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Parametric generalization of Baskakov operators

Ali Aral[1] and Hasan Erbay[2,*]

[1] *Mathematics Department, Arts and Science Faculty, Kırıkkale University, Yahşihan, 71450 Kırıkkale, Turkey*
[2] *Computer Engineering Department, Engineering Faculty, Kırıkkale University, Yahşihan, 71450 Kırıkkale, Turkey*

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Abstract. Herein we propose a non-negative real parametric generalization of Baskakov operators and call them α-Baskakov operators. We show that α-Baskakov operators can be expressed in terms of divided differences. Then, we obtain the nth order derivative of α-Baskakov operators in order to obtain its new representation as powers of independent variable x.
In addition, we obtain Korovkins-type approximation properties of α-Baskakov operators. Moreover, by using the modulus of continuity, we obtain the rate of convergence. Numerical results presented show that depending on the value of the parameter α, an approximation to a function improves compared to classical Baskakov operators.

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1. Introduction

Polynomial approximation theory deals with the approximation of real-valued continuous functions by algebraic polynomials. In 1957, Baskakov [4] introduced a sequence of positive linear operators, called Baskakov operators, on the unbounded interval $[0,\infty)$ for suitable functions defined on the interval $[0,\infty)$. Later, Baskakov operators are studied by many researches. In 1984, Pethe [11] studied approximation properties of Baskakov operators. In 1994, Gupta [8] studied the rate of convergence of Baskakov type operators. In 1998, Mihesan [10] constructed the generalization of Baskakov operators and the convergence rate of the generalization obtained in [12]. Moreover, the preservation properties of Baskakov-Kantorovich operators are considered in [13].

On the other hand, q-analogues to Baskakov operators were introduced by Aral and Gupta in [2]. The same authors introduced another q-analogues to Baskakov operators and studied the convergence rate in weighted norm and some shape preserving properties in [3].

In this paper, motivated by the α-Bernstein operator by Chen and et al. [5], we propose a non-negative real parametric generalization of Baskakov operators and call

*Corresponding author. *Email addresses:* `aliaral73@yahoo.com` (A. Aral), `hasan_erbay@yahoo.com` (H. Erbay)

them α-Baskakov operators. When $\alpha = 1$, the operators reduce to classical Baskakov operators. We also represent higher order derivatives of α-Baskakov operators in terms of forward divided differences. Then, we express α-Baskakov operators in terms of forward divided differences. Later, we study convergence and approximation properties of α-Baskakov operators in weighted spaces including Voronovskaya type formulation. α-Baskakov operators converge uniformly in the polynomial weighted space for any $\alpha \in [0, 1]$. Even though convergence is independent of the parameter α, the approximation errors depend on α, the larger the value of α, the smaller the upper bound for the approximation error. We present some numerical results that correct the theoretical results.

The rest of the paper is organised as follows. In Section 2, we recall classical Baskakov operators. Later, we define α-Baskakov operators, establish their moments and represent them in terms of divided differences. Section 3 presents convergence properties of α-Baskakov operators.

2. α-Baskakov operator

Recall that for every $f \in C_B[0, \infty)$, classical Baskakov operators are defined as

$$\mathbf{B}_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) P_{n,k}(x), \tag{1}$$

where $n \geq 1$, $x \in [0, \infty)$ and

$$P_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{k+n}}. \tag{2}$$

Now, for every $f \in C_B[0, \infty)$, we define the parametric generalization of Baskakov operators as

$$\mathbf{L}_{n,\alpha}(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \mathbf{P}_{n,k}^{(\alpha)}(x), \tag{3}$$

where $n \geq 1$, $x \in [0, \infty)$ and

$$\begin{aligned} \mathbf{P}_{n,k}^{(\alpha)}(x) = &\frac{x^{k-1}}{(1+x)^{n+k-1}} \left\{ \frac{\alpha x}{1+x} \binom{n+k-1}{k} \right. \\ &\left. - (1-\alpha)(1+x) \binom{n+k-3}{k-2} + (1-\alpha)x \binom{n+k-1}{k} \right\}, \end{aligned} \tag{4}$$

with

$$\binom{n-3}{-2} = \binom{n-2}{-1} = 0.$$

We call these operators α-Baskakov operators.

