Almost Picard operators

Cite as: AIP Conference Proceedings **2183**, 060003 (2019); https://doi.org/10.1063/1.5136158
Published Online: 06 December 2019

Ishak Altun and Hatice Aslan Hancer

Almost Picard Operators

Ishak Altun[a)] and Hatice Aslan Hancer[b)]

Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey

[a)]ishakaltun@yahoo.com
[b)]haticeaslanhancer@gmail.com

Abstract. The concept of Picard operator is one of the most important concept of fixed point theory. As known, a self mapping T of a metric space X is called Picard operator (PO) if it has unique fixed point and every Picard iteration sequence converges to this fixed point. There are some weaker forms of PO in the literature as weakly Picard operator (WPO) and pseudo Picard operator (PPO). In this study, we present a new kind of PO as almost Picard operator (APO) and we show the differences from the others. Then we show that every continuous P-contractive self mapping of a compact metric space is APO. Also we present some open problems.
Keywords: Fixed point, Picard operator, complete metric space
PACS: 02.40.Pc, 02.30.Sa.

INTRODUCTION

Let (X, d) be a metric space and $T : X \to X$ be a mapping. It is well known that for $x_0 \in X$ the sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$ for $n \geq 1$ is called Picard iteration sequence with the initial point x_0. Then T is said to be a Picard operator (PO) if it has a unique fixed point and every Picard iteration in X converges to the fixed point. T is said to be a weakly Picard operator (WPO) if it has at least one fixed point and every Picard iteration in X converges to the one of the fixed point.

Example 1 *Let $X = \mathbb{R}$ be endowed with the usual metric and $T : X \to X$ defined by $Tx = 1$ for $x \geq 0$ and $Tx = -1$ for $x < 0$. Then T is WPO but not PO.*

Almost Picard Operator

Here we introduce a new concept for self mapping T on a metric space (X, d) as follows:

Definition 1 *Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be almost Picard operator (APO) if it has a unique fixed point and every Picard iteration in X has a subsequence which converges to the fixed point.*

It is clear that every Picard operator is an almost Picard operator. The following example shows that the converse may not be true:

Example 2 *Let $X = [-1, \frac{3}{2}]$ be endowed with the usual metric. Consider a set $A = \{t_n = (-1)^n + \frac{1}{n} : n \in \mathbb{N}\} \subset X$ and define a mapping $T : X \to X$ as*

$$Tx = \begin{cases} t_{n+1} & , \quad x = t_n \in A \\ 1 & , \quad otherwise \end{cases} .$$

Third International Conference of Mathematical Sciences (ICMS 2019)
AIP Conf. Proc. 2183, 060003-1–060003-4; https://doi.org/10.1063/1.5136158
Published by AIP Publishing. 978-0-7354-1930-8/$30.00

It is clear that T has a unique fixed point which is 1. Now let $x_0 \in X$ be an arbitrary point. If $x_0 \notin A$, then $Tx_0 = 1$ and so we have $x_n = 1$ for all $n \geq 1$. That is $\lim x_n = 1$. If $x_0 \in A$, then there exists $n_0 \in \mathbb{N}$ such that $x_0 = t_{n_0}$ and so we have $x_n = t_{n_0+n}$ for all $n \in \mathbb{N}$. In this case the sequence $\{x_n\}$ does not converge to 1, however, it has a subsequence which converges to 1. Therefore T is an almost Picard operator, but not Picard operator.

In this note we show that every continuous P-contractive self mapping on compact metric space is almost Picard operator. For the sake of completeness we recall the following: Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be

(C_1) contraction if for all $x, y \in X$ there exists $L \in [0, 1)$ such that

$$d(Tx, Ty) \leq Ld(x, y),$$

(C_2) contractive if for all $x, y \in X$ with $x \neq y$

$$d(Tx, Ty) < d(x, y),$$

(C_3) nonexpansive if for all $x, y \in X$

$$d(Tx, Ty) \leq d(x, y).$$

(P_1) P-contraction if for all $x, y \in X$ there exists $L \in [0, 1)$ such that

$$d(Tx, Ty) \leq L\{d(x, y) + |d(x, Tx) - d(y, Ty)|\},$$

(P_2) P-contractive if for all $x, y \in X$ with $x \neq y$

$$d(Tx, Ty) < d(x, y) + |d(x, Tx) - d(y, Ty)|,$$

(P_3) P-nonexpansive if for all $x, y \in X$

$$d(Tx, Ty) \leq d(x, y) + |d(x, Tx) - d(y, Ty)|.$$

Remark 1 *It is well known that $(C_k) \Rightarrow (C_{k+1})$ and it is easy to see that $(P_k) \Rightarrow (P_{k+1})$ for $k \in \{1, 2\}$ but the converse may not be true. Also it is easy to see that $(C_k) \Rightarrow (P_k)$ for $k \in \{1, 2, 3\}$.*

We can find some examples that shows the converse may not be true in general.

