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# PSEUDO PICARD OPERATORS ON GENERALIZED METRIC SPACES

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In this paper, we present a new class of pseudo Picard operators in the setting of generalized metric spaces introduced recently in [M. Jleli and B. Samet: *A generalized metric space and related fixed point theorems,* Fixed Point Theory Appl., (2015) 2015:61]. An example is provided to illustrate the main result.

## 1. INTRODUCTION AND PRELIMINARIES

Let $(X, d)$ be a metric space and $T : X \to X$. Following Rus [**10**], the operator $T$ is said to be a Picard operator (PO), if $T$ has a unique fixed point $x^* \in X$, and for all $x \in X$, the Picard sequence $\{T^n x\}$ converges to $x^*$. Various classes of Picard operators exist in the literature (see, for examples, [**1, 2, 3, 4**]). In [**11**], Rus introduced the concept of weakly Picard operators as follows. The operator $T$ is said to be a weakly Picard operator (WPO), if the set of fixed points of $T$ is nonempty, and for all $x \in X$, the Picard sequence $\{T^n x\}$ converges to a fixed point of $T$. Different classes of weakly Picard operators exist in the literature. As example, we cite the class of almost contractions introduced by Berinde [**2**]. On the other hand, the concept of pseudo Picard operators (PPO) has been recently introduced (see [**9**]). We say that the operator $T$ is pseudo Picard operator, if the set of fixed points of $T$ is nonempty, and for some initial point $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to a fixed point of $T$. We can find many pseudo Picard operators in the literature. As example, we cite the class of $(\alpha, \psi)$-contractions

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introduced by Samet in [**12**] (see also [**13**]), which includes various types of pseudo Picard operators. It is clear that every PO is WPO, and every WPO is PPO. However, the converse is not true as shown in the following examples. Let $X$ be the set of all real numbers, endowed with the standard metric

$$d(x,y) = |x-y|, \quad (x,y) \in X \times X.$$

Let $T, S : X \to X$ be the mappings defined by

$$Tx = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

and

$$Sx = x^2, \quad x \in X.$$

Then $T$ is a WPO but not a PO. However, $S$ is a PPO but not a WPO.

In this paper, we present a new class of pseudo Picard operators on the setting of generalized metric spaces (JS-metric spaces) in the sense of Jleli and Samet [**7**] (see also [**8, 14**] ). For the sake of completeness, we recall briefly some basic concepts of such spaces.

Let $X$ be a nonempty set and $D : X \times X \to [0, \infty]$ be a given mapping. For every $x \in X$, we define the set

$$C(D, X, x) = \left\{ \{x_n\} \subset X : \lim_{n\to\infty} D(x_n, x) = 0 \right\}.$$

We say that $D$ is a JS-metric (in the sense of Jleli and Samet [**7**]) on $X$ if it satisfies the following conditions:

$(D_1)$ For all $(x,y) \in X \times X$,

$$D(x,y) = 0 \Longrightarrow x = y.$$

$(D_2)$ For all $(x,y) \in X \times X$,

$$D(x,y) = D(y,x).$$

$(D_3)$ There exists some constant $C > 0$ such that for every $(x,y) \in X \times X$ and $\{x_n\} \in C(D, X, x)$,

$$D(x,y) \leq C \limsup_{n\to\infty} D(x_n, y).$$

In this case $(X, D)$ is said to be a JS-metric space. The class of JS-metric spaces is larger than many known classes of metric spaces. For examples, every standard metric space, every $b$-metric space, every dislocated metric space (in the sense of Hitzler-Seda [**5**]), and every modular space with the Fatou property is a JS-metric space. For more details see [**7**].

Let $(X, D)$ be a JS-metric space.
We say that a sequence $\{x_n\} \subset X$ converges to some $x \in X$ with respect to $D$ iff

$\{x_n\} \in C(D, X, x)$.
We say that a sequence $\{x_n\} \subset X$ is Cauchy iff $\lim_{n,m\to\infty} D(x_n, x_m) = 0$.
If every Cauchy sequence in $(X, D)$ is convergent, then $(X, D)$ is said to be complete.