Observe that for $\alpha = 1$, we have

$$\begin{aligned}\mathbf{L}_{n,1}(f;x) &= \sum_{k=0}^{\infty} f(\frac{k}{n})\mathbf{P}_{n,k}^{(1)}(x) \\ &= \sum_{k=0}^{\infty} f(\frac{k}{n})\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} \\ &= \mathbf{B}_n(f;x).\end{aligned}$$

So, α-Baskakov operators reduce to the classical Baskakov operators.

Simple calculation yields

$$\frac{k}{n-1}\binom{n+k-2}{k} = \binom{n+k-2}{k-1} \tag{5}$$

and

$$(1+\frac{k}{n-1})\binom{n+k-2}{k} = \binom{n+k-1}{k}. \tag{6}$$

Theorem 1. *The α-Baskakov operator for $f(x)$ can be expressed as*

$$\begin{aligned}\mathbf{L}_{n,\alpha}(f;x) =&(1-\alpha)\sum_{k=0}^{\infty} g_k \binom{n+k-2}{k}\frac{x^k}{(1+x)^{n+k-1}} \\ &+ \alpha\sum_{k=0}^{\infty} f(\frac{k}{n})\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}},\end{aligned} \tag{7}$$

where

$$g_k = f(\frac{k}{n})(1+\frac{k}{n-1}) - f(\frac{k+1}{n})\frac{k}{n-1}. \tag{8}$$

Proof. From the definition of the α-Baskakov operator in equations (3) and (4), one can write

$$\mathbf{L}_{n,\alpha}(f;x) = (1-\alpha)(k_1-k_2) + \alpha\sum_{k=0}^{\infty} f(\frac{k}{n})\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}},$$

where

$$k_1 = \sum_{k=0}^{\infty} f(\frac{k}{n})\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k-1}}.$$

Using equation (6) and adjusting the summation limits yield

$$k_1 = \sum_{k=0}^{\infty} f(\frac{k}{n})(1+\frac{k}{n-1})\binom{n+k-2}{k}\frac{x^k}{(1+x)^{n+k-1}}. \tag{9}$$

And similarly,

$$\begin{aligned} k_2 &= \sum_{k=2}^{\infty} f(\frac{k}{n})\binom{n+k-3}{k-2}\frac{x^{k-1}}{(1+x)^{n+k-2}} \\ &= \sum_{k=1}^{\infty} f(\frac{k+1}{n})\binom{n+k-2}{k-1}\frac{x^{k}}{(1+x)^{n+k-1}}. \end{aligned}$$

Again, using equation (5) and adjusting the summation limits yield

$$k_2 = \sum_{k=0}^{\infty} f(\frac{k+1}{n})\frac{k}{n-1}\binom{n+k-2}{k}\frac{x^{k}}{(1+x)^{n+k-1}}. \tag{10}$$

Subtracting (10) from (9), we obtain

$$\begin{aligned} k_1 - k_2 &= \sum_{k=0}^{\infty} \left\{ f(\frac{k}{n})(1+\frac{k}{n-1}) - f(\frac{k+1}{n})\frac{k}{n-1} \right\} \binom{n+k-2}{k}\frac{x^{k}}{(1+x)^{n+k-1}} \\ &= \sum_{k=0}^{\infty} g_k \binom{n+k-2}{k}\frac{x^{k}}{(1+x)^{n+k-1}}, \end{aligned}$$

which completes the proof. □

Lemma 1. *For $n \in \mathbb{N}$, the α-Baskakov operator has the following identities:*

1. $\mathbf{L}_{n,\alpha}(1;x) = 1$,
2. $\mathbf{L}_{n,\alpha}(t;x) = x + \frac{2}{n}(\alpha-1)x$,
3. $\mathbf{L}_{n,\alpha}(t^2;x) = x^2 + \frac{4\alpha-3}{n}x^2 + \frac{x}{n^2}(n+4\alpha-4)$.

Proof. The proof of the parts follows from a straightforward (yet tedious) calculation, so we just prove part 1 and skip parts 2 and 3.

In part 1 $f(x) = 1$; then it follows from equation (8) that $g_k = 1$. Thus, we have

$$\begin{aligned} \mathbf{L}_{n,\alpha}(1;x) &= (1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{x^{k}}{(1+x)^{n+k-1}} + \alpha\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^{k}}{(1+x)^{n+k}} \\ &= (1-\alpha)\mathbf{B}_{n-1}(1;x) + \alpha\mathbf{B}_n(1;x). \end{aligned}$$

On the other hand, since $\mathbf{B}_{n-1}(1;x) = \mathbf{B}_n(1;x) = 1$, we obtain the desired result. □

From Lemma 1, we obtain the following immediate result.

