f(x), x) = 0$, i.e., $f(x) = x$ holds.
Case 2. By $x_n \leqslant x$ for all n we have

$$
\begin{aligned}
d(f(x), x) &\leqslant d(f(x_n), f(x)) + d(f(x_n), x) \\
&\leqslant k d(x_n, x) + d(x_{n+1}, x) \\
&\leqslant k a_n + a_{n+1} \leqslant (k+1) a_n,
\end{aligned}
$$

for all n. Since $(k+1)a_n \downarrow 0$, then $d(f(x), x) = 0$. Consequently, x is a fixed point of f. □

According to the partial order $\leqslant$, for every two elements x and y in X the following two conditions (1) and (2) are equivalent [7].

(1) x and y have an upper bound or a lower bound in X.

(2) There exists z in X which is comparable to x and y.

These conditions are weaker than the followings:

(3) x and y are comparable.

(4) $x \vee y$ or $x \wedge y$ in X.

(5) X is a lattice.

(6) X is Dedekind (σ-) complete.

When one of the conditions (1) and (2) is added to the hypothesis of Theorem 3.1, the uniqueness of the fixed point is obtained.

Theorem 3.2. *If the condition (1) is added to the hypotheses of Theorem 3.1, then the fixed point x of f is unique. Moreover, for every $u \in X$, $f^n(u) \overset{d,E}{\to} x$.*

Proof. If y is another fixed points of f, we prove that $d(x,y)=0$.

Consider two cases:

Case 1. If x and y are comparable, we have

$$d(x,y)=d(f(x),f(y))\leqslant kd(x,y).$$

Since $k\in[0,1)$, then $d(x,y)=0$.

Case 2. If x and y are incomparable, then there exists z in X comparable to x and y by the above discussion. Monotonicity of f implies that $f^n(z)$ is comparable to $f^n(x)=x$ and $f^n(y)=y$ for all n. Thus,

$$d(x,y)\leqslant d(f^n(x),f^n(z))+d(f^n(z),f^n(y))\leqslant k^n[d(x,z)+d(z,y)],$$

for all n. Since E is Archimedean, then $d(x,y)=0$.

On the other hand, for arbitrary element u in X,

(i) if x is comparable to u, then $f^n(x)=x$ is comparable to $f^n(u)$ and hence

$$d(f^n(u),x)\leqslant k^n d(u,x),$$

for all n, and

(ii) if x is incomparable to u, there exists $z\in X$ which is comparable to x and u such that

$$d(f^n(u),x)\leqslant d(f^n(u),f^n(z))+d(f^n(z),f^n(x))\leqslant k^n[d(u,z)+d(z,x)],$$

for all n since $f^n(z)$ is comparable to $f^n(x)=x$ and $f^n(u)$ for all n.

By E is Archimedean, $k^n d(u,x)\downarrow 0$ in (i) and $k^n[d(u,z)+d(z,x)]\downarrow 0$ in (ii). From both (i) and (ii), $f^n(u)\overset{d,E}{\to} x$. □

Example 3.3. Let $X=\{1,2,3,4\}$. Consider the order $\leqslant$ defined by

$$x\leqslant y,\quad \text{if and only if there exists a positive integer } k \text{ such that}\quad y=kx,$$

for $x,y\in X$. It is well-known that $\mathbb{R}^2$ is an Archimedean Riesz space according to the coordinatewise ordering $\leqslant$ defined by

$$(a_1,a_2)\leqslant(b_1,b_2)\quad\text{if and only if}\quad a_1\leqslant b_1 \text{ and } a_2\leqslant b_2,$$

for $a_1,a_2,b_1,b_2\in\mathbb{R}$. The mapping $d:X\times X\to\mathbb{R}^2$ defined by

$$d(x,y)=\begin{cases}\left(|x-y|,\left|\frac{1}{x}-\frac{1}{y}\right|\right) & ,\quad (x,y)\notin\{(2,4),(4,2)\},\\ (4,4) & ,\quad (x,y)\in\{(2,4),(4,2)\},\end{cases}$$

is $\mathbb{R}^2$-complete vector metric on X.

Let $f:X\to X$ with $f(1)=1$ and $f(x)=x-1$ for $x\neq 1$. Then the function f is order-preserving and the condition (b) in Theorem 3.1 is satisfied. Also, $1\leqslant f(1)$ and the function f verify the inequality (3.1) with $k=2/3$. Moreover, $1\in X$ is comparable with every pair of elements of X. Therefore, f has unique fixed point. But f is not contractive mapping for any real-valued metric on X, thus we can not apply [7, Theorem 2.1] which was given for metric spaces to this example.

Remark 3.4. From now on the condition (1) is going to be given in the hypothesis of existence theorems, because proving the uniqueness is similar to the proof of Theorem 3.2 in next results. Therefore, it should be considered that this condition is only concerned with the uniqueness of the fixed point.

The above discussion is also applicable to order-reversing sequences. The detail is following.