Example 3 *Let $X = [0, 1]$ with the usual metric d and $T : X \to X$, defined by*

$$Tx = \begin{cases} \frac{1}{2} & , \quad x = 0 \\ \frac{x}{2} & , \quad x \neq 0 \end{cases}.$$

Since T is not continuous, then T is not nonexpansive. Now without lost of generality assume $y < x$. Then it is clear that

$$d(Tx, Ty) = \frac{1}{2}d(x, y)$$

for $y > 0$. Also, we have

$$d(T0, Tx) = \left|\frac{1-x}{2}\right| < x + \left|\frac{1-x}{2}\right| = d(0, x) + |d(0, T0) - d(x, Tx)|.$$

Therefore T is P-contractive and so it is P-nonexpansive. Thus $(P_3) \nRightarrow (C_3)$.

Banach fixed point theorem stays that every contraction self mapping of a complete metric space is a Picard operator. In the parallel manner, Popescu [6] proved the following (see also [3]):

Theorem 1 *Let (X, d) be a complete metric space and $T : X \to X$ be a P-contraction mapping. Then T has a unique fixed point and for every $x \in X$ the sequence $\{T^n x\}$ converges to the fixed point of T.*

Therefore, a self mapping satisfying (C_1) or (P_1) of a complete metric space is a Picard operator.

On the other hand, by Edelstein fixed point theorem we know that every contractive self mapping of a compact metric space is a Picard operator. That is, a self mapping satisfying (C_2) of a compact metric space is a Picard operator. In this direction, Altun et al [1] proved that every continuous P-contractive self mapping of a compact metric space has a unique fixed point. In Example 3, although the space (X, d) is compact and the mapping T is P-contractive, it has no fixed point. This shows that the continuity of the mapping T in the result of Altun et al [1] can not be removed. In [1], it is presented an open problem (Problem 2.15 in [1]) that whether continuous P-contractive self mapping of a compact metric space is a Picard operator.

The following theorem partially answered this problem by shown the mentioned mapping is almost Picard operator.

Theorem 2 *Let (X, d) be a compact metric space and $T : X \to X$ be a continuous P-contractive mapping. Then T has a unique fixed point in X and every Picard iteration has a subsequence which converges to the fixed point.*

Proof 1 *By Theorem 2.14 in [1], we know that T has a unique fixed point, say $z \in X$. Now let $x_0 \in X$ and $\{x_n\}$ be the Picard iteration associated x_0. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = z$, then we have $x_n = z$ for all $n \geq n_0$ and so $\lim x_n = z$. Suppose $x_n \neq z$ for all $n \in \mathbb{N}$ and so we have $x_n \neq x_{n+1} = T x_n$ for all $n \in \mathbb{N}$. Hence from (P_2) we have*

$$\begin{aligned} d(x_n, x_{n+1}) &= d(T x_{n-1}, T x_n) \\ &< d(x_{n-1}, x_n) + |d(x_{n-1}, T x_{n-1}) - d(x_n, T x_n)| \\ &= d(x_{n-1}, x_n) + |d(x_{n-1}, x_n) - d(x_n, x_{n+1})| \,. \end{aligned}$$

Now if there exists $m \in \mathbb{N}$ such that $d(x_{m-1}, x_m) \leq d(x_m, x_{m+1})$, then from (1), we have

$$\begin{aligned} d(x_m, x_{m+1}) &< d(x_{m-1}, x_m) + |d(x_{m-1}, x_m) - d(x_m, x_{m+1})| \\ &= d(x_{m-1}, x_m) - d(x_{m-1}, x_m) + d(x_m, x_{m+1}) \\ &= d(x_m, x_{m+1}), \end{aligned}$$

which is a contradiction, so $d(x_{n-1}, x_n) > d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Therefore the sequence of real numbers defined by $a_n = d(x_n, x_{n+1})$ is a decreasing sequence which is bounded below. Hence there exists $a \geq 0$ such that $\lim a_n = a$. Suppose $a > 0$. Since X is compact there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to a point $w \in X$. By the continuity of T we have

