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Matematik Anabilim Dalı Onur ÖKTEN tarafından hazırlanan BERNSTEIN-STANCU-CHLODOWSKY POLİNOMLARININ YAKINSAKLIK ÖZELLİKLERİ adlı Yüksek Lisans Tezinin Anabilim Dalı standartlarına uygun olduğunu onaylarım.

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Bu tezi okuduğumu ve tezin **Yüksek Lisans Tezi** olarak bütün gereklilikleri yerine getirdiğini onaylarız.

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ÖZET

BERNSTEIN-STANCU-CHLODOWSKY POLİNOMLARININ

YAKINSAKLIK ÖZELLİKLERİ

ÖKTEN, Onur

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Bu çalışma iki bölümden oluşmaktadır.

Birinci bölümde, Bernstein-Stancu-Chlodowsky tipli operatörlerin tanımı verilmiş ve sınırsız aralıklar üzerinde f ve f' ‘nin süreklilik modülü yardımı ile yakınsaklık özellikleri incelenmiştir. Ayrıca f ‘nin konveks ve artan olması durumda operatöründe konveks ve artan olduğu gösterilmiştir.

İkinci bölümde, bu operatörün türevi verilmiş ve Lipschitz uzaylarında türevin yakınsaklığı incelenmiştir.

Anahtar kelimeler: Bernstein-Stancu polinomları, süreklilik modülü, monotonluk

ABSTRACT

CONVERGENCE PROPERTIES OF

BERNSTEIN-STANCU-CHLODOWSKY POLYNOMIALS

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This thesis consist of two section.

In first section, Bernstein-Stancu-Chlodowsky operators are given and we investigate approximation properties of these operators on unbounded intervals via the modulus of continuity of f and f'.

Also, it is shown that if f is convex and increasing then these operators are also convex and increasing.

In second section, it is given that derivative of these operators and convergence of derivative of these operators are investigated.

Key Words: Bernstein-Stancu polynomials, modulus of continuity, monotonicity

TEŞEKKÜR

Tezimin hazırlanması esnasında hiçbir yardımı esirgemeyen, biz genç araştırmacılara büyük destek olan ve tez çalışmalarım esnasında, bilimsel konularda daima yardımını gördüğüm değerli hocam, Sayın Doç. Dr. Ali ARAL'a, çalışmalarım esnasında beni daima destekleyen Kırıkkale Üniversitesi Matematik Bölümündeki değerli hocalarıma ve beni yalnız bırakmayan sevgili aileme teşekkür ederim.

İÇİNDEKİLER

Sayfa

1.BERNSTEIN-STANCU-CHLODOWSKY TİPİNDEKİ OPERATÖRLERİN YAKINSAKLIK ÖZELLİKLERİ

1.1. Giriş

Bernstein-Stancu tipli operatörlerin, Gadjiev ve Ghorbanalizadeh tarafından aşağıdaki genelleştirilmesi 2010 yılında tanımlanmıştır [1]:

$$S_{n,\alpha,\beta}(f;x)=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}f\left(\frac{r+\alpha_1}{n+\beta_1}\right)\binom{n}{r}\left(x-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r}.$$

Burada, $\frac{\alpha_2}{n+\beta_2}\leq x\leq\frac{n+\alpha_2}{n+\beta_2}$,$\alpha_k,\beta_k, k=1,2$ pozitif reel sayı ve $0\leq\alpha_2\leq\alpha_1\leq\beta_2\leq\beta_1$ dir. Bu çalışmada, Korovkin teoremi kullanılarak düzgün yakınsaklık teoremi ispatlanmış süreklilik modülü yardımı ile yakınsaklık hızı verilmiştir. Aynı makalede operatörün iki değişkenli durumu da incelenmiştir.

1937 yılında Chlodowsky, Bernstein operatörünün sınırsız aralıklar üzerine bir genelleştirmesini tanımlamıştır [2]. Bu çalışmadan ilham alarak, Gadjiev ve Ghorbanalizadeh tarafından tanımlanan operatörün genişleyen sınırsız aralık üzerine bir genelleşmesi Aral ve Acar tarafından aşağıdaki şekilde tanımlanmıştır [3]:

$$T_{n,\alpha,\beta}(f;x)=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}f\left(\alpha_3x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$\frac{\alpha_2}{n+\beta_2}b_n\leq x\leq\frac{n+\alpha_2}{n+\beta_2}b_n$ ve $(b_n)_{n\geq1}$ pozitif artan, $n\to\infty$ iken $(b_n)\to\infty$ ve $\frac{b_n}{n}\to 0$ özellikte ki bir dizidir.

Bu tezde $T_{n,\alpha,\beta}(f;x)$ operatörünün bazı şekil koruma özellikleri, yani f fonksiyonu $[0,\infty)$ aralığında artan (azalan) olduğunda, $T_{n,\alpha,\beta}(f;x)$ operatörünün de $[0,\infty)$ aralığında artan (azalan) olduğu ve f fonksiyonu $[0,\infty)$ aralığında konveks (konkav) olduğunda, $T_{n,\alpha,\beta}(f;x)$ operatörünün de $[0,\infty)$ aralığında konveks (konkav) olduğu verilmiştir.

Ayrıca $T_{n,\alpha,\beta}(f;x)$ operatörünün hem f ve f' fonksiyonlarının süreklilik modülü ile ilişkisi incelenmiş, hem de $T_{n,\alpha,\beta}(f;x)$ operatörünün ağırlıklı yaklaşım özellikleri verilmiştir.

$T_{n,\alpha,\beta}(f;x)$ operatörünün bazı özel durumları aşağıdaki gibi verilebilir:

1. $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = 0$ ve $b_n = 1$ olduğunda klasik Bernstein polinomlarını,

2. $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = 0$ olduğunda klasik Bernstein Chlodowsky tipli polinomlarını,

3. $\alpha_2 = \alpha_3 = \beta_2 = 0$ ve $b_n = 1$ olduğunda D.D. Stancu tarafından 1968 yılında yayınlanan makaledeki klasik Bernstein-Stancu polinomlarını $[4]$,

4. $\alpha_3 = 0$ ve $b_n = 1$ olduğunda A.D.Gadjiev ve A.M.Ghorbanalizadeh tarafından yayınlanan makaledeki polinomların değiştirilmiş Bernstein-Stancu halini

elde ederiz.

1.2. Temel Kavramlar

Bu bölümde tezde kullanacağımız bazı kavramları tanıtacağız.

$B_2[0,\infty)$ uzayı ile

$$|f(x)| \leq M_f\left(1+x^2\right)$$

eşitsizliğini sağlayan fonksiyonların sınıfını göstereceğiz. Burada M_f, f fonksiyonuna bağlı pozitif sabittir.

Burada $C[0,\infty)$ uzayı, $[0,\infty)$ aralığında sürekli fonksiyonların sınıfı olmak üzere

$$C_2[0,\infty) = B_2[0,\infty) \cap C[0,\infty)$$

ve

$$C_2^*[0,\infty) = \left\{ f \in C_2[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^2} < \infty \right\}$$

ise tezde kullanacağımız ağırlıklı uzaylar olarak adlandıracağımız fonksiyon sınıfıdır.

Tanım 1.2.1:

$f:[a,b] \to \Re$ herhangi bir fonksiyon ve $a \leq x_0 \leq x_1 \leq ... \leq x_n \leq b$ olsun.
Bu durumda $f[x_0] = f(x_0)$ olacak şekilde f fonksiyonuna bağlı bölünmüş farklar tanımı

$$f[x_0, x_1, ..., x_n] = \frac{f[x_1,...,x_n] - f[x_{0,} x_1,...,x_{n-1}]}{x_n - x_0}$$

eşitliği ile verilir.

Tanım 1.2.2:

h, pozitif sabit bir sayı olsun. Δ_h^k ileri fark operatörü ve f 'in tanım aralığındaki $x_0, x_1, ..., x_n$ noktaları ve her $k \in \mathrm{N}$ için,

$$\Delta_h^0 f(x_j) = f(x_j),$$

$$\Delta_h^{k+1} f(x_j) = \Delta_h^k f(x_{j+1}) - \Delta_h^k f(x_j), h \geq 0$$

eşitlikleri yardımı ile verilir.

Bu tanımı göz önünde bulundurarak ileri fark operatörünü toplam ifadesi olarak aşağıdaki şekilde verebiliriz:

$$\Delta_h^k f(x) = \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} f(x_{j+s}), (x_j = x_0 + j.h)$$

olarak gerçeklenir.

Tanım 1.2.3:

Her $x_1, x_2 \in [a,b]$ için

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda) x_2), \lambda \in [0,1]$$

eşitsizliğini sağlayan fonksiyona $[a,b]$ aralığında konveks fonksiyon denir.

Tanım 1.2.4:

X ve Y lineer uzayları birer fonksiyon uzayı ve

$$L : \underset{f}{X} \underset{\rightarrow}{\rightarrow} \underset{L(f)=g}{Y}$$

dönüşümü, L operatörü olsun.

Her $f \geq 0$ için

$$L(f) \geq 0$$

eşitsizliği sağlanıyorsa L operatörüne pozitif operatör denir.

Tanım 1.2.5:

L, lineer pozitif operatör ve her t için $f(t) \leq g(t)$ olsun. Bu durumda

$$L(f(t);x) \leq L(g(t);x)$$

eşitsizliği sağlanıyorsa L operatörüne monoton operatör denir.

Tanım 1.2.6:

Kabul edelim ki f fonksiyonu $[a,b]$ aralığında tanımlanmış sınırlı bir fonksiyon olsun. Keyfi $\delta > 0$ sayısı ve $t, x \in [a,b]$ için

$$\omega(f;\delta) = \sup_{|t-x|<\delta} |f(t) - f(x)|$$

şeklinde tanımlanan $\omega(f;\delta)$ fonksiyonuna, f fonksiyonunun $[a,b]$ aralığındaki süreklilik modülü denir.

Ayrıca süreklilik modülü tanımından aşağıdaki özellikler geçerlidir:

1. δ_1 ve δ_2 pozitif sayıları için $\delta_1 \leq \delta_2$ ise

$$\omega(f;\delta_1) \leq \omega(f;\delta_2)$$

olduğu açıktır. (Monoton özellik)

2. m bir doğal sayı olmak üzere

$$\omega(f;m\delta) \leq m\omega(f;\delta)$$

eşitsizliği sağlanır.

3. Keyfi $\lambda > 0$ reel sayısı için

$$\omega(f;\lambda\delta) \leq (1+\lambda)\omega(f;\delta)$$

eşitsizliği sağlanır.

4. Her $x,t \in [a,b]$ için

$$|f(t) - f(x)| \leq \omega(f;|t-x|)$$

eşitsizliği sağlanır.

Tanım 1.2.7:

$f:[a,b] \to \Re$ fonksiyonu verilsin. Her $\varepsilon > 0$ sayısı ve her $x, y \in [a,b]$ için $|x-y| < \delta$ olduğunda

$$|f(x) - f(y)| < \varepsilon$$

olacak şekilde $\delta = \delta(\varepsilon)$ sayıları bulunabiliyorsa f fonksiyonuna $[a,b]$ üzerinde düzgün süreklidir denir.

Tanım 1.2.8:

$L: X \to Y$ operatörü verilsin. $D(L) \subset X$, L 'nin en geniş tanım kümesi olmak üzere her $f \in D(L)$ için

$$\|L(f;x)\|_Y \leq M\|f\|_X$$

eşitsizliğini sağlayan $M \in \Re^+$ varsa L ye $D(L)$ de sınırlı operatör denir.

$$\|L\|_{X \to Y} = \inf\left\{M : \|L(f;x)\|_Y \leq M\|f\|_X\right\}$$

sayısına L operatörünün normu denir.

1.3. Bernstein-Stancu-Chlodowsky Operatörlerinin Yakınsaklık Özellikleri

Lemma 1.3.1:

Herhangi bir $k \geq 0$ tamsayısı için,

$$T^{(k)}_{n+k,\alpha,\beta}(f;x) = \frac{(n+k)!}{n!}\left(\frac{1}{b_{n+k}}\right)^{k}\left(\frac{n+k+\beta_2}{n+k}\right)^{n+k}\sum_{r=0}^{n}\Delta_h^k f\left(\frac{r+\alpha_1}{n+k+\beta_1}b_{n+k}\right)$$

$$\times\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

eşitliği geçerlidir. Burada $h = \dfrac{b_{n+k}}{n+k+\beta_1}$, f 'e bağlı ileri fark operatöründeki artma miktarıdır $[5]$.

