Research Article

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Fixed point results for multivalued mappings of Ćirić type via F-contractions on quasi metric spaces

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Abstract: In this paper, we present some fixed point results for multivalued mappings with both closed values and proximinal values on left K-complete quasi metric spaces. We also provide a nontrivial example to illustrate our results.

Keywords: fixed point, multivalued mappings, quasi metric space

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1 Introduction and preliminaries

In recent studies, many authors have provided new fixed point results to fill out gaps in the literature by taking into account different conditions on different spaces. Accordingly, in this paper, we want to complete one of these literature gaps on the fixed point theory. Therefore, we provide some new fixed point results by considering a recent contractive technique, which is called F-contraction, for multivalued mappings on some kind of complete quasi metric spaces.

Now, we recall some basic concepts of quasi metric spaces.

Let M be a nonempty set and $\rho\colon M \times M \to \mathbb{R}^+$ be a function. Consider the following conditions: for all $\zeta, \eta, \xi \in M$

- (ρ_1) $\rho(\zeta, \zeta) = 0$,
- (ρ_2) $\rho(\zeta, \eta) \leq \rho(\zeta, \xi) + \rho(\xi,\eta)$,
- (ρ_3) $\rho(\zeta, \eta) = \rho(\eta, \zeta) = 0 \Rightarrow \zeta = \eta$,
- (ρ_4) $\rho(\zeta, \eta) = 0 \Rightarrow \zeta = \eta$.

If (ρ_1),(ρ_2) and (ρ_3) hold, then function ρ is called a quasi metric on M. If a quasi metric ρ also satisfies (ρ_4), then it is called a T_1-quasi metric. We can easily see that every metric is a T_1-quasi metric and every T_1-quasi metric is a quasi metric. If ρ is a quasi metric on M, then ρ^{-1} is also a quasi metric, where

$$\rho^{-1}(\zeta, \eta) := \rho(\eta, \zeta)$$

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for $\zeta, \eta \in M$. On the other hand, if ρ is a quasi metric on M, then function ρ^s is metric on M, where

$$\rho^s(\zeta, \eta) =: \max\{\rho(\zeta, \eta), \rho^{-1}(\zeta, \eta)\}$$

for $\zeta, \eta \in M$. If ρ is a quasi metric on M, then ρ generates a T_0 topology on M. The base of this topology is the family of open balls. We will denote it by τ_ρ. If ρ is T_1-quasi metric, then τ_ρ is T_1 topology on M. The closure of a subset A of M with respect to τ_ρ, $\tau_{\rho-1}$ and τ_{ρ^s} is denoted by $cl_{\tau_\rho}(A)$, $cl_{\tau_{\rho^{-1}}}(A)$ and $cl_{\tau_{\rho^s}}(A)$, respectively.

Let (M, ρ) be a quasi metric space and $\zeta \in M$. The convergence of a sequence $\{\zeta_n\}$ to ζ with respect to τ_ρ is defined by

$$\zeta_n \xrightarrow{\rho} \zeta \Leftrightarrow \rho(\zeta, \zeta_n) \to 0.$$

We will call it ρ-convergence and denote by $\zeta_n \xrightarrow{\rho} \zeta$. Similarly, we can define ρ^{-1}-convergence and ρ^s-convergence.

Definition 1. [1,2] *Let $\{\zeta_n\}$ be a sequence in a quasi metric space (M, ρ). Then, $\{\zeta_n\}$ is called*

- *left K-Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall\, n, k, n \geq k \geq n_0, \rho(\zeta_k, \zeta_n) < \varepsilon,$$

- *right K-Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall\, n, k, n \geq k \geq n_0, \rho(\zeta_n, \zeta_k) < \varepsilon,$$

- *ρ^s-Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall\, n, k \geq n_0, \rho(\zeta_n, \zeta_k) < \varepsilon.$$

Definition 2. [1,2] *Let (M, ρ) be a quasi metric space.*

- *If every left (right) K-Cauchy sequence is ρ-convergent, then (M, ρ) is called left (right) $\mathcal{K}$-complete.*
- *If every left (right) K-Cauchy sequence is ρ^{-1}-convergent, then (M, ρ) is called left (right) $\mathcal{M}$-complete.*
- *If every left (right) K-Cauchy sequence is ρ^s-convergent, then (M, ρ) is called left (right) Smyth complete.*

Let (M, ρ) be a quasi metric space. We will consider the following family:

$$\begin{aligned}
\mathcal{P}(M) &= \{A \subseteq M\colon A \text{ is nonempty}\},\\
C_\rho(M) &= \{A \subseteq M\colon A \text{ is nonempty and } \tau_\rho\text{-closed}\},\\
\mathcal{K}_\rho(M) &= \{A \subseteq M\colon A \text{ is nonempty and } \tau_\rho\text{-compact}\}.
\end{aligned}$$

