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ON A BROAD CATEGORY OF MULTIVALUED WEAKLY PICARD OPERATORS

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Abstract. In the present paper, considering a recent technique which is used by Jleli and Samet [10] for fixed points of single-valued maps, we introduce a new concept of multivalued θ-contractions on metric spaces and prove that some of such mappings are multivalued weakly Picard operators on complete metric space. Finally, we give a nontrivial example to show that the class of multivalued θ-contractions is more general than multivalued contractions in the sense of Nadler [14] on complete metric spaces.
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1. Introduction and preliminaries

Let (X, d) be a metric space. We denote by $P(X)$ the collection of all nonempty subsets of X, by $CB(X)$ the collection of all nonempty closed and bounded subsets of X and by $K(X)$ the collection of all nonempty compact subsets of X. It is well known that $H : CB(X) \times CB(X) \to \mathbb{R}$ defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A) \right\}$$

is a metric on $CB(X)$, where $D(x, B) = \inf \{d(x, y) : y \in B\}$, which is called Pompeiu-Hausdorff metric induced by d. We can find detailed information about the Pompeiu-Hausdorff metric in [1, 3, 5, 9]. An element $x \in X$ is said to be a fixed point of a multivalued mapping $T : X \to P(X)$ if $x \in Tx$. Let $T : X \to CB(X)$ be a mapping, then T is called multivalued contraction if there exists $L \in [0, 1)$ such that

$$H(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$.

In 1969, Nadler [14] proved a fundamental fixed point theorem for multivalued mappings: Every multivalued contraction on complete metric spaces has a fixed point.

Inspired by his result, since then various fixed point results concerning multivalued contractions has been further developed in different directions by many authors (see,[6, 7, 8, 11, 12]).

In 2003, Rus et al [19] introduced the concept of multivalued weakly Picard (MWP) operator on a metric space: $T : X \to P(X)$ is a MWP operator if there exists a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ for any initial point x_0, converges to a fixed point of T. Then Petruşel extensively studied on MWP opeartors in [16]. It is easy to see that every Nadler [14], Reich [17], Rus [18], Petruşel [15], Mizoguchi-Takahashi [13], Berinde and Berinde [4] type multivalued contractions on complete metric spaces are MWP operators.

On the other hand, a new type of contractive maps has been introduced by Jleli and Samet [10]. Throughout this study, we called it as θ-contraction.

Let Θ be the set of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(Θ_1) θ is nondecreasing,

(Θ_2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$,

(Θ_3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

Definition 1.1. ([10]) Let (X, d) be a metric space and $T : X \to X$ be a mapping. Given $\theta \in \Theta$, we say that T is θ-contraction if there exists $k \in (0, 1)$ such that

$$\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k \tag{1.1}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$.

If we consider the different type of mapping θ in Definition 1.1., we obtain some of variety of contractions. For example, let $\theta : (0, \infty) \to (1, \infty)$ be given by $\theta(t) = e^{\sqrt{t}}$. It is clear that $\theta \in \Theta$. Then (1.1) turns to

$$d(Tx, Ty) \leq k^2 d(x, y) \tag{1.2}$$

for all $x, y \in X, Tx \neq Ty$. It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality $d(Tx, Ty) \leq k^2 d(x, y)$ also holds. Therefore T is an ordinary contraction. Similarly, let $\theta : (0, \infty) \to (1, \infty)$ be given by $\theta(t) = e^{\sqrt{te^t}}$. It is clear that $\theta \in \Theta$. Then (1.1) turns to

$$\frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty)-d(x,y)} \leq k^2 \tag{1.3}$$

for all $x, y \in X, Tx \neq Ty$.

In addition, we have concluded that every θ-contraction T is a contractive mapping, i.e.,

$$d(Tx, Ty) < d(x, y)$$

for all $x, y \in X, Tx \neq Ty$. Thus, every θ-contraction is a continuous mapping. On the other side, Example in [10] shows that the mapping T is not ordinary contraction, but it is a θ-contraction with $\theta(t) = e^{\sqrt{te^t}}$. Thus the following theorem, which was given as a corollary by Jleli and Samet is a proper generalization of Banach Contraction Principle.

