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A Boundary Value Problem for Nonhomogeneous Vekua Equation in Wiener-type Domains

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Summary. In this article we take the nonhomogeneous Vekua equation

$$w_{\overline{z}} = Aw + B\overline{w} + F \quad , \quad z \in D$$

subject to the conditions

$$\begin{array}{ll} \operatorname{Re} w \mid_{\partial D} = \varphi & , \quad \varphi \in C^{\alpha}(\partial D) \\ \operatorname{Im} w(z_0) = c_0 & , \quad z_0 \in \overline{D}. \end{array}$$

where $A, B, F \in L_p(D), p > 2$. We want to derive the conditions under which the solution exists in Wiener-type domains.

Key words: Generalized analytic functions, solutions in Wiener sense, Wiener-type domain, capacity, non-homogeneous Vekua equations.
Classification categories: 30 G 20, 36 J 40, 35 J 60.

1. Introduction

Let us consider the boundary value problem

$$w_{\overline{z}} = Aw + B\overline{w} + F \quad , \quad z \in D \tag{1.1}$$

$$\operatorname{Re} w \mid_{\partial D} = \varphi(z) \quad , \quad z \in \partial D \tag{1.2}$$

$$\operatorname{Im} w(z_0) = c_0 \quad , \quad z_0 \in \overline{D} \tag{1.3}$$

in a domain $D \subset \mathbb{C}$ with non-smooth boundary where $A, B, F \in L_p(D)$, $p > 2$, $\varphi \in C^\alpha(\partial D)$ and c_0 is a real constant. The differential equation (1.1) is equivalent to the real system of equations

$$
\begin{aligned}
u_x - v_y &= a(x,y)\,u + b(x,y)\,v + f(x,y) \\
u_y + v_x &= c(x,y)\,u + d(x,y)\,v + g(x,y)
\end{aligned}
\tag{1.4}
$$

if we take $w = u + iv$, where

$$
\begin{aligned}
4A &= a + d + i(c - b) \\
4B &= a - d + i(c + b) \\
2F &= f + ig\ .
\end{aligned}
\tag{1.5}
$$

On the other hand, if $u, v \in C^2(D)$, $a, b, c, d, f, g \in C^1(D)$ and $b_x = -d_y$ then we may eliminate, for example v, from the system (1.4) to find

$$
Lu = H(x,y)
\tag{1.6}
$$

where

$$
L = \Delta + p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y} + k(x,y)\ ,
\tag{1.7}
$$

$$
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
$$

and

$$
H(x,y) = \nabla \cdot (f,g) := f_x + g_y\ .
$$

Thus we have deduced the boundary value problem

$$
\begin{aligned}
&Lu = H(x,y) \\
&u\,|_{\partial D} = \varphi \quad , \quad \varphi \in C^\alpha(\partial D)\ ,
\end{aligned}
\tag{1.8}
$$

in the bounded domain $D \subset \mathbb{C}$ where $x + iy \in D$. We assume there exist constants C_1, C_2 such that the coefficients of L satisfy the inequalities

$$
|p(x,y)| \quad , \quad |q(x,y)| \leqslant \tfrac{C_1}{r^\lambda} \quad , \quad -\tfrac{C_2}{r^{\lambda+1}} \leqslant k(x,y) \leqslant 0
\tag{1.9}
$$

where

$$
r = \sqrt{(x-\xi)^2 + (y-\eta)^2} \quad , \quad (x,y) \in D,\ (\xi,\eta) \notin D
\tag{1.10}
$$

and $0 \leqslant \lambda < 1$.

Definition 1.1: The real valued function $u \in C^2(D)$ satisfying the inequality $Lu \geqslant 0$ (or $Lu \leqslant 0$) is called the *subsolution* (or *supersolution*) of $Lu = 0$ where L is given by (1.7).