Also, we will denote the family of all subsets A of M satisfying the following property by $\mathcal{A}_\rho(M)$:

$$\text{there exists } a = a(\zeta) \in A \text{ such that } \rho(\zeta, A) = \rho(\zeta, a) \text{ for all } \zeta \in M.$$

In fact, $\mathcal{A}_\rho(M)$ is the family of all τ_ρ-proximinal subsets of M. It is clear that, if ρ is a metric on M, then $\mathcal{K}_\rho(M) \subseteq \mathcal{A}_\rho(M) \subseteq C_\rho(M) \subseteq \mathcal{P}(M)$. If ρ is a quasi metric on M, then each one of these classes is independent from each other. However, although there is no connection between these classes on quasi metric spaces, if (M, ρ) is a T_1-quasi metric space, then $\mathcal{A}_\rho(M) \subseteq C_\rho(M)$ (for more details see [3]).

We can find many fixed point results for both single valued and multivalued mappings on quasi metric spaces in the literature (see, for example, [2,4–6]).

Recently, Wardowski [7] considered the following family of functions to give more general contractive condition for the fixed point theory on metric spaces. Let $\mathcal{W}$ be the family of all functions F: $(0, \infty) \to \mathbb{R}$ satisfying the following:

(W1) For all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$.

(W2) For each sequence $\{a_n\}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.

(W3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to\infty} + \alpha^k F(\alpha) = 0$.

Many authors have extended fixed point results on metric spaces by considering family $\mathcal{W}$ (see, for example, [8–11]). Then, some fixed point results for multivalued mappings which are closed values on metric spaces have been obtained by adding the following condition (W4) (see, for example, [12–16]):

(W4) $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

It is clear that, if function F satisfies (W1), then it satisfies (W4) if and only if it is right continuous. We denote by $\mathcal{W}_\star$ the set of all functions F satisfying (W1)–(W4).

Let (M, ρ) be a quasi metric space, S: $M \to \mathcal{P}(M)$ be a multivalued mapping, $F \in \mathcal{W}$ and $\sigma \geq 0$. For $\zeta \in M$ with $\rho(\zeta, S\zeta) > 0$, define the set $F_\sigma^\zeta \subseteq M$ as

$$F_\sigma^\zeta = \{\eta \in S\zeta\colon F(\rho(\zeta, \eta)) \leq F(\rho(\zeta, S\zeta)) + \sigma\}.$$

It is obvious that, if $\sigma_1 \leq \sigma_2$, then $F_{\sigma_1}^\zeta \subseteq F_{\sigma_2}^\zeta$. There are some cases for F_σ^ζ that are investigated by Dağ et al. [3]: for $\zeta \in M$ with $\rho(\zeta, S\zeta) > 0$,

- if $S\zeta \in \mathcal{A}_\rho(M)$, then $F_\sigma^\zeta \neq \varnothing$ for all $\sigma \geq 0$,
- if $S\zeta \in \mathcal{K}_\rho(M)$, then F_σ^ζ may be empty for some $\zeta \in M$ and $\sigma > 0$,
- if $S\zeta \in C_\rho(M)$, then F_σ^ζ may be empty for some $\zeta \in M$ and $\sigma > 0$,
- if $S\zeta \in C_\rho(M)$ (even if $S\zeta \in \mathcal{P}(M)$) and $F \in \mathcal{W}_\star$, then for all $\sigma > 0$, we have $F_\sigma^\zeta \neq \varnothing$.

Now, we recall one of the aspects of development of the multivalued fixed point theory. In 1969, Nadler [17] obtained the first fixed point result for contractive-type multivalued mappings thus and so every multivalued contraction on a complete metric space has a fixed point. This result have lighted the way for new fixed point theorems for multivalued mappings. For example, Feng and Liu [18] proved the following:

Theorem 1. [18] *Let (M, ρ) be a complete metric space and S: $M \to C_\rho(M)$. Assume that there exist $b, c \in (0,1)$ such that for any $\zeta \in M$ there is $\eta \in S\zeta$ satisfying*

$$b\rho(\zeta, \eta) \leq \rho(\zeta, S\zeta)$$

and

$$\rho(\eta, S\eta) \leq c\rho(\zeta, \eta).$$

If $c < b$ and $\zeta \to \rho(\zeta, S\zeta)$ is lower semi-continuous, then S has a fixed point in M.

Now, we will order some studies, which are on fixed point results of multivalued contractive maps in the light of Feng and Liu's [18] theorem, to show their development from a complete metric space to a complete quasi metric space.

First, Minak et al. [15] extended Feng and Liu's theorem as follows:

Theorem 2. [15, Theorem 2.6] *Let (M, ρ) be a complete metric space and S: $M \to C_\rho(M)$ be a multivalued map and $F \in \mathcal{W}_\star$. Assume that there exists $\tau > 0$ such that for any $\zeta \in M$ with $\rho(\zeta, S\zeta) > 0$, there is $\eta \in F_\sigma^\zeta$ satisfying*

$$\tau + F(\rho(\eta, S\eta)) \leq F(\rho(\zeta, \eta)).$$

If $\sigma < \tau$ and $\zeta \to \rho(\zeta, S\zeta)$ is lower semi-continuous, then S has a fixed point in M.