Remark 1. *The classical Baskakov operator reproduces a linear polynomial, that is,*

$$\mathbf{B}_n(at+b;x) = ax+b, \quad a \text{ and } b \text{ constants.}$$

However, α-Baskakov operator does not have this property.

To represent the rth derivative of α-Baskakov operators in terms of forward divided differences, we need the following auxilary results.

Lemma 2. *The higher order forward difference of g_k in equation (8) can be expressed in the form*

$$\Delta^r g_k = (1+\frac{k}{n-1})\Delta^r f(\frac{k}{n}) - \frac{k+r}{n-1}\Delta^r f(\frac{k+1}{n}). \tag{11}$$

Proof. The proof is by induction on r. For the inductive step, let us assume that

$$\Delta^{r-1} g_k = (1+\frac{k}{n-1})\Delta^{r-1} f(\frac{k}{n}) - \frac{k+r-1}{n-1}\Delta^{r-1} f(\frac{k+1}{n}). \tag{12}$$

Then,

$$\begin{aligned}\Delta^r g_k &= \Delta\left(\Delta^{r-1} g_k\right)\\ &= \Delta\left((1+\frac{k}{n-1})\Delta^{r-1} f(\frac{k}{n}) - \frac{k+r-1}{n-1}\Delta^{r-1} f(\frac{k+1}{n})\right).\end{aligned} \tag{13}$$

Thus, applying the difference formula

$$\Delta\left(u_k v_k\right) = v_k \Delta\left(u_k\right) + u_{k+1}\Delta\left(v_k\right)$$

to each term in the expression in (13) yields

$$\begin{aligned}\Delta^r g_k &= \frac{1}{n-1}\Delta^{r-1} f(\frac{k}{n}) + (1+\frac{k+1}{n-1})\Delta^r f(\frac{k}{n})\\ &\quad -\{\frac{1}{n-1}\Delta^{r-1} f(\frac{k+1}{n}) + \frac{k+r}{n-1}\Delta^r f(\frac{k+1}{n})\}\\ &= (1+\frac{k}{n-1})\Delta^r f(\frac{k}{n}) - \frac{k+r}{n-1}\Delta^r f(\frac{k+1}{n}),\end{aligned}$$

which completes the proof. ☐

Theorem 2. *The rth order derivative of α-Baskakov operators can be expressed in terms of higher order forward divided differences as*

$$\begin{aligned}\mathbf{L}_{n,\alpha}^{(r)}(f;x) &= (1-\alpha)\frac{(n+r-2)!}{(n-2)!}\sum_{k=0}^{\infty}\Delta^r g_k P_{n-1+r,k}(x)\\ &\quad +\alpha\frac{(n+r-1)!}{(n-1)!}\sum_{k=0}^{\infty}\Delta^r f(\frac{k}{n})P_{n+r,k}(x).\end{aligned}$$

Proof. Let us assume that, by using equation (7),

$$\mathbf{L}_{n,\alpha}(f;x) = (1-\alpha)T_1 + \alpha T_2,$$

where

$$T_1 = \sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{x^k}{(1+x)^{n+k-1}},$$

and

$$T_2 = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}.$$

Then, the rth derivative of $\mathbf{L}_{n,\alpha}(f;x)$ becomes

$$\mathbf{L}_{n,\alpha}^{(r)}(f;x) = (1-\alpha)T_1^{(r)} + \alpha T_2^{(r)}.$$

We prove that

$$T_1^{(r)} = \frac{(n+r-2)!}{(n-2)!} \sum_{k=0}^{\infty} \Delta^r g_k P_{n-1+r,k}(x). \tag{14}$$

The proof is by induction on r. When $r=1$, then the derivative of T_1 becomes

$$\begin{aligned} T_1^{'} &= \sum_{k=1}^{\infty} g_k k \binom{n+k-2}{k} \frac{x^{k-1}}{(1+x)^{n+k-1}} \\ &\quad - \sum_{k=0}^{\infty} g_k (n+k-1) \binom{n+k-2}{k} \frac{x^k}{(1+x)^{n+k}} \\ &= \sum_{k=0}^{\infty} g_{k+1}(k+1) \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \\ &\quad - \sum_{k=0}^{\infty} g_k (n+k-1) \binom{n+k-2}{k} \frac{x^k}{(1+x)^{n+k}}. \end{aligned}$$