İspat:

$$T_{n,\alpha,\beta}(f;x) = \left(\frac{n+\beta_2}{n}\right)^{n}\sum_{r=0}^{n} f\left(\alpha_3 x + \beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)$$

$$\times\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

operatöründe $\alpha_3 = 0$ ve $\beta_3 = 1$ alındığında klasik anlamdaki Bernstein-Chlodowsky operatörüne dönüşür. Bu durumda;

$$T_{n,\alpha,\beta}(f;x) = \left(\frac{n+\beta_2}{n}\right)^{n}\sum_{r=0}^{n} f\left(\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

dir. $T_{n,\alpha,\beta}(f;x)$ operatöründe $n \to n+k$ yazılırsa;

$$T_{n+k,\alpha,\beta}(f;x) = \left(\frac{n+k+\beta_2}{n+k}\right)^{n+k}\sum_{r=0}^{n+k} f\left(\frac{r+\alpha_1}{n+k+\beta_1}b_{n+k}\right)$$

$$\times \binom{n+k}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n+k-r}$$

olur. $T_{n+k,\alpha,\beta}(f;x)$ operatöründe k-kez türev alırsak

$$T^{(k)}_{n+k,\alpha,\beta}(f;x)=\left(\frac{n+k+\beta_2}{n+k}\right)^{n+k}\sum_{r=0}^{n+k} f\left(\frac{r+\alpha_1}{n+k+\beta_1}b_{n+k}\right)\binom{n+k}{r}P(x)$$

elde edilir. Burada,

$$P(x)=\frac{d^k}{dx^k}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n+k-r}$$

dır. Şimdi $P(x)$ fonksiyonuna Leibnitz kuralını uygularsak

$$f(x)=\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r \text{ ve } g(x)=\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n+k-r}$$

olmak üzere

$$f(x)=\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r$$

$$\Rightarrow \frac{d}{dx}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r=r\left(\frac{1}{b_{n+k}}\right)\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r-1}$$

$$\Rightarrow \frac{d^2}{dx^2}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r=r(r-1)\left(\frac{1}{b_{n+k}}\right)^2\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r-2}$$

$$\Rightarrow \frac{d^3}{dx^3}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r=r(r-1)(r-2)\left(\frac{1}{b_{n+k}}\right)^3\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r-3}$$

yazılabilir ve benzer şekilde devam edilirse,

$$\frac{d^s}{dx^s}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r=\begin{cases}\dfrac{r!}{(r-s)!}\left(\dfrac{1}{b_{n+k}}\right)^s\left(\dfrac{x}{b_{n+k}}-\dfrac{\alpha_2}{n+k+\beta_2}\right)^{r-s}, & r\geq s\\ 0,\ r<s\end{cases}$$

bulunur. Böylece

$$g(x)=\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n+k-r}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n+k-r}=(n+k-r)\left(\frac{-1}{b_{n+k}}\right)\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n+k-r-1}$$

$$\Rightarrow \frac{d^2}{dx^2}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n+k-r}=(n+k-r)(n+k-r-1)$$

$$\times\left(\frac{1}{b_{n+k}}\right)^2\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n+k-r-2}$$

$$\Rightarrow \frac{d^3}{dx^3}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n+k-r}=(n+k-r)(n+k-r-1)(n+k-r-2)$$

$$\times\left(\frac{-1}{b_{n+k}}\right)^3\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n+k-r-3}$$

ifadelerinden aşağıdaki eşitlik kolaylıkla görülebilir:

$$\Rightarrow \frac{d^{k-s}}{dx^{k-s}}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n+k-r}$$

$$=(n+k-r)(n+k-r-1)\ldots(n-r+s+1)\left(\frac{-1}{b_{n+k}}\right)^{k-s}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r+s}$$

$$=\left(\frac{-1}{b_{n+k}}\right)^{k-s}\frac{(n+k-r)!}{(n-r+s)!}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r+s}$$

$$=\begin{cases}\left(\frac{-1}{b_{n+k}}\right)^{k-s}\frac{(n+k-r)!}{(n-r+s)!}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r+s}, & r-s\leq n\\ 0, r-s>n\end{cases}.$$

Bu durumda;

$$P(x)=\sum_{s=0}^{k}\binom{k}{s}\frac{r!}{(r-s)!}\left(\frac{1}{b_{n+k}}\right)^{s}\frac{(n+k-r)!}{(n+s-r)!}\frac{(-1)^{k-s}}{(b_{n+k})^{k-s}}$$

$$\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r-s}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n+s-r}$$

$$=\left(\frac{1}{b_{n+k}}\right)^{k}\sum_{s=0}^{k}\binom{k}{s}\frac{r!}{(r-s)!}\frac{(n+k-r)!}{(n+s-r)!}(-1)^{k-s}$$

$$\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r-s}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n+s-r}$$

olarak bulunur. Bu ifade operatörde yerine yazılırsa

$$T_{n+k,\alpha,\beta}^{(k)}(f;x)=\left(\frac{n+k+\beta_2}{n+k}\right)^{n+k}\sum_{r=0}^{n+k}f\left(\frac{r+\alpha_1}{n+k+\beta_1}b_{n+k}\right)\binom{n+k}{r}$$

$$\times\left(\frac{1}{b_{n+k}}\right)^{k}\sum_{s=0}^{k}\binom{k}{s}\frac{r!}{(r-s)!}\frac{(n+k-r)!}{(n+s-r)!}(-1)^{k-s}$$

$$\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r-s}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n+s-r}$$

$$=\left(\frac{n+k+\beta_2}{n+k}\right)^{n+k}\sum_{r=0}^{n+k}f\left(\frac{r+\alpha_1}{n+k+\beta_1}b_{n+k}\right)\binom{n}{r-s}\frac{(n+k)!}{n!}$$

$$\times\left(\frac{1}{b_{n+k}}\right)^{k}\sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r-s}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n+s-r}$$

elde edilir. Şimdi, r yerine $r+s$ yazarsak bu durumda operatör,

$$T_{n+k,\alpha,\beta}^{(k)}(f;x)$$

$$=\left(\frac{n+k+\beta_2}{n+k}\right)^{n+k}\frac{(n+k)!}{n!}\left(\frac{1}{b_{n+k}}\right)^{k}$$

$$\times\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$\times\sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}f\left(\frac{r+s+\alpha_1}{n+k+\beta_1}b_{n+k}\right)$$

$$= \frac{(n+k)!}{n!}\left(\frac{1}{b_{n+k}}\right)^k \left(\frac{n+k+\beta_2}{n+k}\right)^{n+k} \sum_{r=0}^{n} \Delta_h^k f\left(\frac{r+\alpha_1}{n+k+\beta_1} b_{n+k}\right)$$

$$\times \binom{n}{r}\left(\frac{x}{b_{n+k}} - \frac{\alpha_2}{n+k+\beta_2}\right)^r \left(\frac{n+k+\alpha_2}{n+k+\beta_2} - \frac{x}{b_{n+k}}\right)^{n-r}$$

elde edilir.

Teorem 1.3.1:

f, $[a,b]$ aralığında konveks bir fonksiyon olması için gerek ve yeter şart f 'in ikinci merteben bölünmüş farklarının pozitif olmasıdır $[5]$.

İspat:

f 'in ikinci merteben bölünmüş farkları pozitif olsun.

Bu durumda $a \le x_0 \le x_1 \le x_2 \le b$ olmak üzere

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_1}$$

olduğundan $f[x_0, x_1, x_2] \ge 0$ olması için $f[x_1, x_2] - f[x_0, x_1] \ge 0$ olmalıdır.

Bu durumda

$$f[x_1, x_2] - f[x_0, x_1] \ge 0 \Leftrightarrow f[x_1, x_2] - f[x_0, x_1] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \ge 0$$

$$\Leftrightarrow (x_1 - x_0)(f(x_2) - f(x_1)) \ge (x_2 - x_1)(f(x_1) - f(x_0))$$

$$\Leftrightarrow (x_1 - x_0)f(x_2) + (x_2 - x_1)f(x_0) \ge (x_2 - x_0)f(x_1)$$

$$\Leftrightarrow \frac{(x_1 - x_0)}{(x_2 - x_0)} f(x_2) + \frac{(x_2 - x_1)}{(x_2 - x_0)} f(x_0) \ge f(x_1)$$

yazılabilir. $\lambda = \frac{(x_2 - x_1)}{(x_2 - x_0)}$ için $1 - \lambda = \frac{(x_1 - x_0)}{(x_2 - x_0)}$ olduğundan

$$\lambda x_0 + (1-\lambda)x_2 = x_0 \frac{(x_2 - x_1)}{(x_2 - x_0)} + x_2 \frac{(x_1 - x_0)}{(x_2 - x_0)} = x_1$$

elde edilir. Dolayısıyla

$$f[x_1, x_2] - f[x_0, x_1] \geq 0 \Leftrightarrow (1-\lambda)f(x_2) + \lambda f(x_0) \geq f(\lambda x_0 + (1-\lambda)x_2)$$

$\Leftrightarrow f, [a,b]$ aralığında konvekstir.

1.4. $T_{n,\alpha,\beta}(f;x)$ Operatörünün Şekil Koruma Özellikleri

Bu bölümde, Bernstein-Stancu-Chlodowsky operatörünün monotonluk ve konvekslik özelliklerini vereceğiz.

Teorem 1.4.1:

$f \in C^2[0,\infty)$ ve $f'(x)$, $[0,\infty)$ aralığında monoton artan olsun. Eğer $f(x)$, $[0,\infty)$ aralığında konveks(konkav) fonksiyon ise $n \geq 2$ için $T_{n,\alpha,\beta}(f;x)$ operatörü $[0,\infty)$ aralığında konveks(konkav) dır $[5]$.

İspat:

$T_{n,\alpha,\beta}(f;x)$ operatörünün önce birinci türevini hesaplarsak

$$T_{n,\alpha,\beta}(f;x) = \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} f\left(\alpha_3 x + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right) \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r}$$

olmak üzere,

$$\frac{d}{dx}T_{n,\alpha,\beta}(f;x)$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\alpha_3 f'\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}f\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\frac{r}{b_n}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r-1}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}f\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}(n-r)\left(\frac{-1}{b_n}\right)$$

$$\times\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\alpha_3 f'\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=1}^{n}f\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n-1}{r-1}\frac{n}{b_n}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r-1}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n-1}f\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n-1}{r}\left(\frac{-n}{b_n}\right)\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

yazılabilir. Şimdi ikinci toplamda r yerine $r+1$ yazarsak

$$\frac{d}{dx}T_{n,\alpha,\beta}(f;x)$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\alpha_3 f'\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)$$

$$\times\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n-1}f\left(\alpha_3 x+\beta_3\frac{r+1+\alpha_1}{n+\beta_1}b_n\right)\binom{n-1}{r}\frac{n}{b_n}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n \alpha_3 \sum_{r=0}^{n} f'\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n-1}\left[f\left(\alpha_3 x+\beta_3 \frac{r+1+\alpha_1}{n+\beta_1} b_n\right)-f\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)\right]$$

$$\times\left(\frac{n}{b_n}\right)\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

elde edilir. Benzer şekilde ikinci basamaktan türev alınırsa

$$\frac{d^2}{dx^2} T_{n,\alpha,\beta}(f;x)$$

$$=\left(\frac{n+\beta_2}{n}\right)^n \alpha_3^2 \sum_{r=0}^{n} f''\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \alpha_3 \sum_{r=0}^{n-1}\left[f'\left(\alpha_3 x+\beta_3 \frac{r+1+\alpha_1}{n+\beta_1} b_n\right)-f'\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)\right]$$

$$\times\left(\frac{n}{b_n}\right)\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \alpha_3 \sum_{r=0}^{n} f'\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)\frac{r}{b_n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r-1} \left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n-1}\left[f\left(\alpha_3 x+\beta_3 \frac{r+1+\alpha_1}{n+\beta_1} b_n\right)-f\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)\right]$$

$$\times\left(\frac{n}{b_n}\right)\binom{n-1}{r}\frac{r}{b_n}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r-1} \left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \alpha_3 \sum_{r=0}^{n} f'\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)\left(\frac{n-r}{-b_n}\right)$$

$$\times\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n-1}\left[f\left(\alpha_3 x+\beta_3\frac{r+1+\alpha_1}{n+\beta_1}b_n\right)-f\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\right]$$

$$\times\left(\frac{n}{-b_n^2}\right)\binom{n-1}{r}(n-r-1)\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-2}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n \alpha_3^2\sum_{r=0}^{n} f''\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \alpha_3\sum_{r=0}^{n}\left[f'\left(\alpha_3 x+\beta_3\frac{r+1+\alpha_1}{n+\beta_1}b_n\right)-f'\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\right]$$

$$\times\left(\frac{n}{b_n}\right)\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \alpha_3\sum_{r=1}^{n} f'\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\frac{n}{b_n}\binom{n-1}{r-1}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r-1}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=1}^{n}\left[f\left(\alpha_3 x+\beta_3\frac{r+1+\alpha_1}{n+\beta_1}b_n\right)-f\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\right]$$

$$\times\frac{n(n-1)}{b_n^2}\binom{n-2}{r-1}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r-1}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \alpha_3\sum_{r=0}^{n} f'\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\left(\frac{n}{-b_n}\right)\binom{n-1}{r}$$

$$\times\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n}\left[f\left(\alpha_3 x+\beta_3\frac{r+1+\alpha_1}{n+\beta_1}b_n\right)-f\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\right]$$

$$\times\left(\frac{n(n-1)}{-b_n^2}\right)\binom{n-2}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-2}$$

elde edilir. Şimdi üçüncü ve dördüncü toplamda r yerine $r+1$ yazalım.