In Theorem 2.5 of [15], Minak et al. showed that by taking $\mathcal{K}_\rho(M)$ instead of $C_\rho(M)$, condition (F4) can be removed in Theorem 2.

Then, these results are carried out on a complete quasi metric space by Dağ et al. [3] as follows:

Theorem 3. [3, Theorem 3 (resp. Theorem 1)] *Let (M, ρ) be a left $\mathcal{K}$-complete quasi (resp. T_1-quasi) metric space, $S: M \to C_\rho(M)$ (resp. $S: M \to \mathcal{A}_\rho(M)$) be a multivalued mapping and $F \in \mathcal{W}_\star$ (resp. $F \in \mathcal{W}$). Assume that there exists $\tau > 0$ such that for any $\zeta \in M$ with $\rho(\zeta, S\zeta) > 0$, there is $\eta \in F_\sigma^\zeta$ satisfying*

$$\tau + F(\rho(\eta, S\eta)) \leq F(\rho(\zeta, \eta)).$$

If $\sigma < \tau$ and $\zeta \to \rho(\zeta, S\zeta)$ is lower semi-continuous with respect to τ_ρ, then S has a fixed point in M.

In the same study, Dağ et al. [3] obtained some other results by taking into account left $\mathcal{M}$-completeness of quasi metric spaces (see Theorems 2 and 4 in [3]).

On the other hand, a different kind of generalization of Theorem 1 was presented by Klim and Wardowski [19] as follows:

Theorem 4. [19] *Let (M, ρ) be a complete metric space and $S: M \to C_\rho(M)$. Assume that the following conditions hold:*

(i) *there exists $b \in (0, 1)$ and a function $\varphi: [0, \infty) \to [0, b)$ satisfying*

$$\limsup_{t \to s^+} \varphi(t) < b, \quad \forall\, s \geq 0,$$

(ii) *for any $\zeta \in M$, there is $\eta \in S\zeta$ satisfying*

$$b\rho(\zeta, \eta) \leq \rho(\zeta, S\zeta)$$

and

$$\rho(\eta, S\eta) \leq \varphi(\rho(\zeta, \eta))\rho(\zeta, \eta).$$

Then, S has a fixed point in M provided that $\zeta \to \rho(\zeta, S\zeta)$ is lower semi-continuous.

Then, in the following theorem Altun et al. [20] provided a proper generalization of Theorem 4 by taking into account the F-contractive technique.

Theorem 5. [20, Theorem 10] *Let (M, ρ) be a complete metric space and $S: M \to C_\rho(M)$ and $F \in \mathcal{W}_\star$. Assume that the following conditions hold:*

(i) *there exists $\sigma > 0$ and a function $\tau: (0, \infty) \to (\sigma, \infty)$ such that*

$$\liminf_{t \to s^+} \tau(t) > \sigma \text{ for all } s \geq 0,$$

(ii) *for any $\zeta \in M$ with $\rho(\zeta, S\zeta) > 0$, there exists $\eta \in F_\sigma^\zeta$ satisfying*

$$\tau(\rho(\zeta, \eta)) + F(\rho(\eta, S\eta)) \leq F(\rho(\zeta, \eta)).$$

Then, S has a fixed point in M provided that $\zeta \to \rho(\zeta, S\zeta)$ is lower semi-continuous.

In the same study, Altun et al. [20] also presented Theorem 11 by taking $\mathcal{K}_\rho(M)$ instead of $C_\rho(M)$ and they removed condition (W4) on F. Then, these results are also carried out on a complete quasi metric space by Altun and Dağ [4] as follows:

Theorem 6. [4, Theorem 7 (resp. Theorem 5)] *Let (M, ρ) be a left $\mathcal{K}$-complete quasi (resp. T_1-quasi) metric space and $S: M \to C_\rho(M)$ (resp. $S: M \to \mathcal{A}_\rho(M)$) be a multivalued mapping and $F \in \mathcal{W}_\star$ (resp. $F \in \mathcal{W}$). Assume that the following conditions hold*:

(i) *there exists $\sigma > 0$ and function $\tau: (0, \infty) \to (\sigma, \infty)$ such that*

$$\liminf_{t \to s^+} \tau(t) > \sigma \text{ for all } s \geq 0$$

(ii) *for any $\zeta \in M$ with $\rho(\zeta, S\zeta) > 0$, there exists $\eta \in F_\sigma^\zeta$ satisfying*

$$\tau(\rho(\zeta, \eta)) + F(\rho(\eta, S\eta)) \leq F(\rho(\zeta, \eta)).$$

Then, S has a fixed point in M provided that $\zeta \to \rho(\zeta, S\zeta)$ is lower semi-continuous with respect to τ_ρ.