Theorem 1.2. (Corollary 2.1 of [10]) *Let (X, d) be a complete metric space and $T : X \to X$ be a θ-contraction. Then T has a unique fixed point in X.*

We can find some generalizations of Theorem 1.2. for single valued mappings in [2]. The aim of this paper is to introduce the concept of multivalued θ-contraction, by combining the ideas of Jleli, Samet's and Nadler's, and give some fixed point results for mappings of this type on complete metric spaces.

2. Main results

Definition 2.1. Let (X, d) be a complete metric space and $T : X \to CB(X)$. Given $\theta \in \Theta$, we say that T is multivalued θ-contraction if there exists $k \in (0, 1)$ such that

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^k \tag{2.1}$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

We can easily obtain that every multivalued contraction is also multivalued θ-contraction with $\theta(t) = e^{\sqrt{t}}$.

Our main result is as follows:

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to K(X)$ be a multivalued θ-contraction. Then T is a MWP operator.*

Proof. Let $x_0 \in X$ be an arbitrary point in X. Since Tx is nonempty for all $x \in X$, we can choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T. Let $x_1 \notin Tx_1$. Then, since Tx_1 is closed, $D(x_1, Tx_1) > 0$. On the other hand, from $D(x_1, Tx_1) \leq H(Tx_0, Tx_1)$ and (Θ_1),

$$\theta(D(x_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1)).$$

From (2.1), we can write that

$$\theta(D(x_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1)) \leq [\theta(d(x_1, x_0))]^k. \tag{2.2}$$

Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) = D(x_1, Tx_1).$$

Then, from (2.2)

$$\theta(d(x_1, x_2)) \leq \theta(H(Tx_0, Tx_1)) \leq [\theta(d(x_1, x_0))]^k.$$

If we continue recursively, we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and if $x_n \notin Tx_n$ for all $n \in \mathbb{N}$, then

$$\theta(d(x_n, x_{n+1})) \leq [\theta(d(x_n, x_{n-1}))]^k \tag{2.3}$$

for all $n \in \mathbb{N}$. Otherwise, obviously T has a fixed point. Denote $a_n = d(x_n, x_{n+1})$, for $n \in \mathbb{N}$. Then $a_n > 0$ for all $n \in \mathbb{N}$ and, using (2.3), we have

$$\theta(a_n) \leq [\theta(a_{n-1})]^k \leq [\theta(a_{n-2})]^{k^2} \leq \cdots \leq [\theta(a_0)]^{k^n}.$$

Thus, we obtain

$$1 < \theta(a_n) \leq [\theta(a_0)]^{k^n} \tag{2.4}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.4), we obtain

$$\lim_{n\to\infty} \theta(a_n) = 1. \tag{2.5}$$

From (Θ_2), $\lim_{n\to\infty} a_n = 0^+$ and so from (Θ_3) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(a_n) - 1}{(a_n)^r} = l.$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\left| \frac{\theta(a_n) - 1}{(a_n)^r} - l \right| \leq B.$$

This implies that, for all $n \geq n_0$,

$$\frac{\theta(a_n) - 1}{(a_n)^r} \geq l - B = B.$$

Then, for all $n \geq n_0$,

$$n(a_n)^r \leq An\left[\theta(a_n) - 1\right],$$

where $A = 1/B$.

Suppose now that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\frac{\theta(a_n) - 1}{(a_n)^r} \geq B.$$

This implies that, for all $n \geq n_0$,

$$n(a_n)^r \leq An\left[\theta(a_n) - 1\right],$$

where $A = 1/B$.

Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$n(a_n)^r \leq An\left[\theta(a_n) - 1\right].$$

Using (2.4), we obtain, for all $n \geq n_0$,

$$n(a_n)^r \leq An\left[[\theta(a_0)]^{k^n} - 1\right].$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n(a_n)^r = 0.$$

Thus, there exits $n_1 \in \mathbb{N}$ such that $n(a_n)^r \le 1$ for all $n \ge n_1$. So, we have, for all $n \ge n_1$

$$a_n \le \frac{1}{n^{1/r}}. \tag{2.6}$$

In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \ge n_1$. Using the triangular inequality for the metric and from (2.6), we have

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= a_n + a_{n+1} + \cdots + a_{m-1} = \sum_{i=n}^{m-1} a_i \le \sum_{i=n}^{\infty} a_i \le \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$, passing to limit $n \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$. On the other hand, from (Θ_1) and (2.1), it is easy to conclude that

$$H(Tx, Ty) < d(x, y)$$

for all $x, y \in X$ with $Tx \ne Ty$. Therefore, for all $x, y \in X$

$$H(Tx, Ty) \le d(x, y) \tag{2.7}$$

is satisfied. Then

$$D(x_{n+1}, Tz) \le H(Tx_n, Tz) \le d(x_n, z)$$

Passing to limit $n \to \infty$, we obtain $D(z, Tz) = 0$. Thus, we get $z \in \overline{Tz} = Tz$. Considering the proof technique, we can say that T is a MWP operator.

Remark 2.3. *The following question arises: can we take $CB(X)$ instead of $K(X)$ in Theorem 2.2. under the same conditions? Unfortunately, the answer is negative as shown in the following example.*

Example 2.4. Let $X = [0, 2]$ and

$$d(x, y) = \begin{cases} 0 & , \quad x = y \\ 1 + |x - y| & , \quad x \ne y \end{cases}.$$

Then it is clear that (X, d) is complete metric space, which is also bounded. Since τ_d is discrete topology, all subsets of X are closed. Therefore all subsets of X are closed and bounded. Define a mapping $T : X \to CB(X)$,

$$Tx = \begin{cases} \mathbb{Q} & , \quad x \in X \backslash \mathbb{Q} \\ X \backslash \mathbb{Q} & , \quad x \in \mathbb{Q} \end{cases},$$

where $\mathbb{Q}$ is the set of all rational numbers in X . Therefore T has no fixed points. Now, define $\theta : (0, \infty) \to (1, \infty)$ by

$$\theta(t) = \begin{cases} e^{\sqrt{t}} & , \quad t \le 1 \\ 9 & , \quad t > 1 \end{cases},$$

then we can see that $\theta \in \Theta$. Now we show that

$$\theta(H(Tx,Ty)) \leq [\theta(d(x,y))]^{\frac{1}{2}}$$

for all $x, y \in X$ with $H(Tx,Ty) > 0$. Note that $H(Tx,Ty) > 0 \Leftrightarrow \{x,y\} \cap \mathbb{Q}$ is singleton. Therefore, we have

$$\begin{aligned} H(Tx,Ty) > 0 \quad &\Rightarrow \quad H(Tx,Ty) = 1 \text{ and } d(x,y) = 1 + |x-y| > 1 \\ &\Rightarrow \quad \theta(H(Tx,Ty)) = e \text{ and } [\theta(d(x,y))]^{\frac{1}{2}} = \sqrt{9} = 3 \\ &\Rightarrow \quad \theta(H(Tx,Ty)) \leq [\theta(d(x,y))]^{\frac{1}{2}}. \end{aligned}$$

Therefore, T is a multivalued θ-contraction, but is not a MWP operator.

However, we can take $CB(X)$ instead of $K(X)$, by adding the following weak condition on θ:

(Θ_4) $\theta(\inf A) = \inf \theta(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$.

Note that, if θ satisfies (Θ_1), then it satisfies (Θ_4) if and only if it is right continuous. Let

$$\Xi = \{\theta \mid \theta : (0,\infty) \to (1,\infty) \text{ satisfies } (\Theta_1)\text{-}(\Theta_4)\}.$$

Theorem 2.5. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multivalued θ-contraction with $\theta \in \Xi$. Then T is a MWP operator.*