Bu durumda,

$$\frac{d^2}{dx^2}T_{n,\alpha,\beta}(f;x)$$

$$=\left(\frac{n+\beta_2}{n}\right)^n \alpha_3^2 \sum_{r=0}^{n} f''\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \alpha_3 \sum_{r=0}^{n-1}\left[f'\left(\alpha_3 x+\beta_3 \frac{r+1+\alpha_1}{n+\beta_1}b_n\right)-f'\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1}b_n\right)\right]$$

$$\times\left(\frac{n}{b_n}\right)\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \alpha_3 \sum_{r=0}^{n-1} f'\left(\alpha_3 x+\beta_3 \frac{r+1+\alpha_1}{n+\beta_1}b_n\right)\frac{n}{b_n}\binom{n-1}{r}$$

$$\times\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n-2}\left[f\left(\alpha_3 x+\beta_3 \frac{r+2+\alpha_1}{n+\beta_1}b_n\right)-f\left(\alpha_3 x+\beta_3 \frac{r+1+\alpha_1}{n+\beta_1}b_n\right)\right]$$

$$\times\frac{n(n-1)}{b_n^2}\binom{n-2}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-2}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \alpha_3 \sum_{r=0}^{n-1} f'\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1}b_n\right)\left(\frac{n}{-b_n}\right)\binom{n-1}{r}$$

$$\times\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n-2}\left[f\left(\alpha_3 x+\beta_3 \frac{r+1+\alpha_1}{n+\beta_1}b_n\right)-f\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1}b_n\right)\right]$$

$$\times\frac{n(n-1)}{b_n^2}\binom{n-2}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-2}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n \alpha_3^2 \sum_{r=0}^{n} f''\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n-1}\left[2\alpha_3 f'\left(\alpha_3 x+\beta_3 \frac{r+1+\alpha_1}{n+\beta_1} b_n\right)-2\alpha_3 f'\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)\right]$$

$$\times\left(\frac{n}{b_n}\right)\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n-2}\left\{f\left(\alpha_3 x+\beta_3 \frac{r+2+\alpha_1}{n+\beta_1} b_n\right)-2f\left(\alpha_3 x+\beta_3 \frac{r+1+\alpha_1}{n+\beta_1} b_n\right)\right.$$

$$\left.+f\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)\right\}$$

$$\times\frac{n(n-1)}{b_n^2}\binom{n-2}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-2}$$

yazılabilir.

$$\tau_0=\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n, \tag{1.1}$$

$$\tau_1=\alpha_3 x+\beta_3 \frac{r+1+\alpha_1}{n+\beta_1} b_n, \tag{1.2}$$

$$\tau_2=\alpha_3 x+\beta_3 \frac{r+2+\alpha_1}{n+\beta_1} b_n \tag{1.3}$$

eşitliklerinden,

$$\frac{d}{dx} T_{n,\alpha,\beta}(f;x)$$

$$=\left(\frac{n+\beta_2}{n}\right)^n \alpha_3 \sum_{r=0}^{n} f'(\tau_0)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n-1}\left[f(\tau_1)-f(\tau_0)\right]\left(\frac{n}{b_n}\right)\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

ve

$$\frac{d^2}{dx^2}T_{n,\alpha,\beta}(f;x)$$

$$=\left(\frac{n+\beta_2}{n}\right)^n \alpha_3^2 \sum_{r=0}^{n} f''(\tau_0)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n 2\alpha_3 \sum_{r=0}^{n-1}\left[f'(\tau_1)-f'(\tau_0)\right]\left(\frac{n}{b_n}\right)\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n-2}\left[f(\tau_2)-2f(\tau_1)+f(\tau_0)\right]\frac{n(n-1)}{b_n^2}\binom{n-2}{r}$$

$$\times\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-2}$$

olarak bulunur. Tanım 1.2.1 ve (1.1),(1.2),(1.3)’den;

$$f(\tau_2)-2f(\tau_1)+f(\tau_0)=f(\tau_2)-f(\tau_1)-\left[f(\tau_1)-f(\tau_0)\right]$$

$$=f[\tau_1,\tau_2](\tau_2-\tau_1)-f[\tau_0,\tau_1](\tau_1-\tau_0)$$

$$=f[\tau_1,\tau_2]\left[\left(\alpha_3x+\beta_3\frac{r+2+\alpha_1}{n+\beta_1}b_n\right)-\left(\alpha_3x+\beta_3\frac{r+1+\alpha_1}{n+\beta_1}b_n\right)\right]$$

$$-f[\tau_0,\tau_1]\left[\left(\alpha_3x+\beta_3\frac{r+1+\alpha_1}{n+\beta_1}b_n\right)-\left(\alpha_3x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\right]$$

$$=f[\tau_1,\tau_2]\left[\left(\alpha_3x+\frac{\beta_3rb_n+2\beta_3b_n+\alpha_1\beta_3b_n}{n+\beta_1}\right)-\left(\alpha_3x+\frac{\beta_3rb_n+\beta_3b_n+\alpha_1\beta_3b_n}{n+\beta_1}\right)\right]$$

$$-f[\tau_0,\tau_1]\left[\left(\alpha_3x+\frac{\beta_3rb_n+\beta_3b_n+\alpha_1\beta_3b_n}{n+\beta_1}\right)-\left(\alpha_3x+\frac{\beta_3rb_n+\alpha_1\beta_3b_n}{n+\beta_1}\right)\right]$$

$$=f[\tau_1,\tau_2]\left(\frac{\beta_3b_n}{n+\beta_1}\right)-f[\tau_0,\tau_1]\left(\frac{\beta_3b_n}{n+\beta_1}\right)$$

$$=\left(\frac{\beta_3b_n}{n+\beta_1}\right)\left\{f[\tau_1,\tau_2]-f[\tau_0,\tau_1]\right\}$$

$$=\left(\frac{\beta_3 b_n}{n+\beta_1}\right) f[\tau_0,\tau_1,\tau_2](\tau_2-\tau_0)$$

$$=\left(\frac{\beta_3 b_n}{n+\beta_1}\right) f[\tau_0,\tau_1,\tau_2]\left[\left(\alpha_3 x+\frac{\beta_3 r b_n+2\beta_3 b_n+\alpha_1\beta_3 b_n}{n+\beta_1}\right)-\left(\alpha_3 x+\frac{\beta_3 r b_n+\alpha_1\beta_3 b_n}{n+\beta_1}\right)\right]$$

$$=\left(\frac{\beta_3 b_n}{n+\beta_1}\right) f[\tau_0,\tau_1,\tau_2]\left(\frac{2\beta_3 b_n}{n+\beta_1}\right)$$

$$=2\left(\frac{\beta_3 b_n}{n+\beta_1}\right)^2 f[\tau_0,\tau_1,\tau_2]$$

yazılabilir. Yukarıdaki eşitlik ve Teorem 1.3.1 ‘den

$$f(\tau_2)-2f(\tau_1)+f(\tau_0)\geq 0$$

olup hipotezden $f''(\tau_0)\geq 0$ dır. Ayrıca $f'(x)$ monoton artan olduğu için

$$f'(\tau_1)-f'(\tau_0)\geq 0$$

dır. Dolayısıyla

$$\frac{d^2}{dx^2}T_{n,\alpha,\beta}(f;x)\geq 0$$

olduğundan $T_{n,\alpha,\beta}(f;x)$ operatörü konvekstir.

Teorem 1.4.1 ‘de $\alpha_3=0$ alınırsa aşağıdaki sonuç kolaylıkla elde edilir. Ayrıca $\alpha_3=0$ alındığında f fonksiyonun türevlenebilir olmasına gerek yoktur.

Sonuç 1.4.1:

$f\in C[0,\infty)$ olsun. Eğer $f(x)$, $[0,\infty)$ aralığı üzerinde konveks fonksiyon ise $T_{n,\alpha,\beta}(f;x)$ operatörüde $[0,\infty)$ aralığı üzerinde konvekstir $[5]$.

Teorem 1.4.2:

Eğer $f(x)$, $[0,\infty)$ aralığı üzerinde monoton artan (ya da monoton azalan) ise $T_{n,\alpha,\beta}(f;x)$ operatörüde her $n\geq 2$ için $[0,\infty)$ aralığı üzerinde monoton artan (ya da monoton azalan) 'dır $[5]$.

İspat:

Biliyoruz ki Lemma 1.3.1 'den herhangi bir $k\geq 0$ tamsayısı için,

$$T^{(k)}_{n+k,\alpha,\beta}(f;x)=\frac{(n+k)!}{n!}\left(\frac{1}{b_{n+k}}\right)^{k}\left(\frac{n+k+\beta_2}{n+k}\right)^{n}\sum_{r=0}^{n}\Delta_h^k f\left(\frac{r+\alpha_1}{n+k+\beta_1}b_{n+k}\right)$$

$$\times\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

dir. Bu eşitlikte $k=1$ alınırsa,

$$T'_{n+1,\alpha,\beta}(f;x)=\left(\frac{n+1}{b_{n+1}}\right)\left(\frac{n+1+\beta_2}{n+1}\right)^{n}\sum_{r=0}^{n}\Delta_h^1 f\left(\frac{r+\alpha_1}{n+1+\beta_1}b_{n+1}\right)$$

$$\times\binom{n}{r}\left(\frac{x}{b_{n+1}}-\frac{\alpha_2}{n+1+\beta_2}\right)^{r}\left(\frac{n+1+\alpha_2}{n+1+\beta_2}-\frac{x}{b_{n+1}}\right)^{n-r}$$

olarak yazılabilir.

Ayrıca $\frac{\alpha_2}{n+\beta_2}b_n\leq x\leq\frac{n+\alpha_2}{n+\beta_2}b_n$ eşitsizliğinde n yerine $n+1$ yazılırsa

$$\frac{\alpha_2}{n+1+\beta_2}\leq\frac{x}{b_{n+1}}\leq\frac{n+1+\alpha_2}{n+1+\beta_2}$$

elde edilir. Buradan da

$$\frac{x}{b_{n+1}}-\frac{\alpha_2}{n+1+\beta_2}\geq 0 \text{ ve } \frac{\alpha_2+n+1}{n+1+\beta_2}-\frac{x}{b_{n+1}}\geq 0$$

olduğundan

$$\left(\frac{x}{b_{n+1}}-\frac{\alpha_2}{n+1+\beta_2}\right)^r\left(\frac{n+1+\alpha_2}{n+1+\beta_2}-\frac{x}{b_{n+1}}\right)^{n-r}\geq 0$$

elde edilir. Diğer taraftan $f(x)$ monoton artan (ya da monoton azalan) olup Tanım 1.2.2 'den $k=0$ için,

$$\Delta_h^{k+1}f(x_j)=\Delta_h^k f(x_{j+1})-\Delta_h^k f(x_j)$$

$$\Delta_h^1 f\left(\frac{r+\alpha_1}{n+1+\beta_1}b_{n+1}\right)=\Delta_h^0 f\left(\frac{r+\alpha_1}{n+2+\beta_1}b_{n+2}\right)-\Delta_h^0 f\left(\frac{r+\alpha_1}{n+1+\beta_1}b_{n+1}\right)$$

$$=f\left(\frac{r+\alpha_1}{n+2+\beta_1}b_{n+2}\right)-f\left(\frac{r+\alpha_1}{n+1+\beta_1}b_{n+1}\right)$$

$$\Delta_h^1 f\left(\frac{r+\alpha_1}{n+1+\beta_1}b_{n+1}\right)\geq 0,\left(ya\,da\,\Delta_h^1 f\left(\frac{r+\alpha_1}{n+1+\beta_1}b_{n+1}\right)\leq 0\right)$$

olarak gerçeklenir. Dolayısıyla,

$$T'_{n+1,\alpha,\beta}(f;x)\geq 0,\left(ya\,da\,T'_{n+1,\alpha,\beta}(f;x)\leq 0\right)$$

olduğundan $T_{n,\alpha,\beta}(f;x)$ operatörü monoton artan(ya da monoton azalan) 'dır.

Lemma 1.4.1:

$f\in C[0,\infty)$ olsun. Her $n\in \mathrm{N}$ ve $x\in[0,\infty)$ için aşağıdaki özellikler gerçeklenir $[5]$:

1) $T_{n,\alpha,\beta}(1;x)=1,$

2) $T_{n,\alpha,\beta}(t;x)=\alpha_3 x+\left(\frac{n+\beta_2}{n+\beta_1}\right)\beta_3 x+\left(\frac{\alpha_1-\alpha_2}{n+\beta_1}\right)\beta_3 b_n,$

3) $T_{n,\alpha,\beta}(t^2;x)=\left(\alpha_3 x+\frac{\beta_3\alpha_1}{n+\beta_1}b_n\right)^2+\beta_3\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}b_n\right)$

$$\times\left(2x\alpha_3+\frac{2\beta_3\alpha_1}{n+\beta_1}b_n+\frac{\beta_3(n-1)}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}b_n\right)+\frac{\beta_3}{n+\beta_1}b_n\right).$$

İspat:

1) $f(t)=1$ olsun. Bu durumda,

$$T_{n,\alpha,\beta}(f;x)=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n} f\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\left(\frac{n}{n+\beta_2}\right)^n$$

$$=1$$

elde edilir.

2) $f(t)=t$ olsun. Bu durumda,

$$T_{n,\alpha,\beta}(t;x)=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\left\{\sum_{r=0}^{n}\alpha_3 x\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right.$$

$$\left.+\sum_{r=0}^{n}\left(\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right\}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\left\{\alpha_3 x\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right.$$

$$+\sum_{r=1}^{n}\beta_3\left(\frac{r}{n+\beta_1}\right)b_n\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\sum_{r=0}^{n}\beta_3\left(\frac{\alpha_1}{n+\beta_1}\right)b_n\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\Bigg\}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\Bigg\{\alpha_3 x\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\beta_3\left(\frac{n}{n+\beta_1}\right)b_n\sum_{r=1}^{n}\binom{n-1}{r-1}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\beta_3\left(\frac{\alpha_1}{n+\beta_1}\right)b_n\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\Bigg\}$$

yazılabilir. İkinci toplamda r yerine $r+1$ yazıp, gerekli düzenlemeleri yaparsak

$T_{n,\alpha,\beta}(t;x)$

$$=\left(\frac{n+\beta_2}{n}\right)^n\Bigg\{\alpha_3 x\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\beta_3\left(\frac{n}{n+\beta_1}\right)b_n\sum_{r=0}^{n-1}\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r+1}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$+\beta_3\left(\frac{\alpha_1}{n+\beta_1}\right)b_n\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\Bigg\}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n$$

$$\times\Bigg\{\alpha_3 x\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\beta_3\left(\frac{n}{n+\beta_1}\right)b_n\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)\sum_{r=0}^{n-1}\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$+\beta_3\left(\frac{\alpha_1}{n+\beta_1}\right)b_n\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\Bigg\}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\left\{\alpha_3 x\left(\frac{n}{n+\beta_2}\right)^n+\beta_3\left(\frac{n}{n+\beta_1}\right)b_n\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n}{n+\beta_2}\right)^{n-1}\right.$$

$$\left.+\beta_3\left(\frac{\alpha_1}{n+\beta_1}\right)b_n\left(\frac{n}{n+\beta_2}\right)^n\right\}$$

$$=\alpha_3 x+\beta_3\frac{n}{n+\beta_1}b_n\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n+\beta_2}{n}\right)+\beta_3\frac{\alpha_1}{n+\beta_1}b_n$$

$$=\alpha_3 x+\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x\beta_3-\frac{\alpha_2 b_n\beta_3}{n+\beta_2}\right)+\beta_3\frac{\alpha_1}{n+\beta_1}b_n$$

$$=\alpha_3 x+\left(\frac{n+\beta_2}{n+\beta_1}\right)\beta_3 x+\left(\frac{\alpha_1-\alpha_2}{n+\beta_1}\right)\beta_3 b_n$$

elde ederiz.

