Nuclear Engineering and Technology 52 (2020) 700–707

Contents lists available at ScienceDirect

Nuclear Engineering and Technology

journal homepage: www.elsevier.com/locate/net

Original Article

Half-space albedo problem for İnönü, linear and quadratic anisotropic scattering

R.G. Tureci

Kırıkkale University, Kırıkkale Vocational School, Center Campus, Kırıkkale, 71450, Turkey

ARTICLE INFO

Article history:
Received 22 May 2019
Received in revised form
19 September 2019
Accepted 9 October 2019
Available online 18 October 2019

Keywords:
İnönü-linear-quadratic anisotropy
Half-space albedo
Forward-linear-quadratic anisotropy
Backward-linear-quadratic anisotropy

ABSTRACT

This study is concerned with the investigation of the half-space albedo problem for “İnönü-linear-quadratic anisotropic scattering” by the usage of Modified FN method. The method is based on Case's method. Therefore, Case's eigenfunctions and its orthogonality properties are derived for anisotropic scattering of interest. Albedo values are calculated for various linear, quadratic and İnönü anisotropic scattering coefficients and tabulated in Tables.

1. Introduction

In this study one-speed neutron transport equation is used with İnönü-Linear-Quadratic anisotropic scattering. The one-speed neutron transport equation is the approximate equation of Boltzmann equation or Neutron transport equation in seven variables. Three of them are space variables, other three are velocity and the last one is time variable. This transport equation in seven variables does not have an analytical solution. Therefore, some reasonable approximations have to be made as follows in order to obtain analytical solution:

- Since the neutrons in a reactor are produced via certain nuclear reactions, it is assumed that they all have the same energy or velocity. This approximation is known as the one-speed approximation.
- It's assumed that the space is a planar geometry according to plane geometry approximation.
- If the medium structure is homogeneous, then all neutron interactions have the same cross sections. Therefore, the secondary neutron number, c, becomes a constant number.
- The fourth approximation is the “stationary state approximation”. The time variable drops from the Boltzmann equation by this approximation.

As a result of the above treatments, the number of variables is reduced to two, namely, a position variable and a direction cosine [1–3].

Case has reported a solution for one-speed, time-independent, homogenous medium neutron transport equation, which is known as Case's method [4,5]. The completeness of Case's eigenfunctions allows us to write a solution as a superposition of Case's eigenfunctions which correspond discrete eigenvalues and continuous eigenvalues.

Some semi-analytical methods such as Facile (FN) method [6,7], Complementary (CN) method [8], Hybrid (HN) (with the other name Modified FN) method [9] were developed in the literature. The Placzek lemma [10] is particularly important in these methods in terms of using the full-range orthogonality of Case's method. FN method uses Case's orthogonality relations to find the arbitrary expansion coefficients according to the boundary conditions. These arbitrary expansion coefficients are used in neutron flux. The neutron flux, which contains these coefficients, is written with N mesh points in order to get an equation system according to the angular variable. CN method uses the Green's function and the boundary neutron flux definitions in order to solve the transport equation. CN method is one of the most difficult method but it is an elegant method. Modified FN method, originally known as HN method, uses Case's orthogonality relations in order to find the arbitrary expansion coefficients. Finally, the moments of the neutron flux, which contains these arbitrary expansion coefficients and Case's eigenfunctions, is calculated to get an equation system.

E-mail address: tureci@kku.edu.tr.

https://doi.org/10.1016/j.net.2019.10.008

According to the literature CN method is more convergent than FN method, and Modified FN method (HN method) provides the convergent results like CN method.

2. Linear-quadratic anisotropic scattering

The one-speed, plane geometry, homogenous medium and time independent neutron transport equation is given as

$$\mu\frac{\partial\Psi(x,\mu)}{\partial x}+\Psi(x,\mu)=\frac{c}{2}\int_{-1}^{1}f(\mu,\mu')\Psi(x,\mu')d\mu' \tag{1}$$

where μ is the direction cosine of the incident neutrons, μ' is the direction cosine of the scattered neutrons, $f(\mu,\mu')$ is the scattering function which defines the scattering probability of neutrons, $\Psi(x,\mu)$ is the number of the neutrons at x position and μ direction. c is the secondary neutron number which is a ratio between the cross sections. The scattering function in Eq. (1) is expanded with the Legendre polynomials, which is entitled as Mika's scattering [11] as

$$f(\mu,\mu')=\sum_{n=0}^{N}(2n+1)f_nP_n(\mu)P_n(\mu') \tag{2}$$

where $P_n(\mu)$ and $P_n(\mu')$ are Legendre polynomials and f_n is the scattering coefficient. Since the scattering function defines the probability of each scattering, the sum of these scatterings must equal to unity. Therefore, the definition interval of f_n coefficients must be determined for every scattering situation by physical meaning. The first and second terms in Eq. (2) are defined as isotropic and linear anisotropic scattering respectively. The probability of scattering is proportional to the multiplication of μ and μ', which are $P_1(\mu)$ and $P_1(\mu')$. Linear anisotropic scattering has been studied extensively with the different methods in the literature [12–14]. The third term is defined as quadratic anisotropic scattering and the probability of scattering is proportional to the multiplication $P_2(\mu)$ and $P_2(\mu')$. This scattering has been studied in the literature [15–17].

The scattering function in this study is written as a linear combination of İnönü's scattering [18,19]. Thus the scattering function, which is called as linear-quadratic anisotropic scatterings, is

$$\begin{aligned}f(\mu,\mu')&=a[1+3f_1\mu\mu'+5f_2P_2(\mu)P_2(\mu')]+b\delta(\mu-\mu')+d\,\delta(\mu+\mu')\\&=a f_{\ell q}(\mu,\mu')+b\,\delta(\mu-\mu')+d\delta(\mu+\mu')\end{aligned} \tag{3}$$

where b is the forward scattering coefficient and d is the backward scattering coefficient. If b has a nonzero value then d is zero, and if d has a nonzero value then b is zero. The coefficient a obeys the rule of the sum of probabilities, i.e. $a+b+d=1$. If b and d are equal to zero together in Eq. (3) and $f_{1,2}\neq 0$, then the scatting is linear-quadratic anisotropic scattering.