Since

$$(k+1)\binom{n+k-1}{k+1} = (n-1)\binom{n+k-1}{k}$$

and

$$(n+k-1)\binom{n+k-2}{k} = (n-1)\binom{n+k-1}{k},$$

we have

$$\begin{aligned} T_1^{'} &= (n-1)\sum_{k=0}^{\infty} \left[g_{k+1} - g_k\right] \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \\ &= (n-1)\sum_{k=0}^{\infty} \Delta g_k P_{n,k}(x). \end{aligned}$$

For the inductive step, assume that equation (14) holds for some $r \in \mathbb{N}$. We now prove that the equation holds with r replaced by $r+1$, which means taking the derivative of (14) . Thus,

$$
\begin{aligned}
T_1^{(r+1)} &= \frac{(n+r-2)!}{(n-2)!}\sum_{k=0}^{\infty}\Delta^r g_k P'_{n-1+r,k}(x)\\
&= \frac{(n+r-2)!}{(n-2)!}\sum_{k=1}^{\infty}\Delta^r g_k k\binom{n+r+k-2}{k}\frac{x^{k-1}}{(1+x)^{n+r+k-1}}\\
&\quad -\frac{(n+r-2)!}{(n-2)!}\sum_{k=0}^{\infty}\Delta^r g_k(n+r+k-1)\binom{n+r+k-2}{k}\frac{x^k}{(1+x)^{n+r+k}}\\
&= \frac{(n+r-2)!}{(n-2)!}\sum_{k=0}^{\infty}\Delta^r g_{k+1}(k+1)\binom{n+r+k-1}{k+1}\frac{x^k}{(1+x)^{n+r+k}}\\
&\quad -\frac{(n+r-2)!}{(n-2)!}\sum_{k=0}^{\infty}\Delta^r g_k(n+r+k-1)\binom{n+r+k-2}{k}\frac{x^k}{(1+x)^{n+r+k}}.
\end{aligned}
$$

Since

$$
(k+1)\binom{n+r+k-1}{k+1} = (n+r-1)\binom{n+r+k-1}{k}
$$

and

$$
(n+r+k-1)\binom{n+r+k-2}{k} = (n+r-1)\binom{n+r+k-1}{k},
$$

we obtain

$$
\begin{aligned}
T_1^{(r+1)} &= \frac{(n+r-1)!}{(n-2)!}\sum_{k=0}^{\infty}\left[\Delta^r g_{k+1}-\Delta^r g_k\right]\binom{n+r+k-1}{k}\frac{x^k}{(1+x)^{n+r+k}}\\
&= \frac{(n+r-1)!}{(n-2)!}\sum_{k=0}^{\infty}\Delta^{r+1} g_k P_{n+r,k}(x).
\end{aligned}
$$

Similarly, we obtain

$$
T_2^{(r)} = \frac{(n+r-1)!}{(n-1)!}\sum_{k=0}^{\infty}\Delta^r f(\frac{k}{n})P_{n+r,k}(x).
$$

□

Theorem 3. *The α-Baskakov operator can be expressed in terms of forward divided differences and powers of x as*

$$
\begin{aligned}
\mathbf{L}_{n,\alpha}(f;x) &= (1-\alpha)\sum_{r=0}^{\infty}\frac{(n+r-2)!}{(n-2)!}\left[\Delta^r f(0)-\frac{r}{n-1}\Delta^r f(\frac{1}{n})\right]\frac{x^r}{r!}\\
&\quad +\alpha\sum_{r=0}^{\infty}\frac{(n+r-1)!}{(n-1)!}\Delta^r f(0)\frac{x^r}{r!}.
\end{aligned}
$$

Proof. Taylor expansion of the α-Baskakov operator along with Theorem 2 yields

$$\begin{aligned}
\mathbf{L}_{n,\alpha}(f;x) &= \sum_{r=0}^{\infty} \mathbf{L}_{n,\alpha}^{(r)}(f;x)\Big|_{x=0} \frac{x^r}{r!} \\
&= (1-\alpha)\sum_{r=0}^{\infty} \frac{(n+r-2)!}{(n-2)!}\left[\Delta^r f(0) - \frac{r}{n-1}\Delta^r f(\frac{1}{n})\right] P_{n-1+r,0}(0)\frac{x^r}{r!} \\
&\quad +\alpha\sum_{r=0}^{\infty} \frac{(n+r-1)!}{(n-1)!}\Delta^r f(0) P_{n+r,0}(0)\frac{x^r}{r!}
\end{aligned}$$