3) $f(t)=t^2$ olsun. Bu durumda,

$T_{n,\alpha,\beta}(t^2;x)$

$$=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)^2\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left[\alpha_3^2x^2+2x\alpha_3\beta_3 b_n\left(\frac{r+\alpha_1}{n+\beta_1}\right)+\beta_3^2 b_n^2\left(\frac{r+\alpha_1}{n+\beta_1}\right)^2\right]$$

$$\times\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\left\{\sum_{r=0}^{n}\alpha_3^2x^2\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right.$$

$$+\sum_{r=0}^{n} 2x\alpha_3\beta_3 b_n\left(\frac{r+\alpha_1}{n+\beta_1}\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$\left.+\sum_{r=0}^{n} \beta_3^2 b_n^2\left(\frac{r+\alpha_1}{n+\beta_1}\right)^2\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right\}$$

$$=\left(\frac{n+\beta_2}{n}\right)^{n}\left\{\alpha_3^2 x^2\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right.$$

$$+2x\alpha_3\beta_3 b_n\sum_{r=1}^{n}\left(\frac{r}{n+\beta_1}\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+2x\alpha_3\beta_3 b_n\sum_{r=0}^{n}\left(\frac{\alpha_1}{n+\beta_1}\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$\left.+\sum_{r=0}^{n}\beta_3^2 b_n^2\frac{\left(r^2+2\alpha_1 r+\alpha_1^2\right)}{\left(n+\beta_1\right)^2}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right\}$$

yazılabilir. Gerekli sadeleştirmeler ve düzenlemeler yapılırsa

$T_{n,\alpha,\beta}\left(t^2;x\right)$

$$=\left(\frac{n+\beta_2}{n}\right)^{n}\left\{\alpha_3^2 x^2\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right.$$

$$+2x\alpha_3\beta_3 b_n\left(\frac{n}{n+\beta_1}\right)\sum_{r=1}^{n}\binom{n-1}{r-1}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+2x\alpha_3\beta_3 b_n\left(\frac{\alpha_1}{n+\beta_1}\right)\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$\left.+\sum_{r=0}^{n}\beta_3^2 b_n^2\frac{\left(r^2+2\alpha_1 r+\alpha_1^2\right)}{\left(n+\beta_1\right)^2}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right\}$$

$$=\left(\frac{n+\beta_2}{n}\right)^{n}\left\{\alpha_3^2 x^2\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right.$$

$$+2x\alpha_3\beta_3 b_n\left(\frac{n}{n+\beta_1}\right)\sum_{r=1}^{n}\binom{n-1}{r-1}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+2x\alpha_3\beta_3 b_n\left(\frac{\alpha_1}{n+\beta_1}\right)\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\sum_{r=0}^{n}\beta_3^2 b_n^2\frac{r^2}{(n+\beta_1)^2}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\sum_{r=0}^{n}\beta_3^2 b_n^2\frac{2\alpha_1 r}{(n+\beta_1)^2}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$\left.+\sum_{r=0}^{n}\beta_3^2 b_n^2\frac{\alpha_1^2}{(n+\beta_1)^2}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right\}$$

elde edilir. Yukarıdaki eşitliğin sağ tarafındaki dördüncü toplamda r ekleyip çıkarırsak

$$T_{n,\alpha,\beta}\left(t^2;x\right)$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\left\{\alpha_3^2x^2\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right.$$

$$+2x\alpha_3\beta_3 b_n\left(\frac{n}{n+\beta_1}\right)\sum_{r=1}^{n}\binom{n-1}{r-1}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+2x\alpha_3\beta_3 b_n\left(\frac{\alpha_1}{n+\beta_1}\right)\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\frac{\beta_3^2 b_n^2}{(n+\beta_1)^2}\sum_{r=0}^{n}\left(r^2-r+r\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\frac{\beta_3^2 b_n^2}{(n+\beta_1)^2}2\alpha_1\sum_{r=0}^{n}r\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$\left.+\beta_3^2 b_n^2\frac{\alpha_1^2}{(n+\beta_1)^2}\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right\}$$

elde ederiz. Gerekli düzenlemeler yapılırsa

$$T_{n,\alpha,\beta}\left(t^2;x\right)$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\Bigg\{\alpha_3^2x^2\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+2x\alpha_3\beta_3b_n\left(\frac{n}{n+\beta_1}\right)\sum_{r=1}^{n}\binom{n-1}{r-1}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+2x\alpha_3\beta_3b_n\left(\frac{\alpha_1}{n+\beta_1}\right)\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\frac{\beta_3^2b_n^2}{\left(n+\beta_1\right)^2}\sum_{r=0}^{n}r\left(r-1\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\frac{\beta_3^2b_n^2}{\left(n+\beta_1\right)^2}\sum_{r=0}^{n}r\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\frac{\beta_3^2b_n^2}{\left(n+\beta_1\right)^2}2\alpha_1\sum_{r=0}^{n}r\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\beta_3^2b_n^2\frac{\alpha_1^2}{\left(n+\beta_1\right)^2}\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\Bigg\}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\Bigg\{\alpha_3^2x^2\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+2x\alpha_3\beta_3b_n\left(\frac{n}{n+\beta_1}\right)\sum_{r=1}^{n}\binom{n-1}{r-1}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+2x\alpha_3\beta_3b_n\left(\frac{\alpha_1}{n+\beta_1}\right)\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\frac{\beta_3^2b_n^2}{\left(n+\beta_1\right)^2}\sum_{r=0}^{n}r\left(r-1\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\frac{\beta_3^2 b_n^2(1+2\alpha_1)}{(n+\beta_1)^2}\sum_{r=0}^{n} r\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$\left.+\beta_3^2 b_n^2\frac{\alpha_1^2}{(n+\beta_1)^2}\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right\}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\left\{\alpha_3^2 x^2\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right.$$

$$+2x\alpha_3\beta_3 b_n\left(\frac{n}{n+\beta_1}\right)\sum_{r=1}^{n}\binom{n-1}{r-1}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+2x\alpha_3\beta_3 b_n\left(\frac{\alpha_1}{n+\beta_1}\right)\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\frac{\beta_3^2 b_n^2 n(n-1)}{(n+\beta_1)^2}\sum_{r=2}^{n}\binom{n-2}{r-2}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\frac{n\beta_3^2 b_n^2(1+2\alpha_1)}{(n+\beta_1)^2}\sum_{r=1}^{n}\binom{n-1}{r-1}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$\left.+\beta_3^2 b_n^2\frac{\alpha_1^2}{(n+\beta_1)^2}\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right\}$$

elde edilir. Şimdi ikinci ve beşinci toplamda r yerine $r+1$, dördüncü toplamda r yerine $r+2$ yazılırsa

$$T_{n,\alpha,\beta}(t^2;x)$$

$$=\left(\frac{n+\beta_2}{n}\right)^n$$

$$\times\left\{\alpha_3^2 x^2\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right.$$

$$+2x\alpha_3\beta_3 b_n\left(\frac{n}{n+\beta_1}\right)\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)\sum_{r=0}^{n-1}\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$+2x\alpha_3\beta_3 b_n\left(\frac{\alpha_1}{n+\beta_1}\right)\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\frac{\beta_3^2 b_n^2 n(n-1)}{(n+\beta_1)^2}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^2\sum_{r=0}^{n-2}\binom{n-2}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-2}$$

$$+\frac{n\beta_3^2 b_n^2(1+2\alpha_1)}{(n+\beta_1)^2}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)\sum_{r=0}^{n-1}\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$\left.+\beta_3^2 b_n^2\frac{\alpha_1^2}{(n+\beta_1)^2}\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right\}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\left\{\alpha_3^2x^2\left(\frac{n}{n+\beta_2}\right)^n+2x\alpha_3\beta_3 b_n\frac{n}{n+\beta_1}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n}{n+\beta_2}\right)^{n-1}\right.$$

$$+2x\alpha_3\beta_3 b_n\left(\frac{\alpha_1}{n+\beta_1}\right)\left(\frac{n}{n+\beta_2}\right)^n$$

$$+\frac{\beta_3^2 b_n^2 n(n-1)}{(n+\beta_1)^2}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^2\left(\frac{n}{n+\beta_2}\right)^{n-2}$$

$$+\frac{n\beta_3^2 b_n^2(1+2\alpha_1)}{(n+\beta_1)^2}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n}{n+\beta_2}\right)^{n-1}$$

$$\left.+\beta_3^2 b_n^2\frac{\alpha_1^2}{(n+\beta_1)^2}\left(\frac{n}{n+\beta_2}\right)^n\right\}$$

elde edilir. Bu eşitlik ile

$$T_{n,\alpha,\beta}(t^2;x)$$

$$=\alpha_3^2x^2+2x\alpha_3\beta_3 b_n\left(\frac{n}{n+\beta_1}\right)\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n+\beta_2}{n}\right)$$

$$+2x\alpha_3\beta_3 b_n\left(\frac{\alpha_1}{n+\beta_1}\right)+\frac{\beta_3^2 b_n^2 n(n-1)}{(n+\beta_1)^2}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^2\left(\frac{n+\beta_2}{n}\right)^2$$

$$+\frac{n\beta_3^2 b_n^2(1+2\alpha_1)}{(n+\beta_1)^2}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n+\beta_2}{n}\right)+\beta_3^2 b_n^2\frac{\alpha_1^2}{(n+\beta_1)^2}$$

$$=\left(\alpha_3 x+\frac{\beta_3\alpha_1}{n+\beta_1}b_n\right)^2+\beta_3\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}b_n\right)$$

$$\times\left(2x\alpha_3+\frac{2\beta_3\alpha_1}{n+\beta_1}b_n+\frac{\beta_3(n-1)}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}b_n\right)+\frac{\beta_3}{n+\beta_1}b_n\right)$$

yazılabilir.

Teorem 1.4.3:

$f\in C_2[0,\infty)$ olsun. Bu durumda sonlu herhangi bir $[0,A]$ aralığı üzerinde,

$$\lim_{n\to\infty}T_{n,\alpha,\beta}(f;x)=f(x)$$

dir $[5]$.

İspat:

$f\in C_2[0,\infty)$ olduğu için, öyle bir sabit $c>0$ pozitif sayısı vardır ki $\frac{|f(x)|}{1+x^2}<c$ eşitsizliği sağlanır. Aynı zamanda Lemma 1.4.1 'den açıktır ki,

$T_{n,\alpha,\beta}(1;x)-1=0$,

$$\max_{0\le x\le A}\left|T_{n,\alpha,\beta}(t;x)-x\right|=A\left(\alpha_3+\left(\frac{n+\beta_2}{n+\beta_1}\right)\beta_3-1\right)+\left(\frac{\alpha_1-\alpha_2}{n+\beta_1}\right)\beta_3 b_n,$$

$$\max_{0\le x\le A}\left|T_{n,\alpha,\beta}(t^2;x)-x^2\right|$$

$$=A^2(\alpha_3^2-1)+2A\alpha_3\beta_3\frac{\alpha_1}{n+\beta_1}b_n+\left(\frac{\beta_3\alpha_1}{n+\beta_1}b_n\right)^2$$

$$+\beta_3\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(A-\frac{\alpha_2}{n+\beta_2}b_n\right)$$

$$\times\left(2A\alpha_3+\frac{2\beta_3\alpha_1}{n+\beta_1}b_n+\frac{\beta_3(n-1)}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(A-\frac{\alpha_2}{n+\beta_2}b_n\right)+\frac{\beta_3}{n+\beta_1}b_n\right)$$

dir. Dolayısıyla; yukarıdaki eşitliklerde, $n\to\infty$ iken $\frac{b_n}{n}\to 0$ ve $\alpha_3+\beta_3=1$

şartları göz önünde tutulursa,

$$\lim_{n\to\infty}\max_{0\le x\le A}\left|T_{n,\alpha,\beta}(1;x)-1\right|=0, \tag{1.4}$$

$$\lim_{n\to\infty}\max_{0\le x\le A}\left|T_{n,\alpha,\beta}(t;x)-x\right|=0, \tag{1.5}$$

$$\lim_{n\to\infty}\max_{0\le x\le A}\left|T_{n,\alpha,\beta}(t^2;x)-x^2\right|=0 \tag{1.6}$$

elde edilir. Şimdi $T_{n,\alpha,\beta}(f;x)$ operatörünü dikkate alarak bir $T_n(f;x)$ dizisi

tanımlayalım:

$$T_n(f;x)=\begin{cases} f(x) & ,0\le x\le \frac{\alpha_2}{n+\beta_2}b_n \\ T_{n,\alpha,\beta}(f;x) & ,\frac{\alpha_2}{n+\beta_1}b_n\le x\le \frac{n+\alpha_2}{n+\beta_2}b_n \\ f(x) & ,\frac{n+\alpha_2}{n+\beta_2}b_n\le x<\infty \end{cases}$$

dir. Ayrıca bu diziyi;

$$T_n(f;x)=\begin{cases} f(x) & ,0\le x\le \frac{\alpha_2}{n+\beta_2}b_n \\ T_{n,\alpha,\beta}(f;x) & ,\frac{\alpha_2}{n+\beta_1}b_n\le x\le A \end{cases}$$

şeklinde de tanımlayabiliriz. Açıktır ki;

$$\left\|T_n(f;x)-f\right\|_{C[0,A]}=\max_{\frac{\alpha_2}{n+\beta_1}b_n\le x\le\frac{n+\alpha_2}{n+\beta_2}b_n}\left|T_{n,\alpha,\beta}(f;x)-f(x)\right| \tag{1.7}$$

olup (1.4),(1.5) ve (1.6)'dan

$$\lim_{n\to\infty} \left\| T_n(1;x) - 1 \right\|_{C[0,A]} = 0 \ ,$$

$$\lim_{n\to\infty} \left\| T_n(t;x) - x \right\|_{C[0,A]} = 0 \ ,$$

$$\lim_{n\to\infty} \left\| T_n\left(t^2;x\right) - x^2 \right\|_{C[0,A]} = 0$$

eşitlikleri sağlanır. Korovkin Teoremi ‘nin bütün şartları sağlandığından, her sürekli f fonksiyonu için;

$$\lim_{n\to\infty} \left\| T_n f - f \right\|_{C[0,A]} = 0 \tag{1.8}$$

dır. Dolayısıyla (1.7) ve (1.8) ‘den

$$\lim_{n\to\infty} \max_{\frac{\alpha_2}{n+\beta_1} b_n \le x \le A_n} \left| T_{n,\alpha,\beta}(f;x) - f \right| = 0$$

olup ispat tamamlanır.