İnönü's scattering function is known as the synthetic kernel in the literature. This scattering function defines the forward scattering and backward scattering of neutrons, and moreover, İnönü showed that the solution of the one-speed neutron transport equation with this scattering function can be written by Case's eigenfunctions by using of the transformations which were described by İnönü. The İnönü's scattering function is used for lots of studies [20–26]. The combination of İnönü's scattering function and linear anisotropic scattering was defined as strong anisotropic scattering in the literature [27–29].

Case's method is a powerful method to solve the one-speed neutron transport equation. Mika's anisotropic scattering function can be applied to Case's method. Similarly, İnönü's scattering function also can be applied to Case's method. Furthermore, both of them can be applied to Case's method simultaneously. The aim of this study is to investigate the effect of these scattering functions over Albedo values, in the Case's of simultaneous application of the both.

3. The solution for the anisotropic scattering

If Eq. (3) is used in Eq. (1), then the one-speed neutron transport equation could be written as

$$\begin{aligned}&\mu\frac{\partial\Psi_{Ilq}(x,\mu)}{\partial x}+\Psi_{Ilq}(x,\mu)\\&=\frac{ac}{2}\int_{-1}^{1}f_{lq}(\mu,\mu')\Psi_{Ilq}(x,\mu')d\mu'+cb\Psi_{Ilq}(x,\mu)+cd\Psi_{Ilq}(x,-\mu)\end{aligned} \tag{4}$$

where the subscript Ilq indicates İnönü-linear-quadratic anisotropic scattering. If the following variables, which were redefined by İnönü, are used

$$c'=\frac{ac}{1-bc-dc},\quad q=\left[(1-bc)^2-b^2c^2\right]^{1/2} \tag{5-a}$$

$$x'=qx,\quad \nu'=q\nu \tag{5-b}$$

$$A=\frac{1}{2}\left[1+\left(\frac{1-bc-dc}{1-bc+dc}\right)^{1/2}\right],\quad B=\frac{1}{2}\left[1-\left(\frac{1-bc-dc}{1-bc+dc}\right)^{1/2}\right] \tag{5-c}$$

where q, A and B are the factors which come from the transformation, and ν corresponds to the eigenvalue. Then the solution of Eq. (4) is written as

$$\Psi_{Ilq}(x,\mu)=A\Psi_{lq}(x',\mu)+B\Psi_{lq}(x',-\mu) \tag{6-a}$$

$$\Psi_{Ilq}(x,-\mu)=A\Psi_{lq}(x',-\mu)+B\Psi_{lq}(x',\mu) \tag{6-b}$$

where the lq subscript indicates the linear-quadratic anisotropic scattering. Thus the solution, which includes İnönü-linear-quadratic anisotropic scattering is written in Eq. (6). The neutron flux for the linear-quadratic anisotropic scattering, $\Psi_{lq}(x',\pm\mu)$, is the solution of the one-speed neutron transport equation for $a=1$, $b=0$ and $d=0$; but, $f_1\neq 0$ and $f_2\neq 0$.

The linear-quadratic anisotropic scattering was investigated in Refs. [30–33]. The solution of the neutron transport equation with the linear-quadratic anisotropic scattering,

$$\begin{aligned}\Psi_{lq}(x,\mu)&=a_{0+}\phi_{lq}(\nu_0,\mu)e^{-x/\nu_0}+a_{0-}\phi_{lq}(-\nu_0,\mu)e^{x/\nu_0}\\&+\int_0^1 A(\nu)\phi_{lq}(\nu,\mu)e^{-x/\nu}d\nu+\int_0^1 A(-\nu)\phi_{lq}(-\nu,\mu)e^{x/\nu}d\nu\end{aligned} \tag{7}$$

where Case's eigenfunctions and the orthogonality relations are given as

$$\phi(\pm\nu_0,\mu)=\frac{c\nu_0}{2}\frac{1-\eta_0\pm w\nu_0\mu+3\eta_0\mu^2}{\nu_0\mp\mu} \tag{8-a}$$

$$\phi(\nu,\mu)=\frac{c\nu}{2}P\frac{1-\eta+w\nu\mu+3\eta\mu^2}{\nu-\mu}+\lambda(\nu)\delta(\nu-\mu) \tag{8-b}$$

where P symbol correspond Cauchy Principal Value. The constants w, η_0 and η are defined as

$$w=3f_1(1-c) \tag{8-c}$$

$$\eta_0\equiv\eta(\nu_0)=\frac{5f_2}{4}\left[3\nu_0^2(1-c)(1-cf_1)-1\right] \tag{8-d}$$

$$\eta\equiv\eta(\nu)=\frac{5f_2}{4}\left[3\nu^2(1-c)(1-cf_1)-1\right] \tag{8-e}$$

The normalization integral of the continuous Case's eigenfunctions gives

$$\lambda(\nu)=\left(1+wc\nu^2+3\eta c\nu^2\right)-c\nu\left(1-\eta+w\nu^2+3\eta\nu^2\right)\arctan h(\nu). \tag{8-f}$$