Considering the same direction, in 2009, Ćirić [21] introduced new multivalued nonlinear contractions and established a few nice fixed point theorems for such mappings, one of them is as follows:

Theorem 7. [21] *Let (M, ρ) be a complete metric space and $S: M \to C_\rho(M)$. Assume that the following conditions hold*:

(i) *there exists a function $\varphi: [0, \infty) \to [a, 1)$, $0 < a < 1$, satisfying*

$$\limsup_{t \to s^+} \varphi(t) < 1, \ \forall\, s \geq 0,$$

(ii) *for any $\zeta \in M$, there is $\eta \in S\zeta$ satisfying*

$$\sqrt{\varphi(\rho(\zeta, S\zeta))}\rho(\zeta, \eta) \leq \rho(\zeta, S\zeta)$$

and

$$\rho(\eta, S\eta) \leq \varphi(\rho(\zeta, S\zeta))\rho(\zeta, \eta).$$

Then, S has a fixed point in M provided that $\zeta \to \rho(\zeta, S\zeta)$ is lower semi-continuous.

Then, Altun et al. [14] gave a generalization of Ćirić's theorem as follows:

Theorem 8. [14, Theorem 13] *Let (M, ρ) be a complete metric space, $S: M \to C_\rho(M)$ be a mapping and $F \in \mathcal{W}_\star$. Assume that the following conditions hold*:

(i) *there exists a function $\tau: (0, \infty) \to (0, \sigma]$, $\sigma > 0$ such that*

$$\liminf_{t \to s^+} \tau(t) > 0, \ \forall\, s \geq 0;$$

(ii) *for any $\zeta \in M$ with $\rho(\zeta, S\zeta) > 0$, there is $\eta \in S\zeta$ satisfying*

$$F(\rho(\zeta, \eta)) \leq F(\rho(\zeta, S\zeta)) + \frac{\tau(\rho(\zeta, S\zeta))}{2}$$

and

$$\tau(\rho(\zeta, S\zeta)) + F(\rho(\eta, S\eta)) \leq F(\rho(\zeta, \eta)).$$

If $\zeta \to \rho(\zeta, S\zeta)$ is lower semi-continuous, then S has a fixed point in M.

In the same study, Altun et al. [14] also gave Theorem 14 by considering $\mathcal{K}_\rho(M)$ instead of $C_\rho(M)$ and so they relaxed family $\mathcal{W}_\star$ by taking into account family $\mathcal{W}$.

At this point, we shall provide some quasi metric versions of Theorem 8.

2 Main results

At the beginning, we want to emphasize the following: Let (M, ρ) be a quasi metric space, $S\colon M \to \mathcal{P}(M)$ be a multivalued mapping and $\tau\colon (0, \infty) \to (0, \sigma]$, $(\sigma > 0)$ be a function. Then,

- if $\rho(\zeta, S\zeta) > 0$, $S\zeta \in \mathcal{A}_\rho(M)$ for $\zeta \in M$ and $F \in \mathcal{W}$, then there exists $\eta \in S\zeta$ satisfying

$$F(\rho(\zeta, \eta)) \leq F(\rho(\zeta, S\zeta)) + \frac{\tau(\rho(\zeta, S\zeta))}{2}, \tag{2.1}$$

- if $\rho(\zeta, S\zeta) > 0$, $S\zeta \in C_\rho(M)$ (even if $S\zeta \in \mathcal{P}(M)$) for $\zeta \in M$ and $F \in \mathcal{W}_\star$, then there exists $\eta \in S\zeta$ satisfying (2.1).

Theorem 9. *Let (M, ρ) be a left $\mathcal{K}$-complete quasi metric space, $S\colon M \to C_\rho(M)$ and $F \in \mathcal{W}_\star$. Assume that the following conditions hold*:

(i) *the map $\zeta \to \rho(\zeta, S\zeta)$ is lower semi-continuous with respect to τ_ρ,*
(ii) *there exists a function $\tau\colon (0, \infty) \to (0, \sigma]$, $\sigma > 0$ such that*

$$\liminf_{t\to s^+}\tau(t) > 0, \ \forall\ s \geq 0, \tag{2.2}$$

(iii) *for any $\zeta \in M$ with $\rho(\zeta, S\zeta) > 0$, there is $\eta \in S\zeta$ satisfying* (2.1) *and*

$$\tau(\rho(\zeta, S\zeta)) + F(\rho(\eta, S\eta)) \leq F(\rho(\zeta, \eta)). \tag{2.3}$$

Then, S has a fixed point in M.