Teorem 1.4.4:

$f \in C_2[0,\infty)$ olsun. Bu durumda herhangi bir $A > 0$ ve her $x \in [0, A]$ için,

$$\left| T_{n,\alpha,\beta}(f;x) - f(x) \right| \le C\omega_{1+A}\left(f : \sqrt{\frac{b_n}{n}} \right)$$

eşitsizliği sağlanır. Burada C, n ’den bağımsız bir sabittir, $\omega_{1+A}(f;.)$ ise f fonksiyonunun $[0,1+A]$ aralığındaki süreklilik modülüdür $[5]$.

İspat:

$A > 0$ ve $x \in [0, A]$ için,

$$E_r' = \left\{ r : \alpha_3 x + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \ge 1 + A \right\},$$

$$E_r'' = \left\{ r : \alpha_3 x + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \le 1 + A \right\}$$

olacak şekilde iki nokta cümlesini dikkate alalım. Lemma 1.4.1'i ve $T_{n,\alpha,\beta}(f;x)$ operatörünün lineerlik kuralından,

$$\left| T_{n,\alpha,\beta}(f;x) - f(x) \right|$$

$$= \left| \left(\frac{n+\beta_2}{n} \right)^n \sum_{r=0}^{n} f\left(\alpha_3 x + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right) \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2} \right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n} \right)^{n-r} - 1.f(x) \right|$$

$$= \left| \left(\frac{n+\beta_2}{n} \right)^n \sum_{r=0}^{n} f\left(\alpha_3 x + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right) \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2} \right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n} \right)^{n-r} \right.$$

$$\left. - \left(\frac{n+\beta_2}{n} \right)^n \sum_{r=0}^{n} f(x) \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2} \right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n} \right)^{n-r} \right|$$

$$= \left| \left(\frac{n+\beta_2}{n} \right)^n \sum_{r=0}^{n} \left[f\left(\alpha_3 x + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right) - f(x) \right] \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2} \right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n} \right)^{n-r} \right|$$

$$\le \left(\frac{n+\beta_2}{n} \right)^n \sum_{r \in E_r'}^{n} \left| f\left(\alpha_3 x + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right) - f(x) \right| \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2} \right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n} \right)^{n-r}$$

$$+ \left(\frac{n+\beta_2}{n} \right)^n \sum_{r \in E_r''}^{n} \left| f\left(\alpha_3 x + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right) - f(x) \right| \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2} \right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n} \right)^{n-r}$$

$$\le I_n' + I_n''$$

elde edilir. $f \in C[0,\infty)$ olduğundan öyle bir C_1 pozitif sabit bir sayı vardır ki

$$|f(x)| \le C_1, x \in [0, A] \tag{1.9}$$

eşitsizliği sağlanır.

$f \in B_2[0,\infty)$ ve de $x \in [0, A]$ olduğundan öyle bir C_2 pozitif sabit bir sayı vardır ki

$$\left|f\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)\right| \leq C_2\left[1+\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)^2\right] \tag{1.10}$$

eşitsizliği gerçeklenir.

(1.9) ve (1.10) eşitsizlikleri kullanılırsa aşağıdaki eşitsizlik;

$$\left|f\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)-f(x)\right| \leq\left|f\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)\right|+|f(x)|$$

$$\leq C_2\left[1+\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)^2\right]+C_1$$

$$\leq C_3\left[1+\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)^2+1\right]$$

$$\leq C_3\left[2+\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)^2\right]$$

$C_3=\max\{C_1, C_2\}$ olmak üzere elde edilir. Diğer taraftan $r \in E_r'$ ve $x \in [0, A]$

için açıktır ki;

$$\left|\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n-x\right| \geq 1$$

dir. Bu durumda,

$$\left|f\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)-f(x)\right|$$

$$\leq C_3\left[2+\left(\alpha_3 x+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n-x+x\right)^2\right]$$

$$\leq C_3\left[\left[x(\alpha_3-1)+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right]^2+2x\left[x(\alpha_3-1)+\beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right]+x^2+2\right]$$

$$\leq C_3\left[x(\alpha_3-1)+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right]^2\left(4+4x+x^2\right)$$

$$\leq C_3\left[x(\alpha_3-1)+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right]^2(A+2)^2$$

$$\leq C_4\left[x(\alpha_3-1)+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right]^2$$

$C_4=C_3(A+2)^2$ olmak üzere elde edilir. Yukarıda ki eşitsizliğin her iki tarafına $T_{n,\alpha,\beta}(f;x)$ operatörünü uygularsak

$$\left|T_{n,\alpha,\beta}(f;x)-f(x)\right|$$

$$\leq C_4\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left[x(\alpha_3-1)+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right]^2\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$\leq C_4\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left(\alpha_3x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)^2\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$-2xC_4\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left(\alpha_3x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+x^2C_4\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$\leq C_4\left\{T_{n,\alpha,\beta}\left(t^2;x\right)-2xT_{n,\alpha,\beta}(t;x)+x^2T_{n,\alpha,\beta}(1;x)\right\}$$

elde ederiz. Lemma 1.4.1 ‘den,

$$\left|T_{n,\alpha,\beta}(f;x)-f(x)\right|$$

$$\leq C_4\left\{\left(\alpha_3x+\frac{\beta_3\alpha_1}{n+\beta_1}b_n\right)^2\right.$$

$$+\beta_3\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}b_n\right)$$

$$\times\left(2x\alpha_3+\frac{2\beta_3\alpha_1}{n+\beta_1}b_n+\frac{\beta_3}{n+\beta_1}b_n+\frac{\beta_3(n-1)}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}b_n\right)\right)$$

$$-2x\left[\alpha_3x+\left(\frac{n+\beta_2}{n+\beta_1}\right)\beta_3x+\left(\frac{\alpha_1-\alpha_2}{n+\beta_1}\right)\beta_3b_n\right]+x^2\Bigg\}$$

$$\leq C_4\Bigg\{\alpha_3^2x^2+2\alpha_1\alpha_3\beta_3x\frac{b_n}{n+\beta_1}$$

$$+\beta_3^2\alpha_1^2\frac{b_n^2}{(n+\beta_1)^2}$$

$$+\beta_3\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}b_n\right)$$

$$\times\left(2x\alpha_3+\frac{2\beta_3\alpha_1}{n+\beta_1}b_n+\frac{\beta_3}{n+\beta_1}b_n+\frac{\beta_3(n-1)}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}b_n\right)\right)$$

$$-2x\left[\alpha_3x+\left(\frac{n+\beta_2}{n+\beta_1}\right)\beta_3x+\left(\frac{\alpha_1-\alpha_2}{n+\beta_1}\right)\beta_3b_n\right]+x^2\Bigg\}$$

elde edilir. $0\leq\alpha_2\leq\alpha_1\leq\beta_2\leq\beta_1$ ve $\alpha_3+\beta_3=1$ ifadelerinden

$$\left|T_{n,\alpha,\beta}(f;x)-f(x)\right|$$

$$\leq C_4\Bigg\{2x\alpha_1\frac{b_n}{n+\beta_2}+x\frac{b_n}{n+\beta_2}(1+2\alpha_1)+2x\frac{\alpha_2b_n}{n+\beta_2}$$

$$+2x\frac{\alpha_1b_n}{n+\beta_2}+\alpha_2^2\left(\frac{b_n}{n+\beta_2}\right)^2\beta_3^2+\alpha_1^2\left(\frac{b_n}{n+\beta_2}\right)^2\beta_3^2\Bigg\}$$

$$\leq C_4\left(\frac{b_n}{n}(2A\alpha_1+A(1+2\alpha_1)+2A\alpha_2+2A\alpha_1)+\left(\frac{b_n}{n}\right)^2\left(\alpha_2^2\beta_3^2+\alpha_1^2\beta_3^2\right)\right)$$

olarak bulunur.

Ayrıca $\lim\limits_{n\to\infty}\frac{b_n}{n}=0$ olduğundan yeterince büyük n-ler için $\frac{b_n}{n}\leq\sqrt{\frac{b_n}{n}}$ dir.

Bu durumda,

$$I'_n \le C_4\left(\frac{b_n}{n}\left(2A\alpha_1 + A(1+2\alpha_1) + 2A\alpha_2 + 2A\alpha_1\right) + \frac{b_n}{n}\left(\alpha_2^2\beta_3^2 + \alpha_1^2\beta_3^2\right)\right)$$

$$\le C_4\frac{b_n}{n}\left\{2A\alpha_1 + A(1+2\alpha_1) + 2A\alpha_2 + 2A\alpha_1 + \alpha_2^2\beta_3^2 + \alpha_1^2\beta_3^2\right\}$$

$C_5 = C_4\left\{2A\alpha_1 + A(1+2\alpha_1) + 2A\alpha_2 + 2A\alpha_1 + \alpha_2^2\beta_3^2 + \alpha_1^2\beta_3^2\right\}$ olmak üzere

$$I'_n \le C_5\frac{b_n}{n}$$

elde edilir. Şimdi de I''_n ifadesini hesaplayalım. Bunun için süreklilik modülünün Tanım 1.2.6 ‘daki 3. ve 4. özelliklerini kullanalım. Bu durumda

$$\left|I''_n\right|$$

$$= \left|\left(\frac{n+\beta_2}{n}\right)^n \sum_{r\in E''_r}^{n}\left[f\left(\alpha_3 x + \beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right) - f(x)\right]\binom{n}{r}\left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r}\right|$$

$$= \left(\frac{n+\beta_2}{n}\right)^n \sum_{r\in E''_r}^{n}\left|f\left(\alpha_3 x + \beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right) - f(x)\right|\binom{n}{r}\left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r}$$

$$\le \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n}\omega_{1+A}\left(f : \left|x(\alpha_3 - 1) + \beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right|\right)\binom{n}{r}\left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r}$$

$$\le \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n}\omega_{1+A}\left(f : \frac{\delta_n}{\delta_n}\left|x(\alpha_3 - 1) + \beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right|\right)$$

$$\times\binom{n}{r}\left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r}$$

$$\le \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n}\left[1 + \frac{1}{\delta_n}\left|x(\alpha_3 - 1) + \beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right|\right]\omega_{1+A}(f : \delta_n)$$

$$\times\binom{n}{r}\left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r}$$

$$\leq \left(\frac{n+\beta_2}{n}\right)^n \frac{1}{\delta_n} \sum_{r=0}^{n} \left| x(\alpha_3 - 1) + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right| \omega_{1+A}(f:\delta_n)$$

$$\times \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r}$$

$$+ \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} \omega_{1+A}(f:\delta_n) \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r}$$

$$\leq \left(\frac{n+\beta_2}{n}\right)^n \frac{1}{\delta_n} \omega_{1+A}(f:\delta_n) \sum_{r=0}^{n} \left| x(\alpha_3 - 1) + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right|$$

$$\times \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r} + \omega_{1+A}(f:\delta_n)$$

elde edilir. Burada δ_n n'e bağlı bir dizidir. Yukarıdaki eşitsizliğe Cauchy Schwartz-Bunyakovsky Eşitsizliği'ni uygularsak

$$I_n''$$

$$\leq \omega_{1+A}(f:\delta_n) \frac{1}{\delta_n}$$

$$\times \left\{ \left[\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} \left(x(\alpha_3 - 1) + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right)^2 \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r} \right]^{1/2} \right.$$

$$\left. \times \left[\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} 1^2 \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r} \right]^{1/2} \right\}$$

$$+ \omega_{1+A}(f:\delta_n)$$

$$\leq \omega_{1+A}(f:\delta_n) \frac{1}{\delta_n}$$

$$\times \left[\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} \left(x(\alpha_3 - 1) + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right)^2 \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r} \right]^{1/2}$$

$$+\omega_{1+A}\left(f:\delta_n\right)$$

$$\leq \omega_{1+A}\left(f:\delta_n\right)\frac{1}{\delta_n}\left\{\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n}\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)^2\right.$$

$$\times\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$-2x\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n}\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)$$

$$\times\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$\left.+x^2\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right\}^{1/2}$$

$$+\omega_{1+A}\left(f:\delta_n\right)$$

$$\leq \omega_{1+A}\left(f:\delta_n\right)\frac{1}{\delta_n}\sqrt{T_{n,\alpha,\beta}\left(t^2;x\right)-2xT_{n,\alpha,\beta}\left(t;x\right)+x^2T_{n,\alpha,\beta}\left(1;x\right)}+W_{1+A}\left(f:\delta_n\right)$$

olarak yazabiliriz. Lemma 1.4.1’den

$$I_n'' \leq \omega_{1+A}\left(f:\delta_n\right)\frac{1}{\delta_n}$$

$$\times\left\{\alpha_3^2x^2+2\alpha_1\alpha_3\beta_3 x\frac{b_n}{n+\beta_1}\right.$$

$$+\beta_3^2\alpha_1^2\frac{b_n^2}{\left(n+\beta_1\right)^2}$$

$$+\beta_3\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}b_n\right)$$

$$\times\left(2x\alpha_3+\frac{2\beta_3\alpha_1}{n+\beta_1}b_n+\frac{\beta_3}{n+\beta_1}b_n+\frac{\beta_3\left(n-1\right)}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}b_n\right)\right)$$