$$0 < a = \lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = \lim_{k\to\infty} d(x_{n_k}, T x_{n_k}) = d(w, Tw),$$

which shows that $w \neq Tw$. Now from (P_2) we have

$$d(T^2 w, Tw) < d(Tw, w) + \left| d(T^2 w, Tw) - d(Tw, w) \right|$$

and so

$$d(T^2 w, Tw) < d(Tw, w).$$

Therefore we have

$$\begin{aligned} 0 &< a = \lim_{k\to\infty} a_{n_k+1} \\ &= \lim_{k\to\infty} d(x_{n_k+1}, x_{n_k+2}) \\ &= \lim_{k\to\infty} d(T x_{n_k}, T^2 x_{n_k}) \\ &= d(T^2 w, Tw) < d(Tw, w) = a, \end{aligned}$$

which is a contradiction. This shows that $a = 0$ and hence $d(w, Tw) = 0$. That is, $z = w$ and so $\lim_{k\to\infty} x_{n_k} = z$. This shows that T is an almost Picard operator.

Problem 1 *Does the mapping (mentioned in Theorem 2) Picard operator?*

P-Nonexpansive Mappings

For nonexpansive mappings, Schauder proved that every nonexpansive self mapping of a nonempty, closed and convex subset C of a Banach space X satisfying $T(C)$ is a subset of a compact set of C, has a fixed point. Here we can prove the following:

Theorem 3 *Let C be a nonempty, closed and convex subset of a Banach space X, $T : C \to C$ be a continuous P-nonexpansive mapping. If $T(C)$ is a subset of a compact set of C, then T has a fixed point.*

Proof 2 *Let $x_0 \in C$ and define*

$$T_n x = \left(1 - \frac{1}{n}\right) T x + \frac{1}{n} x_0$$

for $n \in \{2, 3, \cdots\}$. Since C is convex and $x_0 \in C$, then $T_n : C \to C$ for all $n \in \{2, 3, \cdots\}$. Also for all $x, y \in C$, we have

$$\begin{aligned} \|T_n x - T_n y\| &= \left(1 - \frac{1}{n}\right) \|T x - T y\| \\ &\leq \left(1 - \frac{1}{n}\right) \{\|x - y\| + |\|x - T x\| - \|y - T y\||\} \end{aligned}$$

for all $n \in \{2, 3, \cdots\}$. That is, every T_n is a P-contraction. Therefore by Theorem 1 each T_n has a unique fixed point $z_n \in C$, that is,

$$z_n = T_n z_n = \left(1 - \frac{1}{n}\right) T z_n + \frac{1}{n} x_0$$

for all $n \in \{2, 3, \cdots\}$. On the other hand, since $T(C)$ lies in a compact subset of C, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $T z_{n_k} \to z \in C$ as $k \to \infty$. Hence

$$z_{n_k} = \left(1 - \frac{1}{n_k}\right) T z_{n_k} + \frac{1}{n_k} x_0 \to z \quad as \quad k \to \infty.$$

By the continuity of T, we have $T z_{n_k} \to T z$ as $k \to \infty$ and therefore $z = T z$.

In 1965, Browder [2], Göhde [4] and Kirk [5] independently proved the following fixed point theorem for nonexpansive mapping:

Theorem 4 *Let C be a closed, convex and bounded subset of a uniformly convex Banach space and T be nonexpansive self mapping of C. Then T has a fixed point in C.*

Problem 2 *Does Theorem 4 valid for continuous P-nonexpansive mapping?*

REFERENCES

[1] I. Altun, G. Durmaz and M. Olgun, *P*-contractive mappings on metric spaces, *Journal of Nonlinear Functional Analysis*, 1-7, Article ID 43 (2018).

[2] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Natl. Acad. Sci. USA* 54, 1041-1044 (1965).

[3] A. Fulga and A. M. Proca, Fixed points for φ_E -Geraghty contractions, *J. Nonlinear Sci. Appl.* 10 (9), 5125-5131 (2017).

[4] D. Göhde, Zum prinzip der kontraktiven abbildung, *Math. Nachr.* 30, 251-258 (1965).

[5] W. A. Kirk, A fixed point theorem for mappings which dont increase distances, *Amer. Math. Mon.* 72, 1004-1006 (1965).

[6] O. Popescu, A new type of contractive mappings in complete metric spaces, submitted.