By Proposition 2.4 in [**7**], we see that every convergent sequence in $(X, D)$ has a unique limit. That is, if $\{x_n\} \in C(D, X, x) \cap C(D, X, y)$, for some $x, y \in X$, then $x = y$.

Taking into account the convergence concept in JS-metric spaces, we can define PO, WPO and PPO on such spaces.

After introducing the notion of JS-metric spaces, Jleli and Samet [**7**] presented some fixed point results on such spaces, including Banach contraction and Ciric type quasicontraction mappings. In fact, we can see that these types of contraction mappings on complete JS-metric spaces are PPO.

In order to obtain a new class of PPO on JS-metric spaces, we need to present the following class of functions introduced recently in [**6**].

Let $\Theta$ be the set of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

$(\Theta_1)$ $\theta$ is nondecreasing.

$(\Theta_2)$ For each sequence $\{t_n\} \subset (0, \infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \iff \lim_{n\to\infty} t_n = 0^+.$$

$(\Theta_3)$ There exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that

$$\lim_{t\to 0^+} \frac{\theta(t) - 1}{t^r} = l.$$

**Definition 1.1.** **([6])** Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping. Given $\theta \in \Theta$, we say that $T$ is a $\theta$-contraction if there exists $k \in (0, 1)$ such that

$$\theta(d(Tx, Ty)) \le [\theta(d(x, y))]^k , \tag{1}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$.

For different choices of the mapping $\theta \in \Theta$, we can obtain a variety of contractions. For example, let $\theta : (0, \infty) \to (1, \infty)$ be the mapping defined by $\theta(t) = e^{\sqrt{t}}$. It is clear that $\theta \in \Theta$. Then (1) turns to

$$d(Tx, Ty) \le k^2 d(x, y), \quad (x, y) \in X \times X,\ Tx \ne Ty. \tag{2}$$

It is clear that for $x, y \in X$ such that $Tx = Ty$, the inequality (2) holds. Therefore, $T$ is an ordinary contraction. Similarly, let $\theta : (0, \infty) \to (1, \infty)$ be the mapping given by $\theta(t) = e^{\sqrt{te^t}}$. It is clear that $\theta \in \Theta$. Then (1) turns to

$$\frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty)-d(x,y)} \le k^2, \quad (x, y) \in X \times X,\ Tx \ne Ty. \tag{3}$$

Observe that every $\theta$-contraction $T$ is continuous. Indeed, any $\theta$-contraction $T$ satisfies

$$d(Tx,Ty) < d(x,y), \quad (x,y) \in X \times X,\, Tx \neq Ty.$$

On the other side, an example is provided in [**6**] showing that a $\theta$-contraction is not necessarily an ordinary contraction. Thus, the following theorem, which was given as a corollary by Jleli and Samet [**6**], is a proper generalization of Banach Contraction Principle.

**Theorem 1.1. ([6], Corollary 2.1)** *Let $(X,d)$ be a complete metric space and $T : X \to X$ be a $\theta$-contraction. Then $T$ has a unique fixed point in $X$.*

If we examine the proof of the above theorem, we can say that every $\theta$-contraction on a complete metric space is a PO.

## 2. MAIN RESULT

In the sequel $\mathbb{N}$ denotes the set of non-negative intiger numbers. The following lemmas will be useful later.

**Lemma 2.1.** *Let $(X,D)$ be a JS-metric space and $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is Cauchy iff*

$$\forall\, \varepsilon > 0,\, \exists\, N \in \mathbb{N} :\, D(x_n, x_{n+p}) < \varepsilon, \quad n \geq N,\, p \in \mathbb{N}.$$

*Proof.* It follows immediately from the axiom $(D_2)$ of a JS-metric. □

**Lemma 2.2.** *Let $(X,D)$ be a JS-metric space and $\{x_n\}$ be a sequence in $X$. If there exists a real sequence $\{a_n\}$ that converges to $0$ such that*

$$D(x_n, x_{n+p}) \leq a_n, \quad n \geq N,\, p \in \mathbb{N},$$

*then $\{x_n\}$ is Cauchy.*

*Proof.* It follows immediately from Lemma 2.1. □

Let us introduce the following class of operators.