The orthogonality relations are

$$\int_{-1}^{1}\mu\phi_{Iq}(\pm\nu_0,\mu)\phi_{Iq}(\pm\nu_0,\mu)d\mu=N_{Iq}(\pm\nu_0)\ \text{and}\ N_{Iq}(-\nu_0)=-N_{Iq}(\nu_0) \tag{8-g}$$

$$\int_{-1}^{1}\mu\phi_{Iq}(\pm\nu_0,\mu)\begin{Bmatrix}\phi_{Iq}(\nu,\mu)d\mu\\ \phi_{Iq}(\mp\nu_0,\mu)d\mu\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \tag{8-h}$$

$$\int_{-1}^{1}\mu\phi_{Iq}(\nu,\mu)\phi_{Iq}(\nu',\mu)d\mu=N_{Iq}(\nu)\delta(\nu-\nu')\ \text{and}\ N_{Iq}(-\nu)=-N_{Iq}(\nu) \tag{8-i}$$

where

$$N_{Iq}(\nu_0)=\left(\frac{c\nu_0}{2}\right)^2\Bigg\{\frac{2\nu_0}{\nu_0^2-1}\Big[(1-\eta_0)^2+2(1-\eta_0)w+\left(6\eta_0(1-\eta_0)+w^2\nu_0^2\right)+6w\eta_0+9\eta_0^2\Big]-\Big[(1-\eta_0)^2+4(1-\eta_0)w\nu_0^2+3\nu_0^2\left(6\eta_0(1-\eta_0)+w^2\nu_0^2\right)+24w\eta_0\nu_0^4+45\eta^2\nu_0^4\Big]\frac{2}{c\nu_0}\frac{1+wc\nu_0^2+3\eta c\nu_0^2}{1-\eta_0+w\nu_0^2+3\eta_0\nu_0^2}+8(1-\eta_0)w\nu_0+6\nu_0\left(6\eta_0(1-\eta_0)+w^2\nu_0^2\right)+48w\eta_0\nu_0^3+16w\eta_0\nu_0+90\eta_0^2\nu_0^3+30\eta_0^2\nu_0\Bigg\} \tag{8-j}$$

$$N_{Iq}(\nu)=\nu\left[\left(1+wc\nu^2+3\eta c\nu^2\right)-c\nu\left(1-\eta+w\nu^2+3\eta\nu^2\right)\arctan h(\nu)\right]^2+\frac{c^2\pi^2\nu^3}{4}\left(1-\eta+w\nu^2+3\eta\nu^2\right)^2 \tag{8-k}$$

4. Albedo problem

It is well known that albedo is the ratio from a surface between the net outgoing and the net incoming neutron (or any neutral particle). The surface is located at $x=0$. The right side of this surface is the medium which consists of moderator material, and the left side of this surface is vacuum. The neutron flux which is given in Eq. (6) must be convergent to zero when x goes to infinity. Therefore, the arbitrary expansion coefficients, which are the coefficients of the positive exponential terms, must be selected equal to zero, $a'_{0-}=0$ and $A(-\nu')=0$.

$$\Psi_{IIq}(x,-\mu)=A\left[a'_{0+}\phi_{Iq}(-\nu'_0,\mu)\exp\left(-\frac{x'}{\nu'_0}\right)+\int_0^1 A(\nu')\phi_{Iq}(-\nu',\mu)\exp\left(-\frac{x'}{\nu'}\right)d\nu'\right]+B\left[a'_{0+}\phi_{Iq}(\nu'_0,\mu)\exp\left(-\frac{x'}{\nu'_0}\right)+\int_0^1 A(\nu')\phi_{Iq}(\nu',\mu)\exp\left(-\frac{x'}{\nu'}\right)d\nu'\right],\quad \mu\in[-1,1] \tag{9}$$

The suggested neutron flux definitions over the surface are given by power series as

$$\Psi_{IIq}(0,\mu)=\mu^\gamma,\quad \gamma=0,1,2,\ldots \tag{10-a}$$

$$\Psi_{IIq}(0,-\mu)=\sum_{\ell=0}^{N}a_\ell\mu^\ell \tag{10-b}$$

Thus, the albedo relation is found by using Eq. (10)

$$\beta=\frac{\int_0^1\mu\Psi(0,-\mu)d\mu}{\int_0^1\mu\Psi(0,\mu)d\mu}=(\gamma+2)\sum_{\ell=0}^{N}\frac{a_\ell}{\ell+2} \tag{11}$$

Therefore, if a_ℓ coefficients are found as numerical, then the albedo values can be calculated by using Eq. (11).

The first stage is to find a relationship for the arbitrary expansion coefficients, a'_{0+} and $A(\nu')$, by using the neutron flux definitions over the surface which are given in Eq. (10) and Case's orthogonality relations. If Eq. (9) is written by taking $x=0$ and it is multiplied by $\mu\phi_{Iq}(-\nu'_0,\mu)$ and $\mu\phi_{Iq}(-\nu',\mu)$ respectively and integrating over $\mu\in[-1,1]$, then arbitrary expansion coefficients are found as

$$a'_{0+}=\frac{c\nu'_0}{2}\frac{1}{A}\frac{1}{N_{lq}(\nu'_0)}\left[B_\gamma(\nu'_0)-\sum_{\ell}^{N}a_\ell A_\ell(\nu'_0)\right] \tag{12}$$

$$A(\nu')=\frac{c\nu'}{2}\frac{1}{A}\frac{1}{N_{lq}(\nu')}\left[B_\gamma(\nu')-\sum_{\ell}^{N}a_\ell A_\ell(\nu')\right] \tag{13}$$