Let $E, T \subset \mathbb{C}$ be Borel measurable sets, r be the distance defined by (1.10) where $z = x + iy \in T$, $\zeta = \xi + i\eta \in E$. Let $\mathcal{M}$ be the set of all measures defined on the σ-algebra of all subsets of E. Let us define also the real valued function

$$h(z,\zeta) := \left[\log\left(\frac{\gamma}{r}\right)\right]^s \quad , \quad r < \gamma. \tag{1.11}$$

In (1.11), $s \in \mathbb{R}^+$ is a constant and γ is determined so that $Lh \geqslant 0$. Let us define the subset of $\mathcal{M}$ by

$$\mathcal{M}_1 := \left\{ \mu \in \mathcal{M} : \iint_E h(z,\zeta)\, d\mu(\zeta) \leqslant 1 \right\}.$$

Definition 1.2: The *logarithmic* (L,s)*-capacity* of E with respect to T is defined by

$$\mathrm{Cap}_{(L,s)}\, E := \sup_{\mu \in \mathcal{M}_1} \mu(E). \tag{1.12}$$

Now, consider the boundary value problem

$$\left.\begin{array}{lll} Lu = 0 & , \quad z \in D & \\ u\,|_{\partial D} = \varphi & , \quad \varphi \in C^{\alpha}(\partial D) & , \quad 0 < \alpha < 1 \end{array}\right\} \tag{1.13}$$

where $D \subset \mathbb{C}$ is a bounded domain with non-smooth boundary. Let us choose the set $\{D_m\}_1^{\infty}$ of domains with smooth boundaries, such that

$$\overline{D}_m \subset D_{m+1} \subset D \quad m = 1, 2, \ldots \quad , \quad \lim_{m \to \infty} D_m = D. \tag{1.14}$$

Thus we may define the boundary value problem

$$\begin{array}{lll} Lu_m = 0 & , \quad z \in D_m & \\ u_m\,|_{\partial D_m} = \Phi_{0m}(z) & , \quad \Phi_{0m} \in C^{\alpha}(\partial D_m) & , \quad m = 1, 2, \ldots \end{array} \tag{1.15}$$

in D_m which has smooth boundary, where Φ_{0m} is the restriction to the boundary ∂D_m of the Hölder continuous extension Φ_0 of φ into D. This problem has a unique solution u_m (see for example [2]). So we obtain the set of solutions $\{u_m\}_1^{\infty}$.

Definition 1.3: If

$$\lim_{m \to \infty} u_m = u_{\varphi}$$

exists, then u_{φ} is called the *generalized solution of (1.13) in Wiener sense.*

Definition 1.4: Let $z_0 \in \partial D$ be a fixed point and u_{φ} be the generalized solution of (1.13) in Wiener sense. If for each $\varphi \in C^{\alpha}(\partial D)$,

$$\lim_{z \to z_0} u_{\varphi}(z) = \varphi(z_0)$$

holds, then z_0 is called a *regular point*. Otherwise it is called as an *irregular point* of ∂D.

Definition 1.5 A domain is of *Wiener-type* if every point on its boundary is regular in Wiener sense.
Throughout the paper, we assume that the coefficients of the operator L satisfies the inequalities (1.9), r is defined by (1.10) and $B_R(z_0)$ represents the ball with center z_0 and radius R.
Now we will recall

Theorem 1.1: [3]Let us assume that the solution u of $Lu = 0$ in a bounded domain D is continuous in $\overline{D}\setminus\{z_0\}$, $z_0 \in \partial D$, bounded in D and vanishes on $\partial D \cap B_{R_0}(z_0)$. Let $E_R := B_R(z_0)\setminus D$ and $\mathrm{Cap}_{(L,s)}\, E_{4^{-m}} := K_m$ for $0 < 4^{-m} < R_0$, $m = m_0, m_0+1, \ldots$ .. If $\sum_{m=m_0}^{\infty} K_m$ is divergent, then $z_0 \in \partial D$ is a regular point in the sense of Wiener.

Definition 1.6: Let $z_0 \in \partial D$ be a fixed point and u be a subsolution defined in any $D' \subset D$, continuous in $\overline{D'}$ and satisfying $u(z) < 1$ for all $z \in D'$. If there exists a real valued function Ψ such that

$$\begin{array}{ll} (i) & \Psi(r) > 0 for 0 < r < r_0 and \lim_{r\to 0}\Psi(r) = 0 \\ (ii) & u\mid_{D\cap\sigma_1} \leqslant \Psi(r)\, whenever u\mid_{\partial D'\cap\sigma} \leqslant 0 \end{array}$$

where σ and σ_1 are two neighborhoods of z_0, then z_0 is called as Ψ*-regular point* for the boundary value problem (1.13).