**Definition 2.1.** Let $(X,D)$ be a JS-metric space, $\theta \in \Theta$, and $T : X \to X$ be such that

$$(x,y) \in X \times X,\, D(x,y) = 0 \Longrightarrow D(Tx,Ty) = 0 \tag{4}$$

and

$$(x, y) \in X \times X,\ D(Tx, Ty) = \infty \Longrightarrow D(x, y) = \infty. \tag{5}$$

Then $T$ is said to be a $\theta_D$-contraction, if there exists some $k \in (0, 1)$ such that

$$(x, y) \in X \times X,\ D(Tx, Ty) > 0,\ D(x, y) < \infty \Longrightarrow \theta(D(Tx, Ty)) \leq [\theta(D(x, y))]^k. \tag{6}$$

**Remark 2.1.** Observe that from (4), if for some $(x, y) \in X \times X$, we have $D(Tx, Ty) > 0$, then $D(x, y) > 0$. Moreover, from (5), if for some $(x, y) \in X \times X$, we have $D(x, y) < \infty$, then $D(Tx, Ty) < \infty$. Therefore,

$$(x, y) \in X \times X,\ D(Tx, Ty) > 0,\ D(x, y) < \infty \Longrightarrow (D(x, y), D(Tx, Ty)) \in (0, \infty) \times (0, \infty).$$

For every $x \in X$, set

$$\delta(D, T, x) = \sup\{D(T^i x, T^j x) : i, j \in \mathbb{N}\}.$$

Our main result is given by the following theorem.

**Theorem 2.1.** *Let $(X, D)$ be a complete JS-metric space and $T : X \to X$ be a $\theta_D$-contraction for some $\theta \in \Theta$ and $k \in (0, 1)$. If there exists $x_0 \in X$ such that $\delta(D, T, x_0) < \infty$ and $D(T^n x_0, T^n x_0) = 0$, for all $n \in \mathbb{N}$, then $T$ is a PPO.*

*Proof.* Let $x_0 \in X$ be such that

$$\delta(D, T, x_0) < \infty \tag{7}$$

and

$$D(T^n x_0, T^n x_0) = 0, \quad n \in \mathbb{N}. \tag{8}$$

First, assume that

$$D(T^N x_0, T^{N+q} x_0) = 0,$$

for some $(N, q) \in \mathbb{N} \times \mathbb{N} \backslash \{0\}$. From the axiom $(D_1)$, we obtain $T^N x_0 = T^q(T^N x_0)$, that is, $T^N x_0$ is a fixed point of $T^q$. We claim that $T^N x_0$ is the unique fixed point of $T^q$ in the set $\{T^n x_0 : n \in \mathbb{N}\}$. Indeed, suppose that $T^M x_0$, $M \in \mathbb{N}$, is another fixed point of $T^q$ $(M \neq N)$. In this case, we have

$$D(T^{q+N} x_0, T^{q+M} x_0) = D(T^N x_0, T^M x_0) > 0$$

and

$$D(T^{q+N-1} x_0, T^{q+M-1} x_0) \leq \delta(D, T, x_0) < \infty.$$

Using (6), we obtain

$$\theta(D(T^N x_0, T^M x_0)) = \theta(D(T^{q+N} x_0, T^{q+M} x_0)) \leq [\theta(D(T^{q+N-1} x_0, T^{q+M-1} x_0))]^k.$$

Again, we have

$$D(T^{q+N-1} x_0, T^{q+M-1} x_0) > 0$$

and

$$D(T^{q+N-2} x_0, T^{q+M-2} x_0) \leq \delta(D, T, x_0) < \infty.$$

Using (6), we obtain

$$\theta(D(T^{q+N-1} x_0, T^{q+M-1} x_0)) \leq [\theta(D(T^{q+N-2} x_0, T^{q+M-2} x_0))]^k.$$