where $A_n(\xi)$ and $B_\gamma(\xi)$ are defined as

$$A_n(\xi)=\frac{2}{c\xi}\int_0^1\mu^{n+1}\phi_{lq}(\xi,-\mu)d\mu \tag{14}$$

and

$$B_n(\xi)=\frac{2}{c\xi}\int_0^1\mu^{n+1}\phi_{lq}(\xi,\mu)d\mu \tag{15}$$

$$\sum_{\ell=0}^{N}a_\ell\left[\frac{1}{m+\ell+2}+\left(\frac{c'\nu'_0}{2}\right)^2\frac{A_\ell(\nu'_0)A_m(\nu'_0)}{N_{lq}(\nu'_0)}+\frac{B}{A}\left(\frac{c'\nu'_0}{2}\right)^2\frac{A_\ell(\nu'_0)B_m(\nu'_0)}{N_{lq}(\nu'_0)}+\int_0^1\left(\frac{c'\nu'}{2}\right)^2\frac{A_\ell(\nu')A_m(\nu')}{N_{lq}(\nu')}d\nu'+\frac{B}{A}\int_0^1\left(\frac{c'\nu'}{2}\right)^2\frac{A_\ell(\nu')B_m(\nu')}{N_{lq}(\nu')}d\nu'\right]=\left(\frac{c'\nu'_0}{2}\right)^2\frac{B_\gamma(\nu'_0)A_m(\nu'_0)}{N_{lq}(\nu'_0)}+\frac{B}{A}\left(\frac{c'\nu'_0}{2}\right)^2\frac{B_\gamma(\nu'_0)B_m(\nu'_0)}{N_{lq}(\nu'_0)}+\int_0^1\left(\frac{c'\nu'}{2}\right)^2\frac{B_\gamma(\nu')A_m(\nu')}{N_{lq}(\nu')}d\nu'+\frac{B}{A}\int_0^1\left(\frac{c'\nu'}{2}\right)^2\frac{B_\gamma(\nu')B_m(\nu')}{N_{lq}(\nu')}d\nu' \tag{20}$$

and n (or γ) corresponds to the integer numbers according to N number in Eq. (10) which is the approximation number. ξ correspond to ν'_0 and ν' for discrete and the continuous eigenvalues respectively. These integrals have separately recurrence relations:

$$A_n(\xi)=\frac{1-\eta_\xi}{n+1}-\frac{w\xi}{n+2}+\frac{3\eta_\xi}{n+3}-\xi A_{n-1}(\xi) \tag{16}$$

$$A_0(\xi)=1-\eta_\xi-\frac{w\xi(1-2\xi)}{2}+\frac{\eta_\xi}{2}\left(2-3\xi+6\xi^2\right)-\xi\left(1-\eta_\xi+w\xi^2+3\eta_\xi\xi^2\right)\ln\left(1+\frac{1}{\xi}\right) \tag{17}$$

$$B_n(\xi)=\xi B_{n-1}(\xi)-\frac{1-\eta_\xi}{n+1}-\frac{w\xi}{n+2}-\frac{3\eta_\xi}{n+3} \tag{18}$$

$$B_0(\xi)=\frac{2}{c}\left(1+cw\xi^2+3c\eta_\xi\xi^2\right)-(1-\eta_\xi)-\frac{w\xi(1+2\xi)}{2}-\frac{\eta_\xi}{2}\left(2+3\xi+6\xi^2\right)-\xi\left(1-\eta_\xi+w\xi^2+3\eta_\xi\xi^2\right)\ln\left(1+\frac{1}{\xi}\right) \tag{19}$$

where η_ξ is related to the kind of eigenvalue. η_0 is related to the discrete eigenvalues and η is related to the continuous eigenvalues. If $a_{0+}{}'$ and $A(\nu')$ coefficients, which are found in Eqs. 12 and 13, are put into Eq. (9) and this relation is written in the interval $\mu\in[0,1]$, then the relation defines the outgoing neutron flux from the surface.

$$\Psi_{llq}(0,-\mu)=A\left[a'_{0+}\phi_{lq}(-\nu'_0,\mu)+\int_0^1A(\nu')\phi_{lq}(-\nu',\mu)d\nu'\right]+B\left[a'_{0+}\phi_{lq}(\nu'_0,\mu)+\int_0^1A(\nu')\phi_{lq}(\nu',\mu)d\nu'\right],\quad \mu\in[0,1] \tag{9a}$$

In FN method to get a system of linear equations, Eq. (9a) is written with N+1 mesh points for the angular variable. The N represents the number of meshes. In Modified FN method to get a system of linear equations, Eq. (9a) is multiplied with μ^{m+1} and integrated over $\mu\in[0,1]$. Thus, Eq. (20) is written by replacing the expansion coefficients Eq.12 and 13 and by using Eqs. 14 and 15 in this integrated equation.

Eq. (20) can be rewritten as

$$\sum_{\ell=0}a_\ell T_{m\ell}=U_{m\gamma} \tag{21}$$

m and ℓ indices change together from zero to N, thus we get a square matrix. a_ℓ coefficients are calculated by using ordinary matrix operations.

$$A=T^{-1}\bullet U \tag{22}$$

where the elements of A matrix are a_ℓ coefficients. T^{-1} is the inverse matrix of T, and its elements are $T_{m\ell}$. U matrix is the column matrix and its elements are $U_{m\gamma}$. Finally a_ℓ coefficients are used in Eq. (11) and so the albedo values are found. Here $T_{m\ell}$ is

$$T_{m\ell}=\frac{1}{m+\ell+2}+\left(\frac{c'\nu'_0}{2}\right)^2\frac{A_\ell(\nu'_0)A_m(\nu'_0)}{N_{lq}(\nu'_0)}+\frac{B}{A}\left(\frac{c'\nu'_0}{2}\right)^2\frac{A_\ell(\nu'_0)B_m(\nu'_0)}{N_{lq}(\nu'_0)}+\int_0^1\left(\frac{c'\nu'}{2}\right)^2\frac{A_\ell(\nu')A_m(\nu')}{N_{lq}(\nu')}d\nu'+\frac{B}{A}\int_0^1\left(\frac{c'\nu'}{2}\right)^2\frac{A_\ell(\nu')B_m(\nu')}{N_{lq}(\nu')}d\nu'$$

and $U_{m\gamma}$ is

$$U_{m\gamma} = \left(\frac{c'\nu'_0}{2}\right)^2 \frac{B_\gamma(\nu'_0)A_m(\nu'_0)}{N_{lq}(\nu'_0)} + \frac{B}{A}\left(\frac{c'\nu'_0}{2}\right)^2 \frac{B_\gamma(\nu'_0)B_m(\nu'_0)}{N_{lq}(\nu'_0)} + \int_0^1 \left(\frac{c'\nu'}{2}\right)^2 \frac{B_\gamma(\nu')A_m(\nu')}{N_{lq}(\nu')}d\nu' + \frac{B}{A}\int_0^1 \left(\frac{c'\nu'}{2}\right)^2 \frac{B_\gamma(\nu')B_m(\nu')}{N_{lq}(\nu')}d\nu'$$

5. Numerical results

Case's eigenfunction is normalized to 1. The discrete eigenvalue corresponds to $\nu \notin [-1,1]$ and represented as ν_0. By using the normalization condition of the discrete eigenfunction we get a distribution which is a transedental equation.

$$\Lambda(\nu'_0) = ln\left(\frac{1+1/\nu'_0}{1-1/\nu'_0}\right) - \frac{2}{c\nu'_0}\frac{1+wc\nu'^2_0+3\eta_0 c\nu'^2_0}{1-\eta_0+w\nu'^2_0+3\eta_0\nu'^2_0} = 0 \tag{23}$$

The numerical solution of this equation gives the discrete eigenvalues, $\pm\nu_0$.