Since $P_{n-1+r,0}(0) = P_{n+r,0}(0) = 1$,

$$\begin{aligned}
&= (1-\alpha)\sum_{r=0}^{\infty} \frac{(n+r-2)!}{(n-2)!}\left[\Delta^r f(0) - \frac{r}{n-1}\Delta^r f(\frac{1}{n})\right] \frac{x^r}{r!} \\
&\quad +\alpha\sum_{r=0}^{\infty} \frac{(n+r-1)!}{(n-1)!}\Delta^r f(0) \frac{x^r}{r!}.
\end{aligned}$$

This completes the proof. $\square$

Using the following forward divided difference formula, one can write the α-Baskakov operator as in the following corollary:

$$n^r \frac{\Delta^r f(0)}{r!} = f\left[0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{r}{n}\right]. \tag{15}$$

Corollary 1. *The α-Baskakov operator can be expressed as*

$$\begin{aligned}
\mathbf{L}_{n,\alpha}(f;x) = (1-\alpha)&\sum_{r=0}^{\infty} \frac{(n+r-2)!}{n^r(n-2)!}\left(f\left[0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{r}{n}\right] \right.\\
&\left. - \frac{r}{n-1} f\left[\frac{1}{n}, \frac{2}{n}, \cdots, \frac{r}{n}, \frac{r+1}{n}\right]\right) x^r \\
&+\alpha\sum_{r=0}^{\infty} \frac{(n+r-1)!}{n^r(n-1)!} f\left[0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{r}{n}\right] x^r.
\end{aligned}$$

3. Convergence properties of α-Baskakov operators

Throughout this section, we argue that α-Baskakov operators can be used to approximate functions defined on the unbounded infinite interval $[0,\infty)$. Recall that an immediate analog of the Bohman-Korovkin theorem does not hold in the unbounded interval, so some restrictions are needed. Now, we present these restrictions and notations.

Let $B_2[0,\infty)$ be the space of all functions f defined on the unbounded interval $[0,\infty)$ satisfying the inequality

$$|f(x)| \leq M_f(1+x^2),$$

where M_f is a positive constant only depending on function f. Also, let us define the spaces

$$C_2[0,\infty) = B_2[0,\infty) \cap C[0,\infty)$$

and

$$C_2^*[0,\infty) = \left\{ f \in C_2[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^2} = k_f < \infty \right\}$$

and endow them with the norm

$$\|f\|_2 = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

Remember that, as it follows from articles [6] and [7] by Gadjiev, the Korovkin-type theorems for positive linear operators do not hold in the space $C_2[0,\infty)$ but hold in the space of $C_2^*[0,\infty)$ with the previous norm. In addition to the assumption of the function f being positive, from now on, we assume that g_k is positive as well.

Theorem 4. *For each $f \in C_2^*[0,\infty)$, we have*

$$\lim_{n\to\infty} \|\mathbf{L}_{n,\alpha}(f) - f\|_2 = 0.$$

Proof. From [7], we observe that it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty} \|\mathbf{L}_{n,\alpha}(t^\nu;x) - x^\nu\|_2 = 0, \quad \nu = 0,1,2. \tag{16}$$

Since $\mathbf{L}_{n,\alpha}(1;x) = 1$, condition (16) holds for $\nu = 0$. From Lemma 1 we have

$$\begin{aligned}\|\mathbf{L}_{n,\alpha}(t;x) - x\|_2 &= \frac{2}{n}(1-\alpha) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\ &\le \frac{2}{n}(1-\alpha),\end{aligned}$$

which implies that the condition in (16) holds for $\nu = 1$.

Similarly, we can write

$$\begin{aligned}\left\|\mathbf{L}_{n,\alpha}(t^2;x) - x^2\right\|_2 &= \frac{|4\alpha-3|}{n} \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} + \frac{n+4\alpha-4}{n^2} \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\ &\le \frac{|4\alpha-3|}{n} + \frac{n+4\alpha-4}{n^2}\end{aligned}$$

which implies that the condition in (16) holds for $\nu = 2$.