Therefore,

$$\theta(D(T^N x_0, T^M x_0)) \leq [\theta(D(T^{q+N-2} x_0, T^{q+M-2} x_0))]^{k^2}.$$

Continuing this process, by induction we obtain the following contradiction

$$\theta(D(T^N x_0, T^M x_0)) \leq [\theta(D(T^N x_0, T^M x_0))]^{k^q} < \theta(D(T^N x_0, T^M x_0)),$$

which proves our claim. Since

$$T^q T^{N+1} x_0 = T^q T T^N x_0 = T(T^q T^N x_0) = T T^N x_0 = T^{N+1} x_0,$$

then $T^{N+1} x_0$ is also a fixed point of $T^q$. By uniqueness of the fixed points of $T^q$ in the set $\{T^n x_0 : n \in \mathbb{N}\}$, we obtain

$$T(T^N x_0) = T^{N+1} x_0 = T^N x_0,$$

that is, $T^N x_0$ is a fixed point of $T$.

Now, assume that

$$D(T^n x_0, T^{n+q} x_0) > 0, \quad (n, q) \in \mathbb{N} \times \mathbb{N} \backslash \{0\}.$$

Therefore, from (7)

$$0 < D(T^n x_0, T^{n+q} x_0) \leq \delta(D, T, x_0) < \infty, \quad (n, q) \in \mathbb{N} \times \mathbb{N} \backslash \{0\}. \tag{9}$$

Using (9) and (6), we obtain

$$\begin{array}{rcl} \theta(D(T^n x_0, T^{n+q} x_0)) & \leq & [\theta(D(T^{n-1} x_0, T^{n+q-1} x_0))]^k \\ & \leq & [\theta(D(T^{n-2} x_0, T^{n+q-2} x_0))]^{k^2} \\ & \vdots & \\ & \leq & [\theta(D(x_0, T^q x_0))]^{k^n} \\ & \leq & [\theta(\delta(D, T, x_0))]^{k^n}. \end{array}$$

Thus we have

$$1<\theta(D(T^n x_0, T^{n+q}x_0)) \leq [\theta(\delta(D,T,x_0))]^{k^n}, \quad (n,q)\in \mathbb{N}\times\mathbb{N}\backslash\{0\}. \tag{10}$$

Fix $q \in \mathbb{N}\backslash\{0\}$ and passing to the limit as $n\to\infty$, we get

$$\lim_{n\to\infty}\theta(D(T^n x_0, T^{n+q}x_0)) = 1,$$

which yields from the axiom $(\Theta_2)$,

$$\lim_{n\to\infty} D(T^n x_0, T^{n+q}x_0) = 0.$$

By the axiom $(\Theta_3)$, there exist $r\in(0,1)$ and $l\in(0,\infty]$ such that

$$\lim_{n\to\infty}\frac{\theta(D(T^n x_0, T^{n+q}x_0)) - 1}{[D(T^n x_0, T^{n+q}x_0)]^r} = l.$$

By the definition of the limit, we infer that there exist a constant $B>0$ and $n_0\in\mathbb{N}$ such that

$$\frac{\theta(D(T^n x_0, T^{n+q}x_0)) - 1}{[D(T^n x_0, T^{n+q}x_0)]^r} \geq \frac{1}{B}, \quad n\geq n_0,$$

which yields

$$n[D(T^n x_0, T^{n+q}x_0)]^r \leq Bn\left(\theta(D(T^n x_0, T^{n+q}x_0)) - 1\right), \quad n\geq n_0.$$

Using (10), we obtain

$$n[D(T^n x_0, T^{n+q}x_0)]^r \leq Bn\left([\theta(\delta(D,T,x_0))]^{k^n} - 1\right), \quad n\geq n_0.$$