Proof. First, assume that S has no fixed point in M. Then, $\rho(\zeta, S\zeta) > 0$ for all $\zeta \in M$. (Note that, if $\rho(\zeta, S\zeta) = 0$ for some $\zeta \in M$, then $\zeta \in cl_{\tau_\rho}(S\zeta) = S\zeta = S\zeta$ since $S\zeta \in C_\rho(M)$.) Therefore, since $\tau(t) > 0$ for all $t > 0$ and $F \in \mathcal{W}_\star$, then for any $\zeta \in M$ there exists $\eta \in S\zeta$ such that (2.1) holds. Let $\zeta_0 \in M$ be an initial point. By assumptions (2.1) and (2.3), we can choose $\zeta_1 \in S\zeta_0$ such that

$$F(\rho(\zeta_0, \zeta_1)) \leq F(\rho(\zeta_0, S\zeta_0)) + \frac{\tau(\rho(\zeta_0, S\zeta_0))}{2} \tag{2.4}$$

and

$$\tau(\rho(\zeta_0, S\zeta_0)) + F(\rho(\zeta_1, S\zeta_1)) \leq F(\rho(\zeta_0, \zeta_1)). \tag{2.5}$$

From (2.4) and (2.5), we get

$$\frac{\tau(\rho(\zeta_0, S\zeta_0))}{2} + F(\rho(\zeta_1, S\zeta_1)) \leq F(\rho(\zeta_0, S\zeta_0)). \tag{2.6}$$

Now, we choose $\zeta_2 \in S\zeta_1$ such that

$$F(\rho(\zeta_1, \zeta_2)) \leq F(\rho(\zeta_1, S\zeta_1)) + \frac{\tau(\rho(\zeta_1, S\zeta_1))}{2}$$

and

$$\tau(\rho(\zeta_1, S\zeta_1)) + F(\rho(\zeta_2, S\zeta_2)) \leq F(\rho(\zeta_1, \zeta_2)).$$

Hence, we get

$$\frac{\tau(\rho(\zeta_1, S\zeta_1))}{2} + F(\rho(\zeta_2, S\zeta_2)) \le F(\rho(\zeta_1, S\zeta_1)).$$

Continuing this process, we can choose a sequence $\{\zeta_n\}$ such that $\zeta_{n+1} \in S\zeta_n$ satisfying

$$F(\rho(\zeta_n, \zeta_{n+1})) \le F(\rho(\zeta_n, S\zeta_n)) + \frac{\tau(\rho(\zeta_n, S\zeta_n))}{2} \tag{2.7}$$

and

$$\frac{\tau(\rho(\zeta_n, S\zeta_n))}{2} + F(\rho(\zeta_{n+1}, S\zeta_{n+1})) \le F(\rho(\zeta_n, S\zeta_n)) \tag{2.8}$$

for all $n \ge 0$.

Now, we will show that $\rho(\zeta_n, S\zeta_n) \to 0$ as $n \to \infty$. From (2.8), we conclude that $\{\rho(\zeta_n, S\zeta_n)\}$ is a decreasing sequence of positive real numbers. Therefore, there exists $\delta \ge 0$ such that

$$\lim_{n\to\infty} \rho(\zeta_n, S\zeta_n) = \delta.$$

Suppose $\delta > 0$. Then, since F is right continuous, taking the limit inferior on both sides of (2.8) and having in mind assumption (2.2), we have

$$\liminf_{\rho(\zeta_n, S\zeta_n)\to\delta^+} \frac{\tau(\rho(\zeta_n, S\zeta_n))}{2} + F(\delta) \le F(\delta),$$

which is a contradiction. Thus, $\delta = 0$, that is,

$$\lim_{n\to\infty} \rho(\zeta_n, S\zeta_n) = 0. \tag{2.9}$$

Now, we shall show that $\{\zeta_n\}$ is a left K-Cauchy sequence in ζ. Let

$$\alpha = \liminf_{\rho(\zeta_n, S\zeta_n)\to\delta^+} \frac{\tau(\rho(\zeta_n, S\zeta_n))}{2} > 0.$$

and $0 < q < \alpha$. Then, there exists $n_0 \in \mathbb{N}$ such that $\frac{\tau(\rho(\zeta_n, S\zeta_n))}{2} > q$ for all $n \ge n_0$. Thus, from (2.8),

$$q + F(\rho(\zeta_{n+1}, S\zeta_{n+1})) \le F(\rho(\zeta_n, S\zeta_n))$$

for each $n \ge n_0$. Hence, by induction, for all $n \ge n_0$

$$\begin{aligned} F(\rho(\zeta_{n+1}, S\zeta_{n+1})) &\le F(\rho(\zeta_n, S\zeta_n)) - q \\ &\vdots \\ &\le F(\rho(\zeta_{n_0}, S\zeta_{n_0})) - (n+1-n_0)q. \end{aligned} \tag{2.10}$$