$$-2x\left[\alpha_3 x+\left(\frac{n+\beta_2}{n+\beta_1}\right)\beta_3 x+\left(\frac{\alpha_1-\alpha_2}{n+\beta_1}\right)\beta_3 b_n\right]+x^2\Bigg\}$$

$$+\,\omega_{1+A}\left(f:\delta_n\right)$$

elde edilir. $0\le\alpha_2\le\alpha_1\le\beta_2\le\beta_1$ ve $\alpha_3+\beta_3=1$ ifadelerinden

$$I_n''\le\omega_{1+A}\left(f:\delta_n\right)\frac{1}{\delta_n}$$

$$\times\Bigg\{2x\alpha_1\frac{b_n}{n+\beta_2}+x\frac{b_n}{n+\beta_2}\left(1+2\alpha_1\right)+2x\frac{\alpha_2 b_n}{n+\beta_2}$$

$$+2x\frac{\alpha_1 b_n}{n+\beta_2}+\alpha_2^2\left(\frac{b_n}{n+\beta_2}\right)^2\beta_3^2+\alpha_1^2\left(\frac{b_n}{n+\beta_2}\right)^2\beta_3^2\Bigg\}^{1/2}$$

$$+\,\omega_{1+A}\left(f:\delta_n\right)$$

$$\le\omega_{1+A}\left(f:\delta_n\right)\frac{1}{\delta_n}\Bigg\{\frac{b_n}{n+\beta_2}\left[2A\alpha_1+A\left(1+2\alpha_1\right)+2A\alpha_2+2A\alpha_1\right]$$

$$+\left(\frac{b_n}{n+\beta_2}\right)^2\left(\alpha_2^2\beta_3^2+\alpha_1^2\beta_3^2\right)\Bigg\}^{1/2}$$

$$+\,\omega_{1+A}\left(f:\delta_n\right)$$

$$\le\omega_{1+A}\left(f:\delta_n\right)\frac{1}{\delta_n}\Bigg\{\frac{b_n}{n}\left[2A\alpha_1+A\left(1+2\alpha_1\right)+2A\alpha_2+2A\alpha_1\right]$$

$$+\left(\frac{b_n}{n}\right)^2\left(\alpha_2^2\beta_3^2+\alpha_1^2\beta_3^2\right)\Bigg\}^{1/2}$$

$$+\,\omega_{1+A}\left(f:\delta_n\right)$$

eşitsizliği yazılabilir.

Ayrıca $\lim\limits_{n\to\infty}\frac{b_n}{n}=0$ olduğundan yeterince büyük n-ler için $\frac{b_n}{n}\le\sqrt{\frac{b_n}{n}}$ dir.

Bu durumda

$$I_n'' \leq \omega_{1+A}\left(f:\delta_n\right)\frac{1}{\delta_n}\left\{\frac{b_n}{n}\left[2A\alpha_1 + A(1+2\alpha_1)+2A\alpha_2 + 2A\alpha_1\right]\right.$$

$$\left.+\frac{b_n}{n}\left(\alpha_2^2\beta_3^2 + \alpha_1^2\beta_3^2\right)\right\}^{1/2}$$

$$+\omega_{1+A}\left(f:\delta_n\right)$$

$$\leq \omega_{1+A}\left(f:\delta_n\right)\frac{1}{\delta_n}\sqrt{\frac{b_n}{n}\left\{2A\alpha_1 + A(1+2\alpha_1)+2A\alpha_2 + 2A\alpha_1 + \alpha_2^2\beta_3^2 + \alpha_1^2\beta_3^2\right\}}$$

$$+\omega_{1+A}\left(f:\delta_n\right)$$

$C_6 = \left\{2A\alpha_1 + A(1+2\alpha_1)+2A\alpha_2 + 2A\alpha_1 + \alpha_2^2\beta_3^2 + \alpha_1^2\beta_3^2\right\}$ olmak üzere,

$$I_n'' \leq \omega_{1+A}\left(f:\delta_n\right)\frac{1}{\delta_n}\sqrt{C_6\,\frac{b_n}{n}} + \omega_{1+A}\left(f:\delta_n\right)$$

$$\leq \omega_{1+A}\left(f:\delta_n\right)\frac{1}{\delta_n}\sqrt{\frac{b_n}{n}}\sqrt{C_6} + \omega_{1+A}\left(f:\delta_n\right)$$

$$\leq \omega_{1+A}\left(f:\delta_n\right)\left(\frac{1}{\delta_n}\sqrt{\frac{b_n}{n}}\sqrt{C_6} + 1\right)$$

elde edilir.

Özel olarak $\delta_n = \sqrt{\frac{b_n}{n}}$ alınırsa

$$I_n'' \leq \omega_{1+A}\left(f:\sqrt{\frac{b_n}{n}}\right)\left(\sqrt{\frac{n}{b_n}}\sqrt{\frac{b_n}{n}}\sqrt{C_6} + 1\right)$$

$$\leq \omega_{1+A}\left(f:\sqrt{\frac{b_n}{n}}\right)\left(\sqrt{C_6} + 1\right)$$

elde edilir. Dolayısıyla

$$\left|T_{n,\alpha,\beta}(f;x)-f(x)\right| \le I'_n + I''_n$$

$$\le C_5\frac{b_n}{n} + \omega_{1+A}\left(f:\sqrt{\frac{b_n}{n}}\right)\left(\sqrt{C_6}+1\right)$$

$C_7 = \max\left(C_5, \sqrt{C_6}+1\right)$ olmak üzere

$$\left|T_{n,\alpha,\beta}(f;x)-f(x)\right| \le C_7\left[\frac{b_n}{n} + \omega_{1+A}\left(f:\sqrt{\frac{b_n}{n}}\right)\right]$$

eşitsizliği gerçeklenir. Ayrıca $\lim\limits_{n\to\infty}\frac{b_n}{n}=0$ olduğundan, yeterince büyük n-ler için;

$$\frac{b_n}{n} \le \sqrt{\frac{b_n}{n}} \text{ ve } C_f\sqrt{\frac{b_n}{n}} \le \omega_{1+A}\left(f:\sqrt{\frac{b_n}{n}}\right)$$

olarak yazılabilir. Bu durumda

$$\left|T_{n,\alpha,\beta}(f;x)-f(x)\right| \le C_7\left[\frac{b_n}{n} + \omega_{1+A}\left(f:\sqrt{\frac{b_n}{n}}\right)\right]$$

$$\le C_7\frac{b_n}{n} + C_7\omega_{1+A}\left(f:\sqrt{\frac{b_n}{n}}\right)$$

$$\le C_7\sqrt{\frac{b_n}{n}} + C_7\omega_{1+A}\left(f:\sqrt{\frac{b_n}{n}}\right)$$

$$\le C_7\frac{1}{C_f}\omega_{1+A}\left(f:\sqrt{\frac{b_n}{n}}\right) + C_7\omega_{1+A}\left(f:\sqrt{\frac{b_n}{n}}\right)$$

$$\le \omega_{1+A}\left(f:\sqrt{\frac{b_n}{n}}\right)\left(C_7\frac{1}{C_f}+C_7\right)$$

$C = \left(C_7\frac{1}{C_f}+C_7\right)$ olmak üzere

$$\left|T_{n,\alpha,\beta}(f;x)-f(x)\right|\leq C\omega_{1+A}\left(f:\sqrt{\frac{b_n}{n}}\right)$$

elde edilir ve ispat tamamlanır.

Teorem 1.4.5:

$f, f' \in C_2[0,\infty)$ olsun. Bu durumda herhangi bir $A>0$ ve her $x\in[0,A]$ için,

$$\left|T_{n,\alpha,\beta}(f;x)-f(x)\right|\leq M\sqrt{\frac{b_n}{n}}\omega_1\left(f';\sqrt{\frac{b_n}{n}}\right)$$

eşitsizliği sağlanır. Burada M , n'den bağımsız bir sabittir, $\omega_1(f';.)$ ise f' fonksiyonunun $[0,1+A]$ aralığındaki süreklilik modülüdür $[5]$.

İspat:

Ortalama Değer Teoremi'ne göre ξ, x ve $\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n$ arasında bir nokta olmak üzere,

$$f\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)-f(x)=\left((\alpha_3-1)x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)f'(\xi)$$

$$=\left((\alpha_3-1)x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)f'(x)$$

$$+\left((\alpha_3-1)x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\left(f'(\xi)-f'(x)\right)$$

yazılabilir. Yukarıdaki eşitliğin her iki tarafına $T_{n,\alpha,\beta}(f;x)$ operatörünü uygularsak

$$T_{n,\alpha,\beta}(f;x)-f(x)$$

$$=\left(\frac{n+\beta_2}{n}\right)^n$$

$$\times\sum_{r=0}^{n}\left[\left((\alpha_3-1)x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)f'(x)+\left((\alpha_3-1)x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)(f'(\xi)-f'(x))\right]$$

$$\times\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$=f'(x)\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left((\alpha_3-1)x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left((\alpha_3-1)x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)(f'(\xi)-f'(x))$$

$$\times\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$=f'(x)\left\{\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left(\alpha_3x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right.$$

$$\left.-\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}x\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right\}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left((\alpha_3-1)x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)(f'(\xi)-f'(x))\binom{n}{r}$$

$$\times\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$=f'(x)\{T_{n,\alpha,\beta}(t;x)-xT_{n,\alpha,\beta}(1;x)\}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left((\alpha_3-1)x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)(f'(\xi)-f'(x))$$

$$\times\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$=I_1+I_2$$

elde ederiz. O halde

$$\left|T_{n,\alpha,\beta}(f;x)-f(x)\right|\leq |I_1|+|I_2|$$

olur. İlk önce $|I_1|$ i hesaplayalım. Lemma 1.4.1'den

$$I_1=f'(x)\{T_{n,\alpha,\beta}(t;x)-xT_{n,\alpha,\beta}(1;x)\}$$

$$=f'(x)\left\{\alpha_3 x+\left(\frac{n+\beta_2}{n+\beta_1}\right)\beta_3 x+\left(\frac{\alpha_1-\alpha_2}{n+\beta_1}\right)\beta_3 b_n-x.1\right\}$$

$$|I_1|=\left|f'(x)\left\{\alpha_3 x+\left(\frac{n+\beta_2}{n+\beta_1}\right)\beta_3 x+\left(\frac{\alpha_1-\alpha_2}{n+\beta_1}\right)\beta_3 b_n-x\right\}\right|$$

$$\leq |f'(x)|\left|\alpha_3 x+\left(\frac{n+\beta_2}{n+\beta_1}\right)\beta_3 x+\left(\frac{\alpha_1-\alpha_2}{n+\beta_1}\right)\beta_3 b_n-x\right|$$

elde edilir. $0\leq\alpha_2\leq\alpha_1\leq\beta_2\leq\beta_1, \alpha_3+\beta_3=1$ için ve f fonksiyonunun $x\in[0,A]$ aralığında f' fonksiyonu sınırlı olduğundan,

$$|I_1|\leq |f'(x)|\left|\alpha_3 x+\left(\frac{n+\beta_2}{n+\beta_1}\right)\beta_3 x+\left(\frac{\alpha_1-\alpha_2}{n+\beta_1}\right)\beta_3 b_n-x\right|$$

$$\leq |f'(x)|\left|x(\alpha_3+\beta_3)+(\alpha_1-\alpha_2)\left(\frac{b_n}{n}\right)-x\right|$$

$$\leq |f'(x)|(\alpha_1-\alpha_2)\left(\frac{b_n}{n}\right)$$

eşitsizliği elde edilir.

Ayrıca $\lim_{n\to\infty}\frac{b_n}{n}=0$ olduğundan, yeterince büyük n-ler için;

$$C_f\sqrt{\frac{b_n}{n}}\leq \omega_1\left(f':\sqrt{\frac{b_n}{n}}\right)$$

olarak yazabiliriz. Bu durumda

$$|I_1|\leq |f'(x)|(\alpha_1-\alpha_2)\left(\frac{b_n}{n}\right)$$

$$\leq |f'(x)|(\alpha_1-\alpha_2)\sqrt{\frac{b_n}{n}}\sqrt{\frac{b_n}{n}}$$

$$\leq |f'(x)|(\alpha_1-\alpha_2)\frac{1}{C_f}\sqrt{\frac{b_n}{n}}\,\omega_1\left(f':\sqrt{\frac{b_n}{n}}\right)$$