Passing to the limit as $n\to\infty$, we obtain

$$\lim_{n\to\infty} n[D(T^n x_0, T^{n+q}x_0)]^r = 0,$$

which implies that

$$D(T^n x_0, T^{n+q}x_0) < \left(\frac{1}{n}\right)^{1/r}, \quad n\geq n_1,\ q\in\mathbb{N}\backslash\{0\},$$

for some $n_1\in\mathbb{N}$. On the other hand, since $D(T^n x_0, T^n x_0)=0$, for all $n\in\mathbb{N}$, the above inequality holds also for $q=0$. Thus we deduce that

$$D(T^n x_0, T^{n+q}x_0) < \left(\frac{1}{n}\right)^{1/r}, \quad n\geq n_1,\ q\in\mathbb{N},$$

which implies from Lemma 2.2 that $\{T^n x_0\}$ is a Cauchy sequence (since $0<r<1$).

Since $(X,D)$ is a complete JS-metric space, there is some $z\in X$ such that

$$\{T^n x_0\}\in C(D,X,z). \tag{11}$$

Consider now the set

$$\Lambda = \{n \in \mathbb{N} : D(T^n x_0, Tz) = 0\}.$$

We distinguish two cases.
Case 1. If $|\Lambda| = \infty$. In this case there is a sub-sequence $\{T^{\varphi(n)}x_0\}$ of $\{T^n x_0\}$ such that

$$D(T^{\varphi(n)}x_0, Tz) = 0, \quad n \in \mathbb{N}.$$

By the axiom $(D_1)$ of a JS-metric, we have

$$T^{\varphi(n)}x_0 = Tz, \quad n \in \mathbb{N}.$$

Therefore, we have

$$\{T^{\varphi(n)}\} \in C(D, X, z) \cap C(D, X, Tz),$$

which yields by the uniqueness of the limit

$$z = Tz.$$

Case 2. If $|\Lambda| < \infty$. In this case, for some $n_2 \in \mathbb{N}$,

$$D(T^n x_0, Tz) > 0, \quad n \geq n_2.$$

On the other hand, from (11), there is some $n_3 \in \mathbb{N}$ such that

$$D(T^{n-1}x_0, z) < \infty, \quad n \geq n_3.$$

Using the above facts and (6), we obtain

$$\theta(D(T^n x_0, Tz)) \leq [\theta(D(T^{n-1}x_0, z))]^k, \quad n \geq \max\{n_2, n_3\}.$$

Therefore,

$$\ln\theta(D(T^n x_0, Tz)) \leq k \ln\theta(D(T^{n-1}x_0, z)) \leq \ln\theta(D(T^{n-1}x_0, z)), \quad n \geq \max\{n_2, n_3\}.$$

Since $\theta$ is a nondecreasing function, we obtain

$$D(T^n x_0, Tz) \leq D(T^{n-1}x_0, z), \quad n \geq \max\{n_2, n_3\}.$$

Passing to the limit as $n \to \infty$ and using (11), we obtain

$$\{T^n x_0\} \in C(D, X, z) \cap C(D, X, Tz),$$

which yields

$$z = Tz.$$

Thus, in all cases, we proved that the Picard sequence $\{T^n x_0\}$ converges to a fixed point of $T$. $\square$

## 3. AN EXAMPLE

In this section, we provide an example showing the importance of Theorem 2.1.

**Example 3.1.** Consider the set $X = \mathbb{N}$ and the mapping $D : X \times X \to [0, \infty]$ defined by

$$D(x, y) = \begin{cases} 0 & \text{if} \quad x = y, \\ x + y & \text{if} \quad x \neq y. \end{cases}$$

It is easy to observe that $D$ is a metric on $X$, and so it is a JS-metric on $X$.