The Mika's part of the scattering function is

$$f(\mu,\mu') = 1 + 3f_1\mu\mu' + 5f_2P_2(\mu)P_2(\mu') \tag{24}$$

where $f_1 \in [-1/3, 1/3]$ for pure-linearly anisotropic scattering, $f_2 = 0$, and also $f_2 \in [-0.2, 0.4]$ for pure-quadratic anisotropic scattering, $f_1 = 0$. The maximum and the minimum values of the direction cosines are $+1$ and -1 respectively. If f_2 equal to zero, i.e. the scattering is pure-linear anisotropic scattering, then the scattering function is

$$f(\mu,\mu') = 1 + 3f_1\mu\mu'. \tag{25}$$

Eq. (25) is defined in interval $[0,1]$ according to the rule of the sum of probabilities. In other words, the summation equals one. The minimum value of $\mu\cdot\mu'$ is -1, and the maximum value is $+1$. Thus, the definition interval is determined for f_1 which is $f_1 \in [-1/3, 1/3]$. If f_1 equals zero in Eq. (24) which corresponds to

Table 1
The quadratic anisotropic scattering coefficients and the corresponding discrete eigenvalues according to f_1 for fixed secondary neutron number. $c = 0.8$

		$f_1 = 0.1$		$f_1 = 0.2$		$f_1 = 0.3$	
		$f_2 \in [-0.14, 0.06]$		$f_2 \in [-0.08, 0.12]$		$f_2 \in [-0.02, 0.18]$	
b or d	c'	f_2	$\pm\nu'_0$	f_2	$\pm\nu'_0$	f_2	$\pm\nu'_0$
0.2	0.7619047619048	−0.14	1.3503367489803	−0.08	1.41334586878385	−0.01	1.4867844987356
0.4	0.7058823529412		1.2419111919982		1.29509510276002		1.3567170391812
0.6	0.6153846153846		1.1291274965985		1.16975744761954		1.2166527676182
0.8	0.4444444444444		1.0262359306891		1.04702282933634		1.0723781218970
0.2	0.7619047619048	−0.12	1.3521486330776	−0.01	1.42066290480843	0.05	1.4938635927823
0.4	0.7058823529412		1.2438736414053		1.30288836413616		1.3641417225914
0.6	0.6153846153846		1.1312089296472		1.17784635913769		1.2242044885072
0.8	0.4444444444444		1.0278887618233		1.05374548049964		1.0786818616781
0.2	0.7619047619048	−0.1	1.3540147671943	0.1	1.43395481263645	0.1	1.5003196844828
0.4	0.7058823529412		1.2458903282655		1.31687706972736		1.3708625968616
0.6	0.6153846153846		1.1333393250400		1.19206935333486		1.2309538595848
0.8	0.4444444444444		1.0295808620024		1.06525719311210		1.0842024211896
0.2	0.7619047619048	0.06	1.3712190807226	0.12	1.43664749310913	0.18	1.5118821198339
0.4	0.7058823529412		1.2642676247910		1.31968466887168		1.3827801177440
0.6	0.6153846153846		1.1523638009773		1.19487937971909		1.2427251544163
0.8	0.4444444444444		1.0445661970991		1.06748354209275		1.0935813926334

Table 2
The albedo values for $c = 0.8, f_1 = 0.1, \gamma = 0$ varying f_2, forward and backward.

$f_2 = -0.14$								
N	$d = 0$				$b = 0$			
	$b = 0.2$	$b = 0.4$	$b = 0.6$	$b = 0.8$	$d = 0.2$	$d = 0.4$	$d = 0.6$	$d = 0.8$
1	0.2875967	0.2434477	0.1873575	0.1118563	0.3597839	0.3920832	0.4227940	0.4553374
2	0.2868021	0.2428643	0.1870181	0.1117550	0.3588239	0.3912132	0.4221210	0.4549991
3	0.2867814	0.2428470	0.1870063	0.1117507	0.3588000	0.3911897	0.4221006	0.4549872
4	0.2867802	0.2428459	0.1870055	0.1117503	0.3587987	0.3911883	0.4220993	0.4549863
5	0.2867801	0.2428458	0.1870054	0.1117503	0.3587985	0.3911881	0.4220991	0.4549862
6	0.2867800	0.2428458	0.1870054	0.1117502	0.3587985	0.3911881	0.4220991	0.4549862
7	0.2867800	0.2428458	0.1870054	0.1117502	0.3587985	0.3911880	0.4220991	0.4549862
$f_2 = 0.06$								
	$d = 0$				$b = 0$			
	$b = 0.2$	$b = 0.4$	$b = 0.6$	$b = 0.8$	$d = 0.2$	$d = 0.4$	$d = 0.6$	$d = 0.8$
1	0.2896088	0.2456510	0.1897130	0.1140646	0.3616931	0.3940646	0.4247977	0.4571087
2	0.2888531	0.2451073	0.1894057	0.1139769	0.3608007	0.3932910	0.4242337	0.4568503
3	0.2888233	0.2450826	0.1893890	0.1139709	0.3607663	0.3932572	0.4242047	0.4568335
4	0.2888215	0.2450810	0.1893877	0.1139703	0.3607643	0.3932552	0.4242027	0.4568322
5	0.2888213	0.2450808	0.1893876	0.1139703	0.3607641	0.3932549	0.4242025	0.4568321
6	0.2888213	0.2450807	0.1893876	0.1139702	0.3607641	0.3932549	0.4242025	0.4568320
7	0.2888213	0.2450807	0.1893875	0.1139702	0.3607641	0.3932549	0.4242024	0.4568320

Table 3
The albedo values for $c = 0.8, f_1 = 0.2, \gamma = 0$ varying f_2, forward and backward.