Theorem 3.5. *Let (X, d, E) be an E-complete vector metric space equipped with an order relation $\leqslant$ satisfying (1) and E is Archimedean. Let f be an order-preserving mapping (according to $\leqslant$) from X to itself, and one of the followings is satisfied:*

(a) *the mapping f is vectorially continuous; or*

(b) *$x \leqslant x_n$ for all n whenever (x_n) is order-reversing sequence and $x_n \overset{d,E}{\to} x$.*

If there exists an element u in X such that $f(u) \leqslant u$ and if f satisfies the condition (3.1), then f has a fixed point in X.

Proof. Let $u = x_0$. Define the sequence (x_n) by $x_n = f(x_{n-1})$ for all n. Note that $x_{n+1} = f^{n+1}(x_0)$ for all n. Since the mapping f is order-preserving, then for arbitrary n

$$x_0 \geqslant f(x_0) = x_1 \geqslant \cdots \geqslant f(x_{n-1}) = f^n(x_0) = x_n,$$

holds. The rest of proof is similar to the proof of Theorem 3.1. □

Let us give fixed point theorems for order-reversing mappings.

Theorem 3.6. *Let (X, d, E) be an E-complete vector metric space equipped with an order relation $\leqslant$ satisfying (1) and E is Archimedean. Let f be an order-reversing mapping from X to itself, and one of the followings is satisfied:*

(a) *the mapping f is vectorially continuous; or*

(b) *if (x_n) is a sequence in X satisfying $x_n \lesseqgtr x_{n+1}$ for all n and $x_n \overset{d,E}{\to} x$, then there exists a subsequence (x_{n_m}) of (x_n) such that $x_{n_m} \lesseqgtr x$ for all n_m.*

If there exists an element u in X such that $f(u) \lesseqgtr u$ and if f satisfies the condition (3.1), then f has a unique fixed point in X.

Proof. Let $u = x_0$. Define the sequence (x_n) by $x_n = f(x_{n-1})$ for all n. Since $x_{n+1} = f^{n+1}(x_0)$ for all n, by hypotheses, x_n and x_{n+1} are comparable for all n. We can assume without loss of generality that $x_0 \leqslant f(x_0)$. Since the mapping f is order-reversing, then for arbitrary n

$$\begin{aligned} x_0 &\leqslant f(x_0) = x_1 \\ &\geqslant f(x_1) = f^2(x_0) = x_2 \\ &\leqslant f(x_2) = f^3(x_0) = x_3 \\ &\geqslant f(x_3) = f^4(x_0) = x_4 \\ &\vdots \end{aligned}$$

holds. By induction,

$$d(x_{n+1}, x_n) \leqslant k^n d(x_1, x_0),$$

for all n since every successive terms of (x_n) are comparable. As the proof of Theorem 3.1, for all n and p

$$d(x_n, x_{n+p}) \leqslant \frac{k^n}{1-k} d(x_0, x_1).$$

Then the sequence (x_n) is E-Cauchy since E is Archimedean. By the E-completeness of X, there exists x in X such that $x_n \overset{d,E}{\to} x$.

We show that x is fixed point of f. If f is vectorially continuous, by the proof of Theorem 3.1 it is clear that x is fixed point of f. Although the sequence (x_n) is not monotone, but every successive terms of (x_n) are comparable and convergent sequence to x. Supposing the hypotheses in (b), there is a subsequence (x_{n_m}) of (x_n) whose each term is comparable to x. Since there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x) \leqslant a_n$ for all n, we have

$$\begin{aligned} d(x, f(x)) &\leqslant d(x, x_{n_m+1}) + d(f(x), x_{n_m+1}) \\ &\leqslant d(x, x_{n_m+1}) + d(f(x), f(x_{n_m})) \\ &\leqslant d(x, x_{n_m+1}) + kd(x, x_{n_m}) \\ &\leqslant a_{n_m+1} + ka_{n_m} \\ &\leqslant (k+1)a_{n_m} \end{aligned}$$

for all m. Then $f(x) = x$. The uniqueness of the fixed point x is obtained from Theorem 3.2. □

Example 3.7. The set ℓ_∞ of all bounded real sequences is a lattice with the pointwise ordering (that is, $(x_n) \leqslant (y_n)$ in ℓ_∞ if and only if $x_n \leqslant y_n$ for all n). With the same ordering and the usual operations, the set ℓ_1 of all real sequences (a_n) satisfying $\sum_{n=1}^{\infty} |a_n| < \infty$ is an Archimedean Riesz space. The mapping $d : \ell_\infty \times \ell_\infty \to \ell_1$ defined by

$$d((x_n), (y_n)) = \left(\frac{1}{2^n} |x_n - y_n| \right),$$

is ℓ_1-complete vector metric on ℓ_∞. Define a sequence $(x_n) = ((x_m^n))$ of ℓ_∞ with

$$x_m^n = \begin{cases} 2 & , \quad m = 1, \\ 1/2 & , \quad 2 \leqslant m \leqslant n+1, \\ 1 & , \quad m > n+1. \end{cases} .$$

The sequence (x_n) is order-reversing and $x_n \overset{d,\ell_1}{\to} \mathbf{1} = (1, 1, \ldots)$. Since the all terms of (x_n) are not comparable with $\mathbf{1}$, the condition (b) is not satisfied. However, the function $f : \ell_\infty \to \ell_\infty$; $f(x) = x/2$ is monotone and vectorially continuous. For every $x \in \ell_\infty$, $f(x) \leqslant x$ and the function f verify the inequality (3.1) with $k = 1/2$. Therefore, f has the unique fixed point $\mathbf{0} = (0, 0, \ldots)$.