Since $0 < \tau(t) \le \sigma$ for all $t > 0$. From (2.7), we get

$$F(\rho(\zeta_n, \zeta_{n+1})) \le F(\rho(\zeta_n, S\zeta_n)) + \sigma.$$

Thus, by (2.10), for all $n \ge n_0$

$$\begin{aligned} F(\rho(\zeta_n, \zeta_{n+1})) &\le F(\rho(\zeta_n, S\zeta_n)) + \sigma \\ &\le F(\rho(\zeta_{n_0}, S\zeta_{n_0})) - (n-n_0)q + \sigma. \end{aligned} \tag{2.11}$$

From (2.11), we get $\lim_{n\to\infty} F(\rho(\zeta_n,\zeta_{n+1})) = -\infty$. Thus, from (W2) we have $\lim_{n\to\infty}\rho(\zeta_n,\zeta_{n+1})) = 0$. Therefore, from (W3) there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} [\rho(\zeta_n, \zeta_{n+1})]^k F(\rho(\zeta_n, \zeta_{n+1})) = 0.$$

By (2.11), for all $n \geq n_0$

$$[\rho(\zeta_n, \zeta_{n+1})]^k F(\rho(\zeta_n, \zeta_{n+1})) - [\rho(\zeta_n, \zeta_{n+1})]^k F(\rho(\zeta_{n_0}, S\zeta_{n_0})) \leq -[\rho(\zeta_n, \zeta_{n+1})]^k [(n-n_0)q + \sigma] \leq 0. \quad (2.12)$$

Letting $n \to \infty$ in (2.12), we obtain that

$$\lim_{n\to\infty} [\rho(\zeta_n, \zeta_{n+1})]^k [(n-n_0)q + \sigma] = 0. \quad (2.13)$$

From (2.13), there exits $n_1 \in \mathbb{N}$ with $n_1 > n_0$ such that

$$[\rho(\zeta_n, \zeta_{n+1})]^k [(n-n_0)q + \sigma] \leq 1$$

for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$\rho(\zeta_n, \zeta_{n+1}) \leq \frac{1}{[(n-n_0)q + \sigma]^{\frac{1}{k}}}. \quad (2.14)$$

In order to show that $\{\zeta_n\}$ is a left K-Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the quasi metric and from (2.14), we have

$$\begin{aligned}\rho(\zeta_n, \zeta_m) &\leq \rho(\zeta_n, \zeta_{n+1}) + \rho(\zeta_{n+1}, \zeta_{n+2}) + \dots + \rho(\zeta_{m-1}, \zeta_m) = \sum_{i=n}^{m-1} \rho(\zeta_i, \zeta_{i+1}) \\ &\leq \sum_{i=n}^{\infty} \rho(\zeta_i, \zeta_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{[(i-n_0)q + \sigma]^{1/k}}.\end{aligned}$$

By the convergence of the series

$$\sum_{i > n_0 - \frac{\sigma}{q}} \frac{1}{[(i-n_0)q + \sigma]^{1/k}},$$

passing to limit $n, m \to \infty$, we get $\rho(\zeta_n, \zeta_m) \to 0$. This yields that $\{\zeta_n\}$ is a left K-Cauchy sequence in (M, ρ). Since (M, ρ) is a left K-complete quasi metric space, the sequence $\{\zeta_n\}$ is ρ-convergent to a point $\xi \in M$, that is, $\rho(\xi, \zeta_n) \to 0$ as $n \to \infty$.

On the other hand, since $\lim_{n\to\infty}\rho(\zeta_n, S\zeta_n) = -0$ and function $\zeta \to \rho(\zeta, S\zeta)$ is lower semi-continuous with respect to τ_ρ, then

$$0 \leq \rho(\xi, S\xi) \leq \liminf_{n\to\infty} \rho(\zeta_n, S\zeta_n) = 0.$$

Hence, $\rho(\xi, S\xi) = 0$, which is a contradiction. Therefore, S has a fixed point in M. □

The following result is left $\mathcal{M}$-complete version of Theorem 9.

Theorem 10. *Let (M, ρ) be a left $\mathcal{M}$-complete quasi metric space, $S\colon M \to C_\rho(M)$ and $F \in \mathcal{W}_\star$. If we replace (i′) the map $\zeta \to \rho(\zeta, S\zeta)$ is lower semi-continuous with respect to $\tau_{\rho^{-1}}$, instead of condition (i) at Theorem 9, then S has a fixed point in M.*

Proof. Assume that S has no fixed point. Like in the proof of Theorem 9, we can construct a sequence $\{\zeta_n\}$, which is left K-Cauchy. As (M,ρ) is left $\mathcal{M}$-complete, there is $\xi \in M$ such that $\{\zeta_n\}$ is $\tau_{\rho^{-1}}$-convergent to ξ, that is, $\rho(\zeta_n, \xi) \to 0$ as $n \to \infty$. Since $\zeta \to \rho(\zeta, S\zeta)$ is lower semi-continuous with respect to $\tau_{\rho^{-1}}$, we have

$$0 \leq \rho(\xi, S\xi) \leq \liminf_{n\to\infty} \rho(\zeta_n, S\zeta_n)=0.$$

Therefore, $\rho(\xi, S\xi) = 0$, which is a contradiction. Therefore, S has a fixed point in M. □