$f_2 = -0.08$

N	$d = 0$				$b = 0$			
	$b = 0.2$	$b = 0.4$	$b = 0.6$	$b = 0.8$	$d = 0.2$	$d = 0.4$	$d = 0.6$	$d = 0.8$
1	0.2691058	0.2262191	0.1724083	0.1013803	0.3422094	0.3765424	0.4100238	0.4468974
2	0.2682567	0.2256016	0.1720538	0.1012768	0.3411726	0.3756014	0.4092971	0.4465348
3	0.2682329	0.2255817	0.1720403	0.1012718	0.3411448	0.3755736	0.4092727	0.4465202
4	0.2682315	0.2255804	0.1720393	0.1012713	0.3411432	0.3755720	0.4092711	0.4465191
5	0.2682314	0.2255803	0.1720392	0.1012713	0.3411431	0.3755718	0.4092709	0.4465190
6	0.2682313	0.2255803	0.1720392	0.1012713	0.3411430	0.3755718	0.4092709	0.4465190
7	0.2682313	0.2255802	0.1720391	0.1012712	0.3411430	0.3755718	0.4092709	0.4465190

$f_2 = 0.12$

	$d = 0$				$b = 0$			
	$b = 0.2$	$b = 0.4$	$b = 0.6$	$b = 0.8$	$d = 0.2$	$d = 0.4$	$d = 0.6$	$d = 0.8$
1	0.2713509	0.2286496	0.1749641	0.1037134	0.3443460	0.3787399	0.4122137	0.4487824
2	0.2705373	0.2280691	0.1746399	0.1036226	0.3433772	0.3779003	0.4116034	0.4485049
3	0.2705032	0.2280409	0.1746208	0.1036156	0.3433375	0.3778608	0.4115689	0.4484846
4	0.2705012	0.2280391	0.1746194	0.1036151	0.3433352	0.3778585	0.4115667	0.4484830
5	0.2705010	0.2280388	0.1746192	0.1036150	0.3433350	0.3778582	0.4115665	0.4484829
6	0.2705010	0.2280388	0.1746192	0.1036150	0.3433350	0.3778582	0.4115664	0.4484828
7	0.2705010	0.2280388	0.1746192	0.1036150	0.3433349	0.3778582	0.4115664	0.4484828

the pure-quadratic anisotropic scattering, then the scattering function is

$$f(\mu,\mu') = 1 + 5f_2 P_2(\mu) P_2(\mu') \tag{26}$$

The definition interval f_2 is determined as $f_2 \in [-0.2, 0.4]$ by using the same rule. But, there is a linear dependent relation between f_1 and f_2 in the presence of both of the scatterings. This linear dependent relation is determined as

$$\frac{3f_1 - 1}{5} \le f_2 \le \frac{3f_1}{5} \tag{27}$$

by using the same rule and taking the set of intersections between f_1 and f_2. This interval was written for f_1 because the linear anisotropic scattering is more dominant scattering than the quadratic anisotropic scattering. Mathematica software v.12.0 [34] was used in all calculations. Table 1 represents the discrete eigenvalues for varying f_1, f_2, b and d.

Tables 2–4 represent the albedo values which are calculated by Eq. (11) for $c = 0.8 f_1 = 0.1$, $\gamma = 0$ and varying selected f_2 values. Table 5 represents the albedo values for varying γ values.

6. Conclusion

In this study; one-speed, homogeneous medium, plane geometry and time-independent neutron transport equation for half-space albedo has been investigated for the anisotropic scattering which contains İnönü's scattering, linear and quadratic anisotropic scatterings in Mika's anisotropic scattering function. To write the solution, Case's eigenfunctions and the orthogonality properties should be written for the scattering function of interest. There is also a linear dependent relation in order to determine the

Table 4
The albedo values for $c = 0.8, f_1 = 0.3, \gamma = 0$ varying f_2, forward and backward.

$f_2 = -0.02$

N	$d = 0$				$b = 0$			
	$b = 0.2$	$b = 0.4$	$b = 0.6$	$b = 0.8$	$d = 0.2$	$d = 0.4$	$d = 0.6$	$d = 0.8$
1	0.2486893	0.2073941	0.1563186	0.0903775	0.3227452	0.3594641	0.3961745	0.4379663
2	0.2477801	0.2067398	0.1559487	0.0902720	0.3216204	0.3584413	0.3953859	0.4375763
3	0.2477530	0.2067172	0.1559333	0.0902664	0.3215883	0.3584089	0.3953569	0.4375585
4	0.2477514	0.2067158	0.1559322	0.0902659	0.3215866	0.3584070	0.3953551	0.4375573
5	0.2477513	0.2067156	0.1559321	0.0902658	0.3215864	0.3584068	0.3953549	0.4375571
6	0.2477512	0.2067156	0.1559321	0.0902658	0.3215864	0.3584068	0.3953549	0.4375571
7	0.2477512	0.2067156	0.1559321	0.0902658	0.3215864	0.3584068	0.3953549	0.4375571