This completes the proof of the theorem. □

Lemma 3. *For $n \in \mathbb{N}$, the α-Baskakov operator has the following identities:*

1. $n\mathbf{L}_{n,\alpha}((t-x)^2;x) = x(1+x) + \frac{4x}{n}(\alpha-1)$,

2. $$n^2\mathbf{L}_{n,\alpha}((t-x)^4;x) = 3x^2(1+x)^2 - \frac{x}{n}(10x^4+36x^2+25x-1) + \frac{\alpha x^2}{n}(16x^2+48x+32) + \frac{16x}{n^2}(\alpha-1).$$

Theorem 5. *For any function $f \in C_2[0,\infty)$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|\mathbf{L}_{n,\alpha}(f;x) - f(x)|}{(1+x^2)^{3/2}} = 0.$$

Proof. Since f is continuous and also uniformly continuous on any closed interval, there exists a positive number σ, depending on ε and f, for $|t-x| < \sigma$, and we have

$$|f(t) - f(x)| < \varepsilon$$

Since $f \in B_2[0,\infty)$, for $|t-x| \geq \sigma$,

$$|f(t) - f(x)| \leq A_f(\sigma)\left\{(t-x)^2 + (1+x^2)\,|t-x|\right\},$$

where $A_f(\sigma)$ is a positive constant depending on f and σ.

Combining the above results yields

$$|f(t) - f(x)| < \varepsilon + A_f(\sigma)\left\{(t-x)^2 + (1+x^2)\,|t-x|\right\},$$

where $t, x \in [0,\infty)$. Thus, we obtain

$$\begin{aligned}|\mathbf{L}_{n,\alpha}(f;x) - f(x)| &< \varepsilon + A_f(\sigma)\left\{\mathbf{L}_{n,\alpha}((t-x)^2;x) + (1+x^2)\mathbf{L}_{n,\alpha}(|t-x|;x)\right\} \\ &< \varepsilon + A_f(\sigma)\left\{\mathbf{L}_{n,\alpha}((t-x)^2;x) + (1+x^2)(\mathbf{L}_{n,\alpha}((t-x)^2;x))^{1/2}\right\}.\end{aligned}$$

From Lemma 3, we obtain the desired result. □

Theorem 6. *Let $f \in C_2[0,\infty)$ and also let $f^{''} \in C_2[0,\infty)$. Then*

$$\lim_{n\to\infty} n\left[\mathbf{L}_{n,\alpha}(f;x) - f(x)\right] = x(x+1)f^{''}(x) + 2(1-\alpha)f^{'}(x).$$

Proof. We observe that

$$\mathbf{L}_{n,\alpha}(1;x) = 1.$$

Moreover, for any $x \geq 0$, using Lemma 1, we have

$$\lim_{n\to\infty} n\mathbf{L}_{n,\alpha}(t-x;x) = 2(1-\alpha).$$

Using Lemma 3, we obtain

$$\lim_{n\to\infty} n\mathbf{L}_{n,\alpha}((t-x)^2;x) = x(1+x),$$

and

$$\lim_{n\to\infty} n^2\mathbf{L}_{n,\alpha}((t-x)^4;x) = 3x^2(x+1)^2.$$

Thus, the proof follows from [1, Proposition 5.1]. □

Recall that the weighted modulus of smoothness is denoted by $\Omega(f;\sigma)$ and defined by

$$\Omega(f;\sigma) = \sup_{0\leq h<\sigma, x\in[0,\infty)} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}. \tag{17}$$

for $f \in C_2^\star[0,\infty)$ (see [9]).

We know that for every function $f \in C_2^\star[0,\infty)$, the weighted modulus of smoothness has the following properties:

$$\lim_{\sigma\to 0} \Omega(f;\sigma) = 0 \tag{18}$$

and

$$\Omega(f;\lambda\sigma) \leq 2(1+\lambda)(1+\sigma^2)\Omega(f;\sigma), \qquad \lambda > 0. \tag{19}$$

Thus, combining these two results yields

$$\begin{aligned} |f(y)-f(x)| &\leq (1+(y-x)^2)(1+x^2)\Omega(f;|y-x|) \\ &\leq 2(1+\frac{|y-x|}{\sigma})(1+\sigma^2)\Omega(t;\lambda\sigma)(1+(y-x)^2)(1+x^2). \end{aligned} \tag{20}$$