If we take $\mathcal{A}_\rho(M)$ instead of $C_\rho(M)$ in the aforementioned theorems, although we do not need condition (W4) on F, we need the space to be a T_1-quasi metric space. Notice that, if $\rho(\zeta, S\zeta) = 0$, then since $S\zeta \in \mathcal{A}_\rho(M)$, there exists $a \in S\zeta$ such that $\rho(\zeta, a) = \rho(\zeta, S\zeta) = 0$. So, $a = \zeta \in S\zeta$ because ρ is a T_1-quasi metric. Hence, the proofs of the following theorems are obvious.

Theorem 11. *Let (M, ρ) be a left $\mathcal{K}$-complete T_1-quasi metric space, S: $M \to \mathcal{A}_\rho(M)$ and $F \in \mathcal{W}$. Assume that conditions (i), (ii), and (iii) of Theorem 9 hold, then S has a fixed point in M.*

Theorem 12. *Let (M, ρ) be a left $\mathcal{M}$-complete T_1-quasi metric space, S: $M \to \mathcal{A}_\rho(M)$ and $F \in \mathcal{W}$. Assume that conditions (i′), (ii), and (iii) of Theorem 10 hold, then S has a fixed point in M.*

Now, we present a nontrivial example.

Example 1. Let $M = \left\{\frac{1}{n^2} \colon n \in \mathbb{N}\setminus\{0\}\right\} \cup \{0\}$ and

$$\rho(\zeta, \eta) = \begin{cases} \zeta - \eta, & \zeta \geq \eta, \\ \dfrac{\eta - \zeta}{2}, & \zeta < \eta. \end{cases}$$

It is clear that (M, ρ) is a left $\mathcal{K}$-complete. Let S: $M \to C_\rho(M)$ be defined by

$$S\zeta = \begin{cases} \left\{0, \dfrac{1}{(n+1)^2}\right\}, & \zeta = \dfrac{1}{n^2}, n > 1, \\ \{\zeta\}, & \zeta \in \{0, 1\}. \end{cases}$$

In this case,

$$f(\zeta) = \rho(\zeta, S\zeta) = \begin{cases} 0, & \zeta \in \{0, 1\}, \\ \dfrac{2n+1}{n^2(n+1)^2}, & \zeta = \dfrac{1}{n^2}. \end{cases}$$

Since $f^{-1}((-\infty, \alpha]) \in C_\rho(M)$ for all $\alpha \in \mathbb{R}$, then f is lower-semicontinuous with respect to τ_ρ. For $\tau(t) = \ln 2$ and $\sigma = 1$, condition (ii) is also satisfied. Now, we claim that condition (iii) is satisfied with

$$F(\alpha) = \begin{cases} \dfrac{\ln \alpha}{\sqrt{\alpha}}, & \alpha \leq 1, \\ \sqrt{\alpha - 1}, & \alpha > 1. \end{cases}$$

It can be seen that $F \in \mathcal{W}_\star$. If $\rho(\zeta, S\zeta) > 0$, then $\zeta = \frac{1}{n^2}$, $n > 1$. Therefore, we choose $\eta = \frac{1}{(n+1)^2} \in S\zeta$ and so (2.1) is clearly satisfied since $\rho(\zeta, \eta) = \rho(\zeta, S\zeta)$. Also, by standard calculation we can see that

$$\rho(\eta, S\eta)^{\frac{1}{\sqrt{\rho(\eta,S\eta)}}}\rho(\zeta,\eta)^{\frac{-1}{\sqrt{\rho(\zeta,\eta)}}} \leq \frac{1}{2}$$

and so we have

$$\ln 2 + F(\rho(\eta, S\eta)) \leq F(\rho(\zeta,\eta))$$

since the diameter of M with respect to ρ is not greater than 1. Thus, all conditions of Theorem 9 are satisfied and so S has a fixed point in M.

Now, we show that Theorem 7 cannot be applied to this example even if we consider the usual metric on M. Suppose that there exist a constant $a \in (0,1)$ and a function φ: $[0, \infty) \to a$, 1) satisfying the assumptions in Theorem 7. Take $\zeta = \frac{1}{n^2}$, then $S\zeta = \left\{0, \frac{1}{(n+1)^2}\right\}$. If $\eta = 0$, then

$$\sqrt{\varphi(\rho(\zeta, S\zeta))}\rho(\zeta,\eta) \leq \rho(\zeta, S\zeta) \Leftrightarrow \varphi\left(\frac{2n+1}{n^2(n+1)^2}\right) \leq \frac{(2n+1)^2}{(n+1)^4}.$$