$f_2 = 0.18$

	$d = 0$				$b = 0$			
	$b = 0.2$	$b = 0.4$	$b = 0.6$	$b = 0.8$	$d = 0.2$	$d = 0.4$	$d = 0.6$	$d = 0.8$
1	0.2512097	0.2100882	0.1591012	0.0928463	0.3251515	0.3619145	0.3985774	0.4399763
2	0.2503296	0.2094663	0.1587584	0.0927522	0.3240933	0.3609976	0.3979131	0.4396769
3	0.2502911	0.2094345	0.1587370	0.0927445	0.3240477	0.3609517	0.3978725	0.4396525
4	0.2502889	0.2094325	0.1587355	0.0927439	0.3240452	0.3609491	0.3978700	0.4396508
5	0.2502886	0.2094322	0.1587353	0.0927438	0.3240450	0.3609489	0.3978698	0.4396506
6	0.2502886	0.2094322	0.1587353	0.0927438	0.3240449	0.3609488	0.3978697	0.4396506
7	0.2502886	0.2094322	0.1587353	0.0927438	0.3240449	0.3609488	0.3978697	0.4396506

Table 5
The albedo values for $c = 0.8, f_1 = 0.1, f_2 = -0.14$, forward and backward for varying.γ

$c = 0.8\ f_1 = 0.1, f_2 = -0.14$										
$b = 0.2$ and $d = 0$ (forward)										
N	$\gamma = 0$	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$	$\gamma = 4$	$\gamma = 5$	$\gamma = 6$	$\gamma = 7$	$\gamma = 8$	$\gamma = 9$
1	0.2875967	0.2692079	0.2590674	0.2526693	0.2482730	0.2450691	0.2426318	0.2407158	0.2391704	0.2378977
2	0.2868021	0.2684446	0.2583050	0.2519013	0.2474983	0.2442881	0.2418450	0.2399240	0.2383743	0.2370977
3	0.2867814	0.2684337	0.2582982	0.2518968	0.2474951	0.2442858	0.2418435	0.2399231	0.2383737	0.2370976
4	0.2867802	0.2684335	0.2582984	0.2518971	0.2474956	0.2442863	0.2418440	0.2399236	0.2383743	0.2370982
5	0.2867801	0.2684335	0.2582985	0.2518972	0.2474957	0.2442864	0.2418441	0.2399237	0.2383744	0.2370983
6	0.2867800	0.2684335	0.2582985	0.2518972	0.2474957	0.2442865	0.2418442	0.2399238	0.2383745	0.2370983
7	0.2867800	0.2684336	0.2582985	0.2518972	0.2474957	0.2442865	0.2418442	0.2399238	0.2383745	0.2370983
$b = 0$ and $d = 0.2$ (backward)										
N	$\gamma = 0$	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$	$\gamma = 4$	$\gamma = 5$	$\gamma = 6$	$\gamma = 7$	$\gamma = 8$	$\gamma = 9$
1	0.3597839	0.3413852	0.3312436	0.3248465	0.3204515	0.3172490	0.3148129	0.3128981	0.3113536	0.3100818
2	0.3588239	0.3410050	0.3311891	0.3250012	0.3207524	0.3176579	0.3153048	0.3134558	0.3119649	0.3107375
3	0.3588000	0.3409927	0.3311512	0.3249313	0.3206515	0.3175291	0.3151516	0.3132811	0.3117714	0.3105274
4	0.3587987	0.3409926	0.3311515	0.3249341	0.3206586	0.3175411	0.3151687	0.3133034	0.3117987	0.3105595
5	0.3587985	0.3409926	0.3311515	0.3249342	0.3206584	0.3175404	0.3151672	0.3133010	0.3117952	0.3105548
6	0.3587985	0.3409927	0.3311516	0.3249343	0.3206584	0.3175405	0.3151674	0.3133013	0.3117957	0.3105555
7	0.3587985	0.3409927	0.3311516	0.3249343	0.3206584	0.3175405	0.3151674	0.3133012	0.3117956	0.3105554

scattering coefficients, f_1 and f_2. This linear dependent relation is found by taking the set of intersections for f_1 and f_2. The solution is written according to the İnönü's results.

Case's method is a powerful method in order to write the solution for the one-speed neutron transport equation. Moreover, Case's method is well defined for Mika's scattering function, İnönü's scattering function and the linear combination of these scattering functions. This is an important result.

Albedo definitions are written for the selected surface neutron distributions. Calculations have been performed with HN (Modified FN) method and the convergent results are obtained like CN method results in the earlier different problems. The method also gives convergent numerical results for this study.

According to the tabulated values:

1 Since the net incoming neutron flux decreases by the increasing γ, the albedo values decrease by the increasing γ number. This expectation seems true according to Table 5.
2 Albedo values decrease by increasing linear anisotropic scattering coefficient. This decline is fast for the increasing b, and slowly for increasing d.
3 Albedo values decrease by increasing forward scattering in the case of certain f_1 and f_2. This is an expected result because the increasing forward scattering gives attribute to incoming flux. While albedo values increase by increasing backward scattering in the case of certain f_1 and f_2. This is an expected result because the increasing backward scattering gives attribute to outgoing flux.
4 The albedo values for the backward scattering is bigger than forward anisotropic scattering in the case of certain f_1 and f_2 because the effect of backward scattering gives the attribute to the outgoing neutron flux.

According to the results, İnönü's scattering or forward-backward scattering becomes the dominant scattering. The variation of the albedo values for the certain forward or backward scattering is changed more quickly than the variation of linear-quadratic anisotropic scattering. The linear anisotropic scattering is more dominant scattering than the quadratic anisotropic scattering for the certain forward or backward scattering. This is an expected result because the linear anisotropic scattering is proportional to the multiplication of $P_1(\mu)$ and $P_1(\mu')$, and the quadratic anisotropic scattering is proportional to the multiplication of $P_2(\mu)$ and $P_2(\mu')$.

Acknowledgments

The author thanks D. Türeci for preparing and checking computer program, D.C. Sahni for useful discussions, and Ş. H. Tanyıldızı for the corrections in English language.