Proof. Let $x_0 \in X$. Since Tx is nonempty for all $x \in X$, we can choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T . Let $x_1 \notin Tx_1$. Then, since Tx_1 is closed, $D(x_1,Tx_1) > 0$. On the other hand, from $D(x_1,Tx_1) \leq H(Tx_0,Tx_1)$ and (Θ_1)

$$\theta(D(x_1,Tx_1)) \leq \theta(H(Tx_0,Tx_1)).$$

From (2.1), we can write that

$$\theta(D(x_1,Tx_1)) \leq \theta(H(Tx_0,Tx_1)) \leq [\theta(d(x_0,x_1))]^k. \tag{2.8}$$

From (Θ_4) we can write (note that $d(x_1,Tx_1) > 0$)

$$\theta(D(x_1,Tx_1)) = \inf_{y \in Tx_1} \theta(d(x_1,y)),$$

and so from (2.8) we have

$$\inf_{y \in Tx_1} \theta(d(x_1,y)) \leq [\theta(d(x_0,x_1))]^k < [\theta(d(x_0,x_1))]^{\frac{k+1}{2}}. \tag{2.9}$$

Then, from (2.9) there exists $x_2 \in Tx_1$ such that

$$\theta(d(x_1,x_2)) \leq [\theta(d(x_0,x_1))]^{\frac{k+1}{2}}.$$

If $x_2 \in Tx_2$ we are finished. Otherwise, by the same way we can find $x_3 \in Tx_2$ such that

$$\theta(d(x_2,x_3)) \leq [\theta(d(x_1,x_2))]^{\frac{k+1}{2}}.$$

We continue recursively, then we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and

$$\theta(d(x_n,x_{n+1})) \leq [\theta(d(x_n,x_{n-1}))]^{\frac{k+1}{2}}$$

for all $n \in \mathbb{N}$. The rest of the proof can be completed as in the proof of Theorem 2.2.

Now we give an example showing that T is multivalued θ-contraction but is not a multivalued contraction.

Example 2.6. Consider the complete metric space (X, d), where $X = \{0, 1, 2, \cdots\}$ and $d : X \times X \to [0, +\infty)$ is given by

$$d(x,y) = \begin{cases} 0 & , \quad x = y \\ \\ x + y & , \quad x \neq y \end{cases} .$$

Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \{0,1\} & , \quad x \in \{0,1,2\} \\ \\ \{0, x-1\} & , \quad x \geq 3 \end{cases} .$$

Let $x \geq 3$ and $y = 0$, then we obtain

$$\lim_{x\to\infty} \frac{H(Tx, Ty)}{d(x, y)} = \lim_{x\to\infty} \frac{x-1}{x} = 1.$$

Thus, T is not multivalued contraction. Therefore, we can not guarantee that T is a MWP operator by Nadler fixed point theorem.

Now we claim that, T is multivalued θ-contraction with $\theta(t) = e^{\sqrt{te^t}}$ and $k = e^{-\frac{1}{2}}$. To see (2.1), we have to show that

$$\frac{H(Tx, Ty)}{d(x, y)} e^{H(Tx,Ty)-d(x,y)} \leq e^{-1}$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

First, observe that

$$H(Tx, Ty) > 0 \Leftrightarrow (x \neq y \text{ and } \{x, y\} \cap \{0, 1, 2\} \text{ is empty or singleton}).$$

Now, without loss of generality we may assume $x > y$ in the following cases. Case 1. If $\{x, y\} \cap \{0, 1, 2\}$ is singleton, then $H(Tx, Ty) = x - 1$ and $d(x, y) \in \{x, x+1, x+2\}$. Thus, we have

$$\frac{H(Tx, Ty)}{d(x, y)} e^{H(Tx,Ty)-d(x,y)} \leq \frac{x-1}{x} e^{-1} \leq e^{-1}.$$

Case 2. If $\{x, y\} \cap \{0, 1, 2\}$ is empty, then $H(Tx, Ty) = x - 1$ and $d(x, y) = x + y$. Thus, we have

$$\frac{H(Tx, Ty)}{d(x, y)} e^{H(Tx,Ty)-d(x,y)} = \frac{x-1}{x+y} e^{-1-y} \leq e^{-1}.$$

This shows that all conditions of Theorem 2.2 (or Theorem 2.5) are satisfied and so T is a MWP operator.

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