Taking limit as $n \to \infty$, we have the following contradiction:

$$0 < a \leq \lim_{n\to\infty} \varphi\left(\frac{2n+1}{n^2(n+1)^2}\right) \leq 0.$$

If $\eta = \frac{1}{(n+1)^2}$, then

$$\begin{aligned}\rho(\eta, S\eta) \leq \varphi(\rho(\zeta, S\zeta))\rho(\zeta,\eta) &\Leftrightarrow \frac{2n+3}{(n+1)^2(n+2)^2} \leq \varphi\left(\frac{2n+1}{n^2(n+1)^2}\right)\frac{2n+1}{n^2(n+1)^2}\\ &\Leftrightarrow \frac{(2n+3)n^2}{(2n+1)(n+2)^2} \leq \varphi\left(\frac{2n+1}{n^2(n+1)^2}\right).\end{aligned}$$

Taking limit supremum as $n \to \infty$, we have the following contradiction:

$$1 \leq \limsup_{n\to\infty} \varphi\left(\frac{2n+1}{n^2(n+1)^2}\right) \leq \limsup_{t\to 0^+} \varphi(t) < 1.$$

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References

[1] I. L. Reilly, P. V. Subrahmanyam, and M. K. Vamanamurthy, *Cauchy sequences in quasi- pseudo-metric spaces*, Monatsh. Math. **93** (1982), 127–140.

[2] I. Altun, M. Olgun, and G. Mınak, *Classification of completeness of quasi metric space and some new fixed point results*, J. Nonlinear Sci. Appl. **22** (2017), 371–384.

[3] H. Dağ, G. Mınak, and I. Altun, *Some fixed point results for multivalued F-contractions on quasi metric spaces*, RACSAM **111** (2017), 177–187.

[4] I. Altun and H. Dağ, *Nonlinear proximinal multivalued contractions on quasi-metric spaces*, J. Fixed Point Theory Appl. **19** (2017), 2449–2460.

[5] W. Shatanawi, *Fixed and common fixed point theorems in frame of quasi metric spaces under contraction condition based on ultra distance functions*, Nonlinear Anal. Model. Control **23** (2018), 724–748.

[6] W. Shatanawi and A. Pitea, *Some coupled fixed point theorems in quasi-partial metric spaces*, Fixed Point Theory Appl. **2013** (2013), 153.

[7] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. **2012** (2012), 94.

[8] A. Al-Rawashdeh, H. Aydi, A. Felhi, S. Sahmim, and W. Shatanawi, *On common fixed points for α-F-contractions and applications*, J. Nonlinear Sci. Appl. **9** (2016), 3445–3458.

[9] M. Cosentino and P. Vetro, *Fixed point results for F-contractive mappings of Hardy-Rogers-type*, Filomat **28** (2014), 715–722.

[10] H. Qawaqneh, M. S. Noorani, and W. Shatanawi, *Fixed point results for Geraghty type generalized F-contraction for weak α-admissible mappings in metric-like spaces*, Eur. J. Pure Appl. Math. **11** (2018), 702–716.

[11] M. Sgrio and C. Vetro, *Multi-valued F-contractions and the solution of certain functional and integral equations*, Filomat **27** (2013), 1259–1268.

[12] I. Altun, G. Durmaz, G. Mınak, and S. Romaguera, *Multivalued almost F-contractions on complete metric spaces*, Filomat **30** (2016), 441–448.

[13] I. Altun, G. Mınak, and H. Dağ, *Multivalued F-contractions on complete metric space*, J. Nonlinear Convex Anal. **16** (2015), 659–666.

[14] I. Altun, M. Olgun, and G. Mınak, *On a new class of multivalued weakly Picard operators on complete metric spaces*, Taiwanese J. Math. **19** (2015), 659–672.

[15] G. Mınak, M. Olgun, and I. Altun, *A new approach to fixed point theorems for multivalued contractive maps*, Carpathian J. Math. **31** (2015), 241–248.

[16] M. Olgun, G. Mınak, and I. Altun, *A new approach to Mizoguchi-Takahashi type fixed point theorems*, J. Nonlinear Convex Anal. **17** (2016), 579–587.

[17] S. B. Nadler, *Multi-valued contraction mappings*, Pac. J. Math. **30** (1969), 475–488.

[18] Y. Feng and S. Liu, *Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings*, J. Math. Anal. Appl. **317** (2006), 103–112.

[19] D. Klim and D. Wardowski, *Fixed point theorems for set-valued contractions in complete metric spaces*, J. Math. Anal. Appl. **334** (2007), 132–139.

[20] I. Altun, G. Mınak, and M. Olgun, *Fixed points of multivalued nonlinear F-contractions on complete metric spaces*, Nonlinear Anal. Model. Control **21** (2016), 201–210.

[21] L. Ćirić, *Multi-valued nonlinear contraction mappings*, Nonlinear Anal. **71** (2009), 2716–2723.