References

[1] B. Davison, Neutron Transport Theory, Oxford University Press, London, 1958.
[2] M.M.R. Williams, Mathematical Methods in Particle Transport Theory, Oxford University Press, London, 1971.
[3] G.I. Bell, S. Glasstone, Nuclear Reactor Theory, Von Nostrand Reinhold, New York, 1972.
[4] K.M. Case, Elementary solutions of the transport equation and their applications, Ann. Phys. 9 (1) (1960) 1–23, https://doi.org/10.1016/0003-4916(60)90060-9.
[5] K.M. Case, P.F. Zweifel, Linear Transport Theory, Addition-Wesley, MA, 1967.
[6] C.E. Siewert, P. Benoist, The F_N method in neutron-transport theory, Part I: theory and applications, Nucl. Sci. Eng. 69 (1979) 156.
[7] P. Grandjean, C.E. Siewert, The F_N method in neutron-transport theory, Part II: applications and numerical results, Nucl. Sci. Eng. 69 (1979) 161.
[8] A. Kavenoky, The CN method of solving the transport equation: application to plane geometry, Nucl. Sci. Eng. 65 (1978) 209–225.
[9] C. Tezcan, A. Kaşkaş, M.Ç. Güleçyüz, The H-N method for solving linear transport equation: theory and Applications, J. Quant. Spectrosc. Radiat. Transfer 78 (2) (2003) 243–254.
[10] K.M. Case, F. de Hoffmann, G. Placzek, Introduction to the Theory of Neutron Diffusion, Los Alamos, New Mexico, 1953.
[11] J. Mika, Neutron transport with anisotropic scattering, Nucl. Sci. Eng. 11 (1961) 415.
[12] N.J. McCormick, I. Kuščer, Half-space neutron transport with linearly anisotropic scattering, J. Math. Phys. 6 (12) (1965) 1939.
[13] V. Protopopescu, N.G. Sjöstrand, On the solution of the dispersion equation for monoenergetic neutron transport with linearly anisotropic scattering, Prog. Nucl. Energy 7 (1) (1981) 47.
[14] A. Kavenoky, The CN Method of solving the transport equation: application to plane geometry, Nucl. Sci. Eng. 65 (1978) 209.
[15] D.C. Sahni, N.G. Sjöstrand, N.S. Garis, Criticality and time eigenvalues for one-speed neutrons in a slab with forward and backward scattering, J. Phys. D Appl. Phys. 25 (10) (1992) 1381.
[16] R.G. Türeci, D. Türeci, Time dependent albedo problem for quadratic anisotropic scattering, Kerntechnik 72 (1–2) (2007) 59.
[17] R.G. Türeci, The albedo problem for pure-quadratic anisotropic scattering with İnönü scattering, Kerntechnik 75 (5) (2010) 292.
[18] E. İnönü, A theorem on anisotropic scattering, Transp. Theory Stat. Phys. 3 (1973) 137–147.
[19] E. İnönü, Scalar and time inversion of the linear mono-energetic Boltzmann equation, Phys. Fluids 19 (1976) 1332–1335.
[20] C.E. Siewert, M.M.R. Williams, The effect of anisotropic scattering on the

critical slab problem in neutron transport theory using a synthetic kernel, J. Phys. D Appl. Phys. 10 (15) (1977) 2031–2040.
[21] M.M.R. Williams, A synthetic scattering kernel for particle transport in absorbing media with anisotropic scattering, J. Phys. D Appl. Phys. 11 (18) (1978) 2455–2463.
[22] D.C. Sahni, N.G. Sjöstrand, Criticality eigenvalues of the one-speed transport equation with strong forward-backward scattering-relationship with rod model, Transp. Theory Stat. Phys. 27 (2) (1998) 137–158.
[23] C. Devaux, C.E. Siewert, Y. Yener, The effect of forward and backward scattering on the spherical albedo for a finite plane-parallel atmosphere, J. Quant. Spectrosc. Radiat. Transfer 21 (6) (1979) 505–509.
[24] C. Tezcan, R. Sever, The critical slab with the backward-forward-isotropic scattering model, Ann. Nucl. Energy 12 (10) (1985) 573–576.
[25] M.M.R. Williams, The energy-dependent backward-forward-isotropic scattering model with some applications to the neutron transport equation, Ann. Nucl. Energy 12 (4) (1985) 167–176.
[26] C. Tezcan, C. Yildiz, The effect of forward, backward and linear anisotropic scattering on the critical slab thickness, J. Quant. Spectrosc. Radiat. Transfer 53 (6) (1995) 687–691.
[27] C. Yildiz, E. Alcan, The effect of strong anisotropic scattering on the critical sphere problem in neutron transport theory using a synthetic kernel, Ann. Nucl. Energy 22 (10) (1995) 671–679.
[28] C. Yildiz, Variation of the albedo and the transmission factor with forward and backward scattering in neutron transport theory- the FN method, Ann. Nucl. Energy 27 (9) (2000) 831–840.
[29] F.S. de Azevedo, E. Sauter, P.H.A. Konzen, M. Thompson, Numerical results for the transport equation with strongly anisotropic scattering in a slab, Ann. Nucl. Energy 79 (2015) 61–67.
[30] R.G. Türeci, D. Türeci, Half-space albedo problem with Modified FN method for linear and quadratic anisotropic scattering, Kerntechnik 82 (2) (2017) 239–245.
[31] R.G. Türeci, D. Türeci, Slab albedo for linearly and quadratically anisotropic scattering kernel with Modified FN method, Kerntechnik 82 (5) (2017) 693–699.
[32] R.G. Türeci, The Half-Space Albedo Calculations with the Linear-Quadratic Anisotropic Scattering with Singular Eigenfunction Method, vol. 13, Süleyman Demirel University Faculty of Arts and Sciences Journal of Science, 2018, pp. 144–153 (2).
[33] R.G. Türeci, Lineer-Kuadratik Anizotropik Saçılma Için Kritiklik Probleminde Yansıtıcı Özdeğerleri, vol. 14, Süleyman Demirel University Faculty of Arts and Sciences Journal of Science, 2019, pp. 1–15 (1).
[34] Mathematica Software, Wolfram researches, UK.