Note It has been proved previously [3] that if $z_0 \in \partial D$ is a Ψ-regular point, than it is also regular in Wiener sense.

2.Existence of the real part of solutions

We will investigate the necessary conditions for the Dirichlet problem (1.8) to have a solution, when $H \in L_p(D)$, H real valued, $p > 2$. This problem may be decomposed into two new problems

$$\left.\begin{array}{lll} LV = 0 & , & z \in D \\ V\mid_{\partial D} = \varphi & , & \varphi \in C^{\alpha}(\partial D)\,; \end{array}\right\} \tag{2.1}$$

and

$$\left.\begin{array}{lll} LW = H & , & z \in D, \\ W\mid_{\partial D} = 0. & & \end{array}\right\} \tag{2.2}$$

to give the solution as $u = V + W$.

The problem (2.1) has been investigated previously [3] in Wiener-type domains. Hence we will deal with (2.2), only. If H were a continuous and bounded function in a domain D with smooth boundary, then the problem (2.2) would have solution $W \in C^2(D) \cap C(\overline{D})$. Otherwise, the classical maximum principle does not hold in general. But it is known that [3], if $H \in L_{p(\lambda)}(D)$, $2 < p(\lambda) < \frac{2}{\lambda}$, then the solutions satisfy

$$\sup_D |W| \leqslant C_3 (\operatorname{meas} D)^{\frac{1}{2}-\frac{1}{p(\lambda)}} \|H\|_{L_{p(\lambda)}(D)} . \tag{2.3}$$

Now we will discuss the generalized solutions of (2.2) in Wiener sense, in the cases where H is a bounded or unbounded function in D.

<u>Case I:</u> *H is continuous and bounded:* First of all, let us consider the domain

$$D_\rho = \{z \in D : \rho > \operatorname{dist}(z, \partial D)\} .$$

Let us choose the subdomains $\{D_k\}_1^\infty$ with smooth boundaries such that

$$D_k \subset D_{k+1} \quad , \quad \overline{D}_k \subset D_\rho \quad , \quad \lim_{k\to\infty} D_k = D_\rho .$$

So, we may define the boundary value problems

$$\begin{array}{lll} LW_k = H & , & z \in D_k \\ W_k = 0 & , & z \in \partial D_k \quad k = 1, 2, \dots . \end{array} \tag{2.4}$$

Let us define functions

$$\Phi_k^+ := e^{2A\delta} e^{A \operatorname{Re}[(1-i)(z-z_{0k})]}$$
$$\Phi_k^- := -e^{2A\delta} e^{A \operatorname{Re}[(1-i)(z-z_{0k})]}$$

where $z_{0k} \in D_\rho$ and $\lim_{k\to\infty} z_{0k} = z_0 \in D_\rho$. It is trivial that

$$\begin{array}{l} \lim_{k\to\infty} \Phi_k^+ = e^{2A\delta} e^{A \operatorname{Re}[(1-i)(z-z_0)]} =: \Phi^+ \\ \lim_{k\to\infty} \Phi_k^- = -e^{2A\delta} e^{A \operatorname{Re}[(1-i)(z-z_0)]} =: \Phi^- , \end{array}$$

δ is the diameter of D and A is a real constant to be chosen. By use of (1.9) and the fact that $r > \rho$, we can find

$$\left.\begin{array}{l} L\Phi^+ \geqslant C_4 A \\ L\Phi^- \leqslant -C_4 A \end{array}\right\} \tag{2.5}$$

where C_4 may depend on δ, ρ, C_1, C_2. On the other hand, let W_k^+ and W_k^- be the classical solutions of the boundary value problems

$$\left.\begin{array}{lll} LW_k^+ = \frac{1}{2}H & , & z \in D_k \\ W_k^+ = \Phi^+ & , & z \in \partial D_k \end{array}\right\} \tag{2.6}$$