$M_1=\dfrac{|f'(x)|(\alpha_1-\alpha_2)}{C_f}$ olmak üzere,

$$|I_1|\leq M_1\sqrt{\frac{b_n}{n}}\,\omega_1\left(f':\sqrt{\frac{b_n}{n}}\right)$$

elde ederiz. Şimdi $|I_2|$ yi hesaplayalım. O halde

$$I_2=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left((\alpha_3-1)x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)(f'(\xi)-f'(x))$$

$$\times\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$|I_2|=\left|\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left((\alpha_3-1)x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)(f'(\xi)-f'(x))\right|$$

$$\times\left|\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right|$$

$$|I_2|\leq\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left|\left((\alpha_3-1)x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\right|\left|(f'(\xi)-f'(x))\right|$$

$$\times\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

yazılabilir. Yukarıdaki eşitsizlikte süreklilik modülünün Tanım 1.2.6 ‘daki 4. özelliğini ve

$$|\xi-x|<\left|(\alpha_3-1)x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right|$$

eşitsizliğini kullanırsak

$$|I_2|$$

$$\leq \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} \left| x(\alpha_3 - 1) + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right| \omega_1\left(f' : |\xi - x|\right)$$

$$\times \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r}$$

$$\leq \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} \left| x(\alpha_3 - 1) + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right| \omega_1\left(f' : \left| x(\alpha_3 - 1) + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right| \right)$$

$$\times \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r}$$

$$\leq \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} \left| x(\alpha_3 - 1) + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right| \omega_1\left(f' : \frac{\delta_n}{\delta_n} \left| x(\alpha_3 - 1) + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right| \right)$$

$$\times \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r}$$

elde ederiz. Şimdi ise süreklilik modülünün Tanım 1.2.6 ‘daki 3. özelliğini kullanalım. Bu durumda

$$|I_2| \leq \left(\frac{n+\beta_2}{n}\right)^n \omega_1(f' : \delta_n)$$

$$\times \sum_{r=0}^{n} \left| x(\alpha_3 - 1) + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right| \left[1 + \frac{1}{\delta_n} \left| x(\alpha_3 - 1) + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right| \right]$$

$$\times \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r}$$

$$\leq \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} \left| x(\alpha_3 - 1) + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n \right| \omega_1(f' : \delta_n)$$

$$\times \binom{n}{r} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \frac{1}{\delta_n}\sum_{r=0}^{n}\left(x(\alpha_3-1)+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)^2 \omega_1(f':\delta_n)$$

$$\times\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

eşitsizliği gerçeklenir. Burada δ_n n'e bağlı bir dizidir. Yukarıdaki eşitsizliğin birinci toplamında Cauchy-Schwartz-Bunyakovsky Eşitsizliği'ni uygularsak

$$|I_2|$$

$$\leq\left\{\left[\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left(x(\alpha_3-1)+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)^2\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right]^{1/2}\right.$$

$$\left.\times\left[\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}1^2\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right]^{1/2}\right\}\omega_1(f':\delta_n)$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \frac{1}{\delta_n}\sum_{r=0}^{n}\left(x(\alpha_3-1)+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)^2 \omega_1(f':\delta_n)$$

$$\times\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

elde ederiz. Teorem 1.4.4'ün ispatından

$$|I_2|$$

$$\leq\left\{\left[\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left(x(\alpha_3-1)+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)^2\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}\right]^{1/2}\right.$$

$$+\left(\frac{n+\beta_2}{n}\right)^n \frac{1}{\delta_n}\sum_{r=0}^{n}\left(x(\alpha_3-1)+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)^2 \omega_1(f':\delta_n)$$

$$\times\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$\leq \omega_1(f':\delta_n)\sqrt{C_6\frac{b_n}{n}}+C_6\frac{b_n}{n}\frac{1}{\delta_n}\omega_1(f':\delta_n)$$

olarak yazılabilir. Özel olarak $\delta_n=\sqrt{\frac{b_n}{n}}$ alınırsa

$$|I_2|\leq \omega_1\left(f':\sqrt{\frac{b_n}{n}}\right)\sqrt{C_6}\sqrt{\frac{b_n}{n}}+\omega_1\left(f':\sqrt{\frac{b_n}{n}}\right)\sqrt{\frac{n}{b_n}}C_6\frac{b_n}{n}$$

$$\leq \omega_1\left(f':\sqrt{\frac{b_n}{n}}\right)\sqrt{C_6}\sqrt{\frac{b_n}{n}}+\omega_1\left(f':\sqrt{\frac{b_n}{n}}\right)\sqrt{\frac{b_n}{n}}C_6$$

$$\leq \omega_1\left(f':\sqrt{\frac{b_n}{n}}\right)\sqrt{\frac{b_n}{n}}\left(\sqrt{C_6}+C_6\right)$$

$M_2=\sqrt{C_6}+C_6$ olmak üzere,

$$|I_2|\leq M_2\sqrt{\frac{b_n}{n}}\omega_1\left(f':\sqrt{\frac{b_n}{n}}\right)$$

eşitsizliği yazılabilir. Bu durumda

$$|T_{n,\alpha,\beta}(f;x)-f(x)|\leq |I_1|+|I_2|$$

$$\leq M_1\sqrt{\frac{b_n}{n}}\omega_1\left(f':\sqrt{\frac{b_n}{n}}\right)+M_2\sqrt{\frac{b_n}{n}}\omega_1\left(f':\sqrt{\frac{b_n}{n}}\right)$$

$$\leq (M_1+M_2)\sqrt{\frac{b_n}{n}}\omega_1\left(f':\sqrt{\frac{b_n}{n}}\right)$$

$M=M_1+M_2$ olmak üzere,

$$|T_{n,\alpha,\beta}(f;x)-f(x)|\leq M\sqrt{\frac{b_n}{n}}\omega_1\left(f';\sqrt{\frac{b_n}{n}}\right)$$

elde edilir.

2.YENİ BERNSTEIN-STANCU-CHLODOWSKY TİPİNDEKİ OPERATÖRLERİN AĞIRLIKLI YAKLAŞIMLARI

2.1.Giriş

Aşağıdaki polinomlar, 1937 yılında I.Chlodowsky olarak tanıtılan genelleştirilmiş Bernstein polinomlarıdır $[2]$. Bu klasik anlamdaki Bernstein-Chlodowsky tipli operatörler,

$$C_n(f;x)=\sum_{r=0}^{n} f\left(\frac{r}{n}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}\right)^{r}\left(1-\frac{x}{b_n}\right)^{n-r},$$

$f,[0,\infty)$ aralığında tanımlı bir fonksiyon, $[0,b_n]\subset[0,\infty)$ alt cümlesi ,$(b_n)_{n\geq 1}$ pozitif artan bir dizi ve bu dizi $n\to\infty$ iken $(b_n)\to\infty$ ve $\frac{b_n}{n}\to 0$ özellikte bir dizi olacak şekilde tanımlanmıştır.

E.A. Gadjieva ve E.İbikli tarafından 1997 yılında yayınlanan makalede, Bernstein- Chlodowsky tipli operatörlerinin ağırlıklı yaklaşım özellikleri ortaya konulmuştur $[6]$. Başka bir genelleştirilmiş Bernstein- Chlodowsky tipli operatörleri A.D. Gadjiev, I.Efendiev ve E.İbikli tarafından 1998 yılında ele alınmıştır $[7]$.

A.D.Gadzhiev'in 1974 ve 1976 yıllarında yayınladığı makalelerindeki gibi, Korovkin tipi teoremlerindeki pozitif lineer operatörler, $C_2[0,\infty)$ uzayında tanımlanmamış fakat $C_2^*[0,\infty)$ uzayından $B_2[0,\infty)$ uzayı üzerine giden operatörün bir normu tanımlanarak aşağıdaki formlar elde edilmiştir $[8,9]$:

Teorem 2.1.1:

L_n, $C_2[0,\infty)$ uzayından $B_2[0,\infty)$ uzayına giden lineer pozitif operatörlerin dizisi olsun. O halde

$$\lim_{n\to\infty}\left\| L_n\left(t^v;x\right)-x^v\right\|_2 = 0,\ v = 0,1,2$$

dır. Bu durumda herhangi bir $f \in C_2^*[0,\infty)$ fonksiyonu için,

$$\lim_{n\to\infty}\left\| L_n f - f\right\|_2 = 0$$

olarak gerçeklenir.

$f \in C_2[0,\infty)$ uzayı olduğunda aşağıdaki teoremi elde ederiz:

Teorem 2.1.2:

$(L_n)_{n\geq 1}$,bir lineer pozitif operatör dizisi olmak üzere Teorem 2.1.1'deki koşullar sağlandığında bir $f^* \in C_2[0,\infty)$ fonksiyonu için,

$$\lim_{n\to\infty}\left\| L_n f^* - f^*\right\|_2 \neq 0$$

olarak gerçeklenir.

$[0,\infty)$ aralığında süreklilik modülü için f fonksiyonunun sürekliliği yetmez. Örneğin, $g(x) = \cos \pi x^2, x \in [0,\infty)$ fonksiyonunu göz önüne alalım. Açıktır ki, bu fonksiyon $[0,\infty)$ aralığında sınırlı ve sürekli fonksiyondur. Fakat süreklilik modülü tanımında $t = \sqrt{n+1}$ ve $x = \sqrt{n}$ alınırsa $n \to \infty$ için $\left|\sqrt{n+1} - \sqrt{n}\right| \to 0$ iken $\left|g\left(\sqrt{n+1}\right) - g\left(\sqrt{n}\right)\right| \to 2$ dir. Bu durumda, düzgün sürekli değildir. Dolayısıyla bu ağır şart yerine $C_2^*[0,\infty)$ uzayı alınmıştır.

2.2.Ağırlıklı Yaklaşımlar

Bu bölümde Teorem 2.1.1'deki verileri kullanarak $T_{n,\alpha,\beta}(f;x)$ operatörünün yaklaşım özelliklerini araştıracağız.

Teorem 2.2.1:

Herhangi bir $f \in C_2^*[0,\infty)$ fonksiyonu için

$$\lim_{n\to\infty} \sup_{x\in\left[\frac{\alpha_2}{n+\beta_2}b_n,\frac{n+\alpha_2}{n+\beta_2}b_n\right]} \frac{\left|T_{n,\alpha,\beta}(f;x)-f(x)\right|}{1+x^2}=0$$

eşitliği elde edilir.

İspat:

Biz öncelikle E.A.Gadjieva ve E.İbikli tarafından 1997 yılında yayınladıkları makaledeki Bernstein-Chlodowsky tipli polinomların ağırlıklı yaklaşım metodllarını kullanacağız. Bu durumda $T_{n,\alpha,\beta}(f;x)$ operatöründe gerekli sadeleştirmeler yaparsak aşağıdaki eşitliği;

$$T^*_{n,\alpha,\beta}(f;x)=\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} f\left(\frac{r}{n}\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

elde ederiz.

Binom açılımı ve $T_{n,\alpha,\beta}(f;x)$ operatörünün tanımından açıktır ki,

$$T_{n,\alpha,\beta}(1;x)=T^*_{n,\alpha,\beta}(1;x)$$

$$=\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}+\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^n$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\left(\frac{n}{n+\beta_2}\right)^n$$

$$=1 \qquad (2.1)$$

dir. Bu durumda

$$\lim_{n\to\infty} \sup_{x\in\left[\frac{\alpha_2}{n+\beta_2}b_n,\frac{n+\alpha_2}{n+\beta_2}b_n\right]} \frac{\left|T_{n,\alpha,\beta}(1;x)-1\right|}{1+x^2}=0 \tag{2.2}$$

elde edilir.

$T^*_{n,\alpha,\beta}(f;x)$ operatörünün tanımında $f(t)=t$ alınırsa

$$T^*_{n,\alpha,\beta}(t;x)=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\frac{r}{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=1}^{n}\binom{n-1}{r-1}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

eşitliği elde edilir. Yukarıdaki eşitlikte r yerine $r+1$ yazılırsa

$T^*_{n,\alpha,\beta}(t;x)$

$$=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n-1}\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r+1}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)\sum_{r=0}^{n-1}\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

elde edilir. Binom açılımından,

$$T^*_{n,\alpha,\beta}(t;x)=\left(\frac{n+\beta_2}{n}\right)^n\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}+\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{r+n-r-1}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n}{n+\beta_2}\right)^{n-1}$$

$$=\left(\frac{n+\beta_2}{n}\right)\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right) \tag{2.3}$$

olarak bulunur. Şimdi benzer şekilde $f(t)=t^2$ alınırsa

$$T^*_{n,\alpha,\beta}(t^2;x)=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\frac{r^2}{n^2}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

yazılabilir. Yukarıdaki eşitlikte $\frac{r}{n^2}$ ekleyip çıkaralım. Bu durumda

$$T^*_{n,\alpha,\beta}(t^2;x)=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\frac{r(r-1)}{n^2}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\frac{r}{n^2}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=2}^{n}\frac{n(n-1)}{n^2}\binom{n-2}{r-2}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=1}^{n}\frac{1}{n}\binom{n-1}{r-1}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

elde edilir. Şimdi birinci toplamda r yerine $r+2$; ikinci toplamda r yerine $r+1$ yazalım. O halde

$T^*_{n,\alpha,\beta}(t^2;x)$

$$=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n-2}\left(\frac{n-1}{n}\right)\binom{n-2}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r+2}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-2}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{n}\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r+1}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\left(\frac{n-1}{n}\right)\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^2\sum_{r=0}^{n-2}\binom{n-2}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-2}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{n}\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-1}$$

olarak bulunur. Binom açılımından

$T^*_{n,\alpha,\beta}(t^2;x)$

$$=\left(\frac{n+\beta_2}{n}\right)^n\left(\frac{n-1}{n}\right)\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^2\left(\frac{n}{n+\beta_2}\right)^{n-2}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)\frac{1}{n}\left(\frac{n}{n+\beta_2}\right)^{n-1}$$

$$=\left(\frac{n+\beta_2}{n}\right)^2\left(\frac{n-1}{n}\right)\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^2+\frac{1}{n}\left(\frac{n+\beta_2}{n}\right)^n\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right) \qquad (2.4)$$

elde edilir. Şimdi $T_{n,\alpha,\beta}(f;x)$ ve $T^*_{n,\alpha,\beta}(f;x)$ operatörlerinin tanımlarını kullanarak ve $f(t)=t$ alarak $T_{n,\alpha,\beta}(f;x)$ operatörünü $T^*_{n,\alpha,\beta}(f;x)$ operatörü türünden bulalım. Bu durumda,

$$T_{n,\alpha,\beta}(t;x)=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left(\alpha_3x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$=\left(\frac{n+\beta_2}{n}\right)^n\alpha_3x\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\left(\beta_3\frac{n}{n+\beta_1}b_n\right)\sum_{r=0}^{n}\frac{r}{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\left(\beta_3\frac{\alpha_1}{n+\beta_1}b_n\right)\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$=\alpha_3xT^*_{n,\alpha,\beta}(1;x)+\left(\beta_3\frac{n}{n+\beta_1}b_n\right)T^*_{n,\alpha,\beta}(t;x)+\left(\beta_3\frac{\alpha_1}{n+\beta_1}b_n\right)T^*_{n,\alpha,\beta}(1;x)$$