Theorem 7. *If $f \in C_2^\star[0,\infty)$ then, for large enough n, we have*

$$\frac{|\mathbf{L}_{n,\alpha}(f;x)-f(x)|}{(1+x^2)^3} \leq 32\,\Omega(f;\frac{1}{\sqrt{n}}).$$

Proof. Using inequality (20), we have

$$|f(y)-f(x)| \leq \begin{cases} 4(1+\sigma^2)^2(1+x^2)\Omega(f;\sigma), & \text{if } |y-x|<\sigma \\ 4(1+\sigma^2)^2(1+x^2)\frac{(y-x)^4}{\sigma^4}\Omega(f;\sigma), & \text{if } |y-x|\geq\sigma \end{cases}.$$

By choosing $\sigma < 1$, we obtain

$$\begin{aligned} |f(y)-f(x)| &\leq 4(1+\sigma^2)^2(1+x^2)\Omega(f;\sigma)(1+\frac{(y-x)^4}{\sigma^4}) \\ &\leq 16(1+x^2)\Omega(f;\sigma)(1+\frac{(y-x)^4}{\sigma^4}). \end{aligned}$$

Using the above inequaity with $y=\frac{k}{n}$, we deduce that

$$\begin{aligned} |\mathbf{L}_{n,\alpha}(f;x)-f(x)| &\leq \left|\sum_{k=0}^{\infty} f(\frac{k}{n})P_{n,k}^{(\alpha)}(x)-f(x)\right| \\ &\leq \sum_{k=0}^{\infty}\left|f(\frac{k}{n})-f(x)\right| P_{n,k}^{(\alpha)}(x) \\ &\leq 8(1+x^2)\Omega(f;\sigma)(1+\frac{1}{\sigma^4}\mathbf{L}_{n,\alpha}((t-x)^4;x). \end{aligned}$$

Thus, using Lemma 3 and choosing $\sigma=\frac{1}{\sqrt{n}}$ yield the desired result. □

4. Numerical results

In this section, we present some numerical results obtained by using Matlab. Figure 1 shows the plot of $f(x) = x^2$ along with $\mathbf{L}_{20,\alpha}(f;x)$ for different values of $\alpha = 0.1, 0.5, 0.7$ and 1.0 on the interval $[0,1]$. Note that when $\alpha = 0.5$, α-Baskakov approximation outperforms others. In this case, the graphs of $f(x)$ and $\mathbf{L}_{20,\alpha}(f;x)$ are almost indistinguishable. However, unlike others, $\alpha = 0.1$ leads to underestimation.

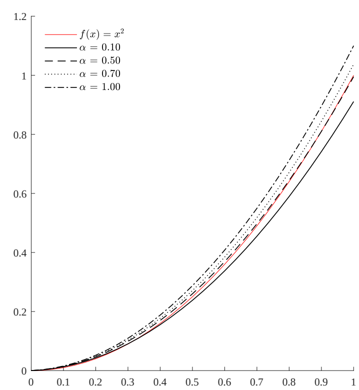

Figure 1: $k \in [0,\infty)$, $x = 0 : 1/10 : 1$, $n = 20$ *and* $f(x) = x^2$

On the other hand, Figure 2 presents similar plots for $f(x) = \exp x$, but $x \in [0,5]$. When we examine the figure, we observe that the best approximation is achieved when $\alpha = 0.1$.

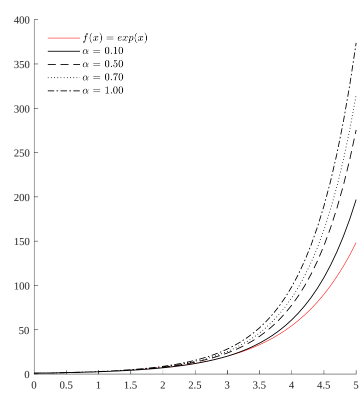

Figure 2: $k \in [0,\infty)$, $x = 0 : 1/10 : 5$, $n = 20$ *and* $f(x) = \exp x$

5. Conclusion

We constructed sequences of non-negative parametric α-Baskakov operators. We expressed α-Baskakov operators in terms of forward divided differences and powers of x, and their higher order derivatives in terms of forward divided differences.

On the other hand, we obtained Korovkin type of approximation properties of the operators. Moreover, we obtained the convergence rate of α-Baskakov operators by using the modulus of continuity. We also presented some simulation results which say, depending on the value of the parameter α, the quaity of approximating a function improves compared to classical Baskakov operators.

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