and

$$\left.\begin{array}{lll} LW_k^- = \frac{1}{2}H & , & z \in D_k \\ W_k^- = \Phi^- & , & z \in \partial D_k \end{array}\right\} \tag{2.7}$$

respectively. Since D_k have smooth boundary, both of these problems have unique solutions. Utilizing (2.5), we find

$$L\left(W_k^+ - \Phi^+\right) \leqslant \frac{1}{2}H - C_4 A.$$

We know that H is bounded in D:

$$|H(z)| \leqslant K \quad , \quad z \in D.$$

Thus

$$L\left(W_k^+ - \Phi^+\right) \leqslant \frac{1}{2}K - C_4 A.$$

Choosing

$$A > \max\left\{1, \frac{K}{2C_4}\right\}$$

we get

$$L\left(W_k^+ - \Phi^+\right) \leqslant 0$$

in D_k. Taking into account that

$$W_k^+(z) - \Phi^+(z) = 0 \quad , \quad z \in \partial D_k,$$

the classical maximum principle leads to

$$W_k^+(z) \geqslant \Phi^+(z)$$

in D_k. Moreover

$$L\left(W_k^+ - W_{k-1}^+\right) = 0 \quad , \quad z \in D_{k-1}$$

and

$$W_k^+(z) - W_{k-1}^+(z) \geqslant 0 \quad , \quad z \in \partial D_{k-1}.$$

Then using the maximum principle in D_{k-1} we find

$$W_k^+(z) \geqslant W_{k-1}^+(z) \quad , \quad z \in \overline{D}_{k-1}.$$

Hence the sequence $\left\{W_k^+\right\}_1^\infty$ is non-decreasing. This sequence is bounded since there exists $\alpha \in \mathbb{R}$ such that

$$\begin{aligned}\sup_{D_k}\left|W_k^+(z)\right| &\leq C_5\left[\max_{D_k}\frac{1}{2}|H(z)| + \max_{\partial D_k}\left|\Phi^+(z)\right|\right] \\ &\leq \frac{1}{2}C_5 K + C_6 A \equiv \alpha\end{aligned}$$

So the sequence $\left\{W_k^+\right\}_1^\infty$ is convergent in every domain D_ρ, $\rho > 0$. In a similar way, it is easy to see that the sequence $\left\{W_k^-\right\}_1^\infty$ is also convergent in D_ρ. On the other hand, if we define

$$W_k := W_k^+ + W_k^-,$$

then W_k are solutions of the boundary value problems

$$LW_k = H\,,\ z \in D_k \tag{2.8}$$

$$W_k = 0\,,\ z \in \partial D_k\,,\ k = 1, 2, \ldots\,.$$

Because of its construction, the sequence $\{W_k^+\}_1^\infty$ is convergent. That is, there exists W defined in D_ρ such that

$$\lim_{k\to\infty} W_k\,(z) = W\,(z)\,.$$

It is well-known by the Schauder interior estimate that [2] the solutions W_k, $k = 1, 2, \ldots$ are equicontinuous together with their first and second derivatives. This means that we have a subsequence $\{W_{k_m}\}_1^\infty$ which can be substituted in (2.8). Taking the limit as $k_m \to \infty$ we find

$$\begin{aligned} LW &= H \quad , \quad z \in D_k \\ W &= 0 \quad , \quad z \in \partial D_k. \end{aligned} \tag{2.9}$$

Definition 2.1: If H is continuous and bounded in D_ρ, then the limiting function W is called *generalized solution* of (2.9).

<u>*Case II:*</u> $H \in L_p\,(D)$, $2 < p < \frac{2}{\lambda}$, $0 < \lambda < 1$: In this case, the generalized solution in Wiener sense cannot be obtained as in Case I.
First of all, let us decompose H as

$$H = H^+ + H^-$$

where

$$H^+\,(z) = \max_{z\in D}\,(H\,(z)\,, 0) \quad , \quad H^-\,(z) = \min_{z\in D}\,(H\,(z)\,, 0)\,.$$

Now, let us consider the boundary value problems

$$\left.\begin{aligned} LW_1 &= H^-\,(z) \quad , \quad z \in D \\ W_1\,(z) &= 0 \qquad\quad , \quad z \in \partial D \end{aligned}\right\} \tag{2.10}$$

and

$$\left.\begin{aligned} LW_2 &= H^+\,(z) \quad , \quad z \in D \\ W_2\,(z) &= 0 \qquad\quad , \quad z \in \partial D. \end{aligned}\right\} \tag{2.11}$$

Thus if the problems (2.10) and (2.11) have generalized solutions in Wiener sense, then the generalized solution of (2.2) in the sense of Wiener is represented by

$$W\,(z) = W_1\,(z) + W_2\,(z)\,.$$

First, let us investigate the existence of the solution of (2.10).