The existence of Theorem 3.1 (b) or Theorem 3.5 (b) imply the existence of Theorem 3.6 (b). Conversely, if $d(x, z) \leqslant d(y, z)$ for any $x, y, z \in X$, $y \leqslant x \leqslant z$ and the existence of Theorem 3.6 (b) are valid, then the existence of Theorem 3.1 (b) and Theorem 3.5 (b) are valid when the iterative sequence is monotone ([8, Remark 1]).

On the light the conclusion previously discussed we attain following result.

Theorem 3.8. *Let (X, d, E) be an E-complete vector metric space equipped with an order relation $\leqslant$ satisfying (1) and E is Archimedean. Let f be a monotone mapping from X to itself, and one of the followings is satisfied:*

(a) *the mapping f is vectorially continuous; or*

(b) *if (x_n) is a sequence in X satisfying $x_n \lesseqgtr x_{n+1}$ for all n and $x_n \overset{d,E}{\to} x$, then there exists a subsequence (x_{n_m}) of (x_n) such that $x_{n_m} \lesseqgtr x$ for all n_m.*

If there exists an element u in X such that $f(u) \lesseqgtr u$ and if f satisfies the condition (3.1), then f has a unique fixed point in X.

Now, we give a fixed point theorem by a condition weaker than the monotonicity of f.

Theorem 3.9. *Let (X, d, E) be an E-complete vector metric space equipped with an order relation $\leqslant$ satisfying (1) and E is Archimedean. Let f be a mapping from X to itself such that for every $x, y \in X$, $f(x) \lesseqgtr f(y)$ whenever $x \leqslant y$, and one of the followings is satisfied:*

(a) *the mapping* f *is vectorially continuous; or*

(b) *if* (x_n) *is a sequence in* X *satisfying* $x_n \lesssim x_{n+1}$ *for all* n *and* $x_n \xrightarrow{d,E} x$*, then there exists a subsequence* (x_{n_m}) *of* (x_n) *such that* $x_{n_m} \lesssim x$ *for all* n_m*.*

If there exists an element u *in* X *such that* $f(u) \lesssim u$ *and if* f *satisfies the condition* (3.1)*, then* f *has a unique fixed point in* X*.*

Proof. Let $u = x_0$. Define the sequence (x_n) by $x_n = f(x_{n-1})$ for all n. We have $x_{n+1} = f^{n+1}(x_0)$ for all n. Since the mapping f transforms to comparable elements into comparable elements, x_n and x_{n+1} are comparable for all n. The rest of proof is similar to the proof of Theorem 3.6. □

Example 3.10. Let $X = \{(x, 0), (0, x) : x \in [0, 1]\}$. Consider the order $\leqslant$ on X defined by

$$\begin{aligned} (x,0) \leqslant (y,0) &\quad \Leftrightarrow \quad x \leqslant y \\ (0,x) \leqslant (0,y) &\quad \Leftrightarrow \quad x \leqslant y, \quad x \neq 0, \quad y \neq 0. \end{aligned}$$

The order $\leqslant$ on $\mathbb{R}^2$ is coordinatewise as in Example 3.3. The mapping $d : X \times X \to \mathbb{R}^2$ defined by

$$\begin{aligned} d((x,0),(y,0)) &= (2\,|x-y|\,,|x-y|), \\ d((0,x),(0,y)) &= \left(|x-y|\,,\frac{1}{2}\,|x-y|\right), \\ d((x,0),(0,y)) &= \left(2x+y, x+\frac{y}{2}\right), \end{aligned}$$

is $\mathbb{R}^2$-complete vector metric on X.

Let $f : X \to X$ with $f((x, 0)) = (0, x)$ for $x \in [0, 1]$ and $f((0, x)) = \left(0, 1 - \frac{x}{2}\right)$ for $x \in (0, 1]$. Then the function f is vectorially continuous, but it is not monotone. Also, for every $x \in [0, 1]$ $(0, x) \lesssim f((0, x))$ and the function f verify the condition (3.1) with $k = 1/2$. The elements $(0, 0)$ and $\left(0, \frac{1}{2}\right)$ are fixed points of f. There are pairs o elements of X have not an upper bound and a lower bound in X as $(1, 0)$ and $(0, 1)$. Thus, f has not unique fixed point in X. Furthermore, we can not apply [8, Theorem 7] which was given for metric spaces to this example since the function f is not a contractive mapping for any real-valued metric on X.

4. Conclusion

In this paper, we have generated nontrivial class by using ordered vector metric spaces. Also, we have given three examples show that theorems in [7] and [8] are specific versions of our theorems. So that, this study will lead to some new applications.

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