Let us prove that $(X, D)$ is a complete JS-metric space. Let $\{x_n\} \subset X$ be a Cauchy sequence with respect to the JS-metric $D$. Then

$$\lim_{n,m\to\infty} D(x_n, x_m) = 0,$$

which yields

$$D(x_n, x_m) < \frac{1}{2}, \quad n \geq m \geq N,$$

for some $N \in \mathbb{N}$. On the other hand, observe that $D(X \times X)$ is a subset of $X$. Taking into account this fact, and using the above inequality, we obtain

$$D(x_n, x_m) = 0, \quad n \geq m \geq N.$$

Thus we have

$$x_n = x_N, \quad n \geq N,$$

which implies that

$$\{x_n\} \in C(D, X, x_N).$$

Therefore, $(X, D)$ is a complete JS-metric space.

Let $T : X \to X$ be defined by

$$Tx = \begin{cases} 0 & \text{if} \quad x \in \{0, 1\}, \\ x - 1 & \text{if} \quad x \geq 2. \end{cases}$$

Observe that $T$ is not a $k$-contraction (in the sense of Jleli and Samet [**7**]), for any $k \in (0, 1)$, that is, there is no $k \in (0, 1)$ such that

$$D(Tx, Ty) \leq kD(x, y), \quad (x, y) \in X \times X.$$

Indeed, suppose that the above inequality is satisfied for some $k \in (0, 1)$, take $(x, y) = (y + 1, y)$, $y \geq 2$, we obtain

$$D(Tx, Ty) = D(y, y - 1) = 2y - 1 \leq kD(x, y) = (2y + 1)k,$$

which yields

$$k \geq \frac{2y - 1}{2y + 1}, \quad y \in \mathbb{N}, \, y \geq 2.$$

Passing to the limit as $y \to \infty$, we get

$$k \geq 1,$$

which is a contradiction. Therefore, using Theorem 3.3 in [**7**], we can not decide that whether $T$ is pseudo Picard operator. Similarly, we can check also that Theorem 4.3 in [**7**] cannot be applied in this case.

Now, we claim that $T$ is a $\theta_D$-contraction with $\theta(t) = e^{\sqrt{te^t}}$ and $k = e^{-\frac{1}{2}}$. To see this we have to show that

$$\theta(D(Tx, Ty)) \leq [\theta(D(x, y))]^k,$$

for all $(x, y) \in X \times X$ with $D(Tx, Ty) > 0$. For this, it is sufficient to show that

$$\frac{D(Tx, Ty)}{D(x, y)} e^{D(Tx,Ty)-D(x,y)} \leq e^{-1}, \tag{12}$$

for all $(x, y) \in X \times X$ with $D(Tx, Ty) > 0$.

First, observe that

$$D(Tx, Ty) > 0 \Longleftrightarrow \text{the set } \{x, y\} \cap \{0, 1\} \text{ is singleton or empty, } x \neq y.$$

Since (12) is symmetric with respect to $x$ and $y$, we may assume $x > y$ in the following cases.

Case 1. If $\{x, y\} \cap \{0, 1\}$ is a singleton.
Then $D(Tx, Ty) = x - 1$ and $D(x, y) = x + y$, and so we have

$$\begin{aligned}
\frac{D(Tx, Ty)}{D(x, y)} e^{D(Tx,Ty)-D(x,y)} &= \left(\frac{x-1}{x+y}\right) e^{x-1-(x+y)} \\
&= \left(\frac{x-1}{x+y}\right) e^{-(y+1)} \\
&\leq \left(\frac{x-1}{x}\right) e^{-1} \\
&\leq e^{-1}.
\end{aligned}$$

Case 2. If $\{x, y\} \cap \{0, 1\} = \emptyset$.
Then $D(Tx, Ty) = x + y - 2$, $D(x, y) = x + y$, and so we have

$$\frac{D(Tx, Ty)}{D(x, y)} e^{D(Tx,Ty)-D(x,y)} = \left(\frac{x+y-2}{x+y}\right) e^{-2} \leq e^{-2} \leq e^{-1}.$$

Thus our claim is proved.

Finally, for $x_0 = 2$, we have

$$\delta(D, T, x_0) = 3 < \infty.$$

Therefore by Theorem 2.1, $T$ is a pseudo Picard operator.

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