We know by the maximum principle that if $H^{-}(z) \leqslant 0$, then $W_1 \geqslant 0$. Now, let us define

$$H_j^{-}(z) = \begin{cases} H^{-}(z) & , \quad H^{-}(z) > -j \\ -j & , \quad H^{-}(z) \leqslant -j \end{cases} \tag{2.12}$$

for $j = 1, 2, \ldots$ and the auxiliary boundary value problems

$$\left.\begin{array}{lll} LW_j^{*}(z) = H_j^{-}(z) & , \quad z \in D & \\ W_j^{*}(z) = 0 & , \quad z \in \partial D & , \quad j = 1, 2, \ldots . \end{array}\right\} \tag{2.13}$$

Let W_j^{*}, $j = 1, 2, \ldots$ be the generalized solutions of (2.13) in Wiener sense. Thus, from (2.12) and (2.13) we have

$$L\left(W_{j+1}^{*}(z) - W_j^{*}(z)\right) = H_{j+1}^{-}(z) - H_j^{-}(z) \leqslant 0\,, \ z \in D$$

$$W_{j+1}^{*}(z) - W_j^{*}(z) = 0\,, \ z \in \partial D\,, \ j = 1, 2, \ldots .$$

Employing the classical maximum principle in D we get

$$W_{j+1}^{*}(z) \geqslant W_j^{*}(z)\,.$$

Thus the sequence $\left\{W_j^{*}\right\}$ is non-decreasing. So, there exists a constant C_7 such that the inequality

$$\begin{aligned} \sup_{z \in D}\left|W_j^{*}(z)\right| &\leqslant C_7\,|D|^{\frac{1}{2} - \frac{1}{p(\lambda)}}\left\|H_j^{-}\right\|_{L_{p(\lambda)}(D)} \\ &\leqslant C_7\,|D|^{\frac{1}{2} - \frac{1}{p(\lambda)}}\left\|H\right\|_{L_{p(\lambda)}(D)} \end{aligned} \tag{2.14}$$

holds, where

$$|D| := \operatorname{meas} D.$$

Since the right-hand side of (2.14) is independent of j, $\left\{W_j^{*}\right\}_1^{\infty}$ is bounded. Hence the limit

$$\lim_{j \to \infty} W_j^{*}(z) = W_1(z)$$

exists. This limiting function W_1 is the generalized solution of the boundary value problem (2.10) in Wiener sense.

To identify the generalized solution of (2.11) in Wiener sense, we will first define the boundary value problems

$$\left.\begin{array}{lll} LW_j^{**}(z) = H_j^{+}(z) & , \quad z \in D & \\ W_j^{**} = 0 & , \quad z \in \partial D & , \quad j = 1, 2, \ldots \end{array}\right\} \tag{2.15}$$

where

$$H_j^{+}(z) = \begin{cases} H^{+}(z) & , \quad H^{+}(z) < j \\ j & , \quad H^{+}(z) \geqslant j. \end{cases}$$

Using the same technique given above for the solutions of (2.13), we can show that the sequence $\left\{W_j^{**}\right\}_1^{\infty}$ of solutions of (2.15) is convergent. Thus the limit

$$\lim_{j\to\infty} W_j^{**}(z) = W_2(z)$$

exists in D. W_2 is the generalized solution of (2.11) in Wiener sense.
Since the boundary value problem (2.2) is linear

$$W(z) = W_1(z) + W_2(z)$$

is the generalized solution of it, in the sense of Wiener. This enables us to find the generalized solution of (1.8) in Wiener sense. Substituting the solution

$$u(z) = V(z) + W(z)$$

in the system of equations (1.4), we find

$$\begin{aligned} v_x &= c(x,y)\,u + d(x,y)\,v + g - u_y \\ v_y &= -a(x,y)\,u - b(x,y)\,v - f + u_x. \end{aligned}$$

It is easy to observe that this system is of exact differentiable type. Imposing the condition

$$\operatorname{Im} w(z_0) = v(x_0,y_0) = c_0\,,\ z_0 \in \overline{D}$$

we find a unique solution. Combining $u(x,y)$ and $v(x,y)$ as

$$w(z) = u(x,y) + iv(x,y)\,,$$

we obtain the existence of the generalized solution of (1.1)-(1.3) in Wiener sense.

3.The Representation of the Solution by T_D Operators:

It is well known [4] that the solution of the boundary value problem defined by (1.1)-(1.3) in a domain D with smooth boundary is given by

$$w(z) = \Phi(z) + T_D\left(Aw + B\overline{w} + F\right)(z)$$

where $\Phi(z)$ is a holomorphic function satisfying the conditions

$$\begin{aligned} \operatorname{Re}\Phi(z) &= \varphi(z) - \operatorname{Re} T_D\left(Aw + B\overline{w} + F\right)(z) \quad , \quad z \in \partial D \\ \operatorname{Im}\Phi(z_0) &= c_0 - \operatorname{Im} T_D\left(Aw + B\overline{w} + F\right)(z_0) \quad , \quad z_0 \in \overline{D}, \end{aligned}$$

if

$$(T_D f)(z) := -\frac{1}{\pi}\iint_D \frac{f(\zeta)}{\zeta - z}\,d\xi\,d\eta\,,\ \zeta = \xi + i\eta\,,\ f \in C^{\alpha}(D)$$

is contractive. In order to extend this result to the domains with non-smooth boundary, we will follow the technique given in [1]. First let us take the set of domains $\{D_m\}_1^{\infty}$ with smooth boundaries, subject to the conditions defined by

(1.14). Let the extension of φ, as a Hölder continuous function into the domain D, be φ_D. Then we may define the boundary value problems

$$\frac{\partial w_m}{\partial \bar{z}} = Aw_m + B\overline{w}_m + F\,,\ z \in D_m$$

$$\operatorname{Re} w_m(z)\,|_{\partial D_m} = \varphi_D(z)\,|_{\partial D_m} := \varphi_{D_m}(z) \tag{3.1}$$

$$\operatorname{Im} w_m(z_{0m}) = c_{0m}\,,\ z_{0m} \in \overline{D}_m\,,\ m = 1, 2, \ldots$$

in D_m, $m = 1, 2, \ldots$ with smooth boundaries where

$$\lim_{m\to\infty} z_{0m} = z_0 \quad , \quad \lim_{m\to\infty} c_{0m} = c_0.$$

Thus the solutions of (3.1) are represented by

$$w_m(z) = \Phi_m(z) + T_{D_m}(Aw_m + B\overline{w}_m + F)(z)\,,\ m = 1, 2, \ldots \tag{3.2}$$

if

$$\left[\|A\|_{L_p(\overline{D}_m)} + \|B\|_{L_p(\overline{D}_m)}\right] \|T_{D_m}\|_{L_p(\overline{D}_m)} \leqslant \frac{1}{K_1 + 1} \tag{3.3}$$

where K_1 is a constant, $\|\cdot\|_{L_p(\overline{D}_m)}$ is the usual norm defined in $L_p\left(\overline{D}_m\right)$ and $\Phi_m(z)$ is a holomorphic function satisfying proper boundary conditions [4]. Hence we have a sequence of functions $\{w_m\}_1^\infty$ as the solutions of the boundary value problem (3.1) in $L_p(D)$. Now we will show that $\{w_m\}_1^\infty$ is a Cauchy sequence.

Theorem 3.1: Under the conditions of (3.3), the solution sequence $\{w_m\}_1^\infty$ of the problem (3.1) is a Cauchy sequence in $L_p(\overline{D}_m)$

Proof. It is evident that $w_m, w_n \in L_p\left(\overline{D}_m\right)$ for $m < n$. If we call

$$Q_m = Aw_m + B\overline{w}_m + F,$$

then we get

$$\begin{aligned}
\|w_m - w_n\|_{L_p(\overline{D}_m)} &\leqslant \|\Phi_m - \Phi_n\|_{L_p(\overline{D}_m)} + \|T_{D_m}(Q_m) - T_{D_n}(Q_n)\|_{L_p(\overline{D}_m)} \\
&\leqslant \|\Phi_m - \Phi_n\|_{L_p(\overline{D}_m)} + \|T_{D_m}(Q_m) - T_{D_m}(Q_n)\|_{L_p(\overline{D}_m)} \\
&\quad + \|T_{D_m}(Q_n) - T_{D_n}(Q_n)\|_{L_p(\overline{D}_m)} \\
&\leqslant \|\Phi_m - \Phi_n\|_{L_p(\overline{D}_m)} + \|T_{D_m}\|_{L_p(\overline{D}_m)} \|Q_m - Q_n\|_{L_p(\overline{D}_m)} \\
&\quad + \left\|T_{D_n\setminus D_m}\right\|_{L_p(\overline{D}_n)} \|Q_n\|_{L_p(\overline{D}_n)} \\
&\leqslant \|\Phi_m - \Phi_n\|_{L_p(\overline{D}_m)} + \|T_{D_m}\|_{L_p(\overline{D}_m)} \Big[\|A\|_{L_p(\overline{D}_m)} + \\
&\quad \|B\|_{L_p(\overline{D}_m)}\Big] \|w_m - w_n\|_{L_p(\overline{D}_m)} + \left\|T_{D_n\setminus D_m}\right\|_{L_p(\overline{D}_n)} \|Q_n\|_{L_p(\overline{D}_n)}\,.
\end{aligned}$$

This inequality may be written as

$$\begin{aligned}\|w_m - w_n\|_{L_p(\overline{D}_m)} \leqslant & \frac{\|\Phi_m - \Phi_n\|_{L_p(\overline{D}_m)}}{1 - \|T_{D_m}\|_{L_p(\overline{D}_m)}\left[\|A\|_{L_p(\overline{D}_m)} + \|B\|_{L_p(\overline{D}_m)}\right]} \\ & + \frac{\left\|T_{D_n \setminus D_m}\right\|_{L_p(\overline{D}_n)} \|Q_n\|_{L_p(\overline{D}_n)}}{1 - \|T_{D_m}\|_{L_p(\overline{D}_m)}\left[\|A\|_{L_p(\overline{D}_m)} + \|B\|_{L_p(\overline{D}_m)}\right]} .\end{aligned}$$

where denominator is away from zero by (3.3). So $\{w_m\}_1^\infty$ is a Cauchy sequence.

Corollary 3.1: Thus the limit

$$\lim_{m\to\infty} w_m = w$$

exists. If we take the limit of the problem (3.1) as $m \to \infty$, we see that

$$\lim_{m\to\infty} w_m(z) = \lim_{m\to\infty} \left[\Phi_m(z) + T_{D_m}(Aw_m + B\overline{w}_m + F)(z)\right]$$

or

$$w(z) = \Phi(z) + T_D(Aw + B\overline{w} + F)(z)$$

is the representation of the solution of (1.1)-(1.3) is a Wiener-type domain.

References:

1. A. O. ÇELEBİ and K. KOCA. A boundary value problem for generalized analytic functions in Wiener-type domains. Complex Variables, 48 (6): 513–526, 2003.
2. D. GILBARG and N. S. TRUDINGER. Elliptic Partial Differential Equations of Second Order. Springer-Verlag, Berlin, Heidelberg, New York, 1983
3. K. KOCA and A. A. NOVRUZOV. Ein singuläres Randwertproblem für elliptische Differ-entialgleichungen in der Ebene. The Scientific Annals of AL. I. CUZA University of IASI, Tom. XLVI f. 2: 373–392, 2000.
4. W. TUTSCHKE. Partielle Differentialgleichungen, klassische, funktional-analytische und komplexe Methoden, volume Band 27. Tuebner Texte zur Math., 1983.