olarak buluruz. (2.1) ve (2.3) ‘deki verileri kullanırsak

$$T_{n,\alpha,\beta}(t;x)=\alpha_3x+\left(\beta_3\frac{n}{n+\beta_1}b_n\right)\left(\frac{n+\beta_2}{n}\right)\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)+\left(\beta_3\frac{\alpha_1}{n+\beta_1}b_n\right)$$

$$=\alpha_3x+\beta_3x\left(\frac{n+\beta_2}{n+\beta_1}\right)-\beta_3b_n\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(\frac{\alpha_2}{n+\beta_2}\right)+\left(\beta_3\frac{\alpha_1}{n+\beta_1}b_n\right)$$

$$=\alpha_3x+\beta_3x\left(\frac{n+\beta_2}{n+\beta_1}\right)+\beta_3b_n\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(\frac{\alpha_1-\alpha_2}{n+\beta_2}\right)$$

elde ederiz. Bu durumda,

$$\lim_{n\to\infty}\sup_{x\in\left[\frac{\alpha_2}{n+\beta_2}b_n,\frac{n+\alpha_2}{n+\beta_2}b_n\right]}\frac{\left|T_{n,\alpha,\beta}(t;x)-x\right|}{1+x^2}$$

$$=\lim_{n\to\infty}\sup_{x\in\left[\frac{\alpha_2}{n+\beta_2}b_n,\frac{n+\alpha_2}{n+\beta_2}b_n\right]}\frac{1}{1+x^2}\left|\alpha_3x+\beta_3x\left(\frac{n+\beta_2}{n+\beta_1}\right)+\beta_3b_n\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(\frac{\alpha_1-\alpha_2}{n+\beta_2}\right)-x\right|$$

olup $0\le\alpha_2\le\alpha_1\le\beta_2\le\beta_1$ ve $\alpha_3+\beta_3=1$ ifadelerinden yararlanırsak

$$\lim_{n\to\infty}\sup_{x\in\left[\frac{\alpha_2}{n+\beta_2}b_n,\frac{n+\alpha_2}{n+\beta_2}b_n\right]}\frac{\left|T_{n,\alpha,\beta}(t;x)-x\right|}{1+x^2}\le\lim_{n\to\infty}\left|\alpha_3+\beta_3\left(\frac{n+\beta_2}{n+\beta_1}\right)-1\right|+\left(\frac{n+\beta_2}{n+\beta_1}\right)\beta_3b_n\left(\frac{\alpha_1-\alpha_2}{n+\beta_2}\right)$$

$$\le(\alpha_3+\beta_3)-1+1.0$$

$$=0 \tag{2.5}$$

olarak gerçeklenir. Benzer şekilde $f(t)=t^2$ alarak $T_{n,\alpha,\beta}(f;x)$ operatörünü $T^*_{n,\alpha,\beta}(f;x)$ operatörü türünden bulalım. Bu durumda,

$T_{n,\alpha,\beta}(t^2;x)$

$$=\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left(\alpha_3x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)^2\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$=\alpha_3^2x^2\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}2x\alpha_3\beta_3\left(\frac{r+\alpha_1}{n+\beta_1}\right)b_n\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\left(\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)^2\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

yazılabilir. Yukarıdaki eşitlikte ikinci ve üçüncü toplamları açıp, düzenlersek

$T_{n,\alpha,\beta}\left(t^2;x\right)$

$$=\alpha_3^2x^2\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+2x\alpha_3\beta_3\left(\frac{n}{n+\beta_1}b_n\right)\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\frac{r}{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+2x\alpha_3\beta_3\left(\frac{\alpha_1}{n+\beta_1}b_n\right)\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\beta_3\frac{n}{n+\beta_1}b_n\right)^2\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\frac{r^2}{n^2}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\beta_3^2\frac{2\alpha_1 n}{(n+\beta_1)^2}b_n^2\right)\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\frac{r}{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

$$+\left(\beta_3\frac{\alpha_1}{n+\beta_1}b_n\right)^2\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}$$

olarak elde ederiz. Şimdi $T^*_{n,\alpha,\beta}\left(f;x\right)$ operatörünün tanımını kullanalım.

Bu durumda,

$T_{n,\alpha,\beta}\left(t^2;x\right)$

$$=\alpha_3^2x^2T^*_{n,\alpha,\beta}\left(1;x\right)+2x\alpha_3\beta_3\left(\frac{n}{n+\beta_1}b_n\right)T^*_{n,\alpha,\beta}\left(t;x\right)+2x\alpha_3\beta_3\left(\frac{\alpha_1}{n+\beta_1}b_n\right)T^*_{n,\alpha,\beta}\left(1;x\right)$$

$$+\left(\beta_3\frac{n}{n+\beta_1}b_n\right)^2T^*_{n,\alpha,\beta}\left(t^2;x\right)+\left(\beta_3^2\frac{2\alpha_1 n}{(n+\beta_1)^2}b_n^2\right)T^*_{n,\alpha,\beta}\left(t;x\right)$$

$$+\left(\beta_3\frac{\alpha_1}{n+\beta_1}b_n\right)^2T^*_{n,\alpha,\beta}\left(1;x\right)$$

olarak bulunur. (2.1),(2.3) ve (2.4) ‘deki verilerden faydalanarak

$T_{n,\alpha,\beta}\left(t^2;x\right)$

$$=\alpha_3^2x^2+\left(2x\alpha_3\beta_3\frac{\alpha_1}{n+\beta_1}b_n\right)+\left(\frac{\beta_3\alpha_1}{n+\beta_1}b_n\right)^2$$

$$+\beta_3\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}b_n\right)$$

$$\times\left(2x\alpha_3+\frac{2\beta_3\alpha_1}{n+\beta_1}b_n+\frac{\beta_3(n-1)}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}b_n\right)+\frac{\beta_3}{n+\beta_1}b_n\right)$$

elde ederiz. Açıktır ki,

$$\lim_{n\to\infty}\sup_{x\in\left[\frac{\alpha_2}{n+\beta_2}b_n,\frac{n+\alpha_2}{n+\beta_2}b_n\right]}\frac{\left|T_{n,\alpha,\beta}\left(t^2;x\right)-x^2\right|}{1+x^2}$$

$$=\lim_{n\to\infty}\sup_{x\in\left[\frac{\alpha_2}{n+\beta_2}b_n,\frac{n+\alpha_2}{n+\beta_2}b_n\right]}\frac{1}{1+x^2}\left|\alpha_3^2x^2+\left(2x\alpha_3\beta_3\frac{\alpha_1}{n+\beta_1}b_n\right)+\left(\frac{\beta_3\alpha_1}{n+\beta_1}b_n\right)^2\right.$$

$$+\beta_3\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}b_n\right)$$

$$\times\left\{2x\alpha_3+\frac{2\beta_3\alpha_1}{n+\beta_1}b_n+\frac{\beta_3}{n+\beta_1}b_n\right.$$

$$\left.+\frac{\beta_3(n-1)}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}b_n\right)\right\}$$

$$\left.-x^2\right|$$

$$=\lim_{n\to\infty}\sup_{x\in\left[\frac{\alpha_2}{n+\beta_2}b_n,\frac{n+\alpha_2}{n+\beta_2}b_n\right]}\left\{\frac{x^2}{1+x^2}\left[\alpha_3^2+\beta_3\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(2\alpha_3+\frac{\beta_3(n-1)}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)\right)-1\right]\right.$$

$$+\frac{x}{1+x^2}\left\{2\alpha_3\beta_3\frac{\alpha_1}{n+\beta_1}b_n-\left(\beta_3\frac{\alpha_2}{n+\beta_1}b_n\right)\left(2\alpha_3+\frac{\beta_3(n-1)}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)\right)\right.$$

$$+\beta_3\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(\frac{2\beta_3\alpha_1}{n+\beta_1}b_n+\frac{\beta_3}{n+\beta_1}b_n-\frac{\beta_3(n-1)}{n}\left(\frac{\alpha_2}{n+\beta_1}b_n\right)\right)\Bigg\}$$

$$+\frac{1}{1+x^2}\left[\left(\frac{\beta_3\alpha_1}{n+\beta_1}b_n\right)^2-\left(\beta_3\frac{\alpha_2}{n+\beta_1}b_n\right)\left(\frac{2\beta_3\alpha_1}{n+\beta_1}b_n-\frac{\beta_3 b_n}{n+\beta_1}\left(\frac{(n-1)}{n}\alpha_2-1\right)\right)\right]\Bigg\}$$

$$=\alpha_3^2+2\alpha_3\beta_3+\beta_3^2-1=0 \qquad (2.6)$$

olarak gerçeklenir.

Şimdi $T_{n,\alpha,\beta}(f;x)$ operatörünü dikkate alarak bir $T_n(f;x)$ dizisi tanımlayalım:

$$T_n(f;x)=\begin{cases} f(x) & ,0\le x\le \dfrac{\alpha_2}{n+\beta_2}b_n \\ T_{n,\alpha,\beta}(f;x) & ,\dfrac{\alpha_2}{n+\beta_1}b_n\le x\le \dfrac{n+\alpha_2}{n+\beta_2}b_n \\ f(x) & ,\dfrac{n+\alpha_2}{n+\beta_2}b_n\le x<\infty \end{cases} \qquad (2.7)$$

dir. Buradan,

$$\lim_{n\to\infty}\left\|T_n(f)-f\right\|_2=\lim_{n\to\infty}\sup_{x\in\left[\frac{\alpha_2}{n+\beta_2}b_n,\frac{n+\alpha_2}{n+\beta_2}b_n\right]}\frac{\left|T_{n,\alpha,\beta}(f;x)-f(x)\right|}{1+x^2} \qquad (2.8)$$

olup (2.2),(2.5) ve (2.6) ‘dan

$$\lim_{n\to\infty}\left\|T_n(t^v;\ )-x^v\right\|_2=0$$

olarak gerçeklenir.

Teorem 2.1.1 ‘deki anlık yaklaşımlarında olduğu gibi bir $f\in C_2^*[0,\infty)$ fonksiyonu için,

$$\lim_{n\to\infty}\left\|T_n(f)-f\right\|_2=0$$

olarak gerçeklenir. Bu durumda (2.8) ‘den istediğimiz sonucu elde ederiz.

Teorem 2.2.2:

$0 \le \alpha_2 \le \alpha_1 \le \beta_2 \le \beta_1$ olsun. Bu durumda herhangi bir $f \in C_2^*[0,\infty)$ için

$$\lim_{n\to\infty} \frac{1}{\sqrt{b_n}} \sup_{x \in \left[\frac{\alpha_2}{n+\beta_2} b_n, \frac{n+\alpha_2}{n+\beta_2} b_n\right]} \frac{\left|T_{n,\alpha,\beta}(f;x) - f(x)\right|}{1+x^2} = 0$$

eşitliği gerçeklenir.

İspat:

$f \in C_2^*[0,\infty)$ olduğundan

$$\lim_{x\to\infty} \frac{|f(x)|}{1+x^2} = K_f$$

olacak şekilde bir K_f sabiti vardır. Aynı zamanda fonksiyonların yeterli

şartları altında

$$\lim_{x\to\infty} \frac{|f(x)|}{1+x^2} = 0$$

olarak da gerçeklenir. $\left(\text{Örneğin}, \varphi(x) = f(x) - K_f\left(1+x^2\right)\right)$.

Bu durumda limit tanımından, yeteri kadar büyük bir $x_0 > 0$ olduğunda $x > x_0$ için

$$\frac{|f(x)|}{1+x^2} < \varepsilon \tag{2.9}$$

olacak şekilde bir ε pozitif sayısı vardır. Bu koşullar altında (2.7)'de tanımlı $T_n(f;x)$ dizisini kullanalım. O halde açıktır ki

$$\frac{1}{\sqrt{b_n}} \sup_{0\le x<\infty} \frac{\left|T_n(f;x) - f(x)\right|}{1+x^2} \le \frac{1}{\sqrt{b_n}} \sup_{0\le x\le x_0} \frac{\left|T_n(f;x) - f(x)\right|}{1+x^2} + \frac{1}{\sqrt{b_n}} \sup_{x>x_0} \frac{\left|T_n(f;x) - f(x)\right|}{1+x^2}$$

dir. Yukarıdaki eşitsizlikte $\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|T_n(f;x)-f(x)\right|}{1+x^2}$ ifadesi için üçgen eşitsizliğini kullanalım. O halde

$$\frac{1}{\sqrt{b_n}}\sup_{0\le x<\infty}\frac{\left|T_n(f;x)-f(x)\right|}{1+x^2}$$

$$\le\frac{1}{\sqrt{b_n}}\sup_{0\le x\le x_0}\frac{\left|T_n(f;x)-f(x)\right|}{1+x^2}+\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|T_n(f;x)-f(x)\right|}{1+x^2}$$

$$\le\frac{1}{\sqrt{b_n}}\sup_{0\le x\le x_0}\frac{\left|T_n(f;x)-f(x)\right|}{1+x^2}+\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|T_n(f;x)\right|}{1+x^2}+\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|f(x)\right|}{1+x^2}$$

elde edilir. $f\in B_2[0,\infty)$ olduğundan

$$\left|f(t)\right|\le K_f\left(1+t^2\right)$$

olacak şekilde bir K_f sabiti vardır.

Yukarıdaki eşitsizliği $\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|T_n(f;x)\right|}{1+x^2}$ ifadesinde yerine yazalım.

Bu durumda

$$\frac{1}{\sqrt{b_n}}\sup_{0\le x<\infty}\frac{\left|T_n(f;x)-f(x)\right|}{1+x^2}$$

$$\le\frac{1}{\sqrt{b_n}}\sup_{0\le x\le x_0}\frac{\left|T_n(f;x)-f(x)\right|}{1+x^2}+\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|T_n(f;x)\right|}{1+x^2}+\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|f(x)\right|}{1+x^2}$$

$$\le\frac{1}{\sqrt{b_n}}\sup_{0\le x\le x_0}\frac{\left|T_n(f;x)-f(x)\right|}{1+x^2}+\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|T_n\left(K_f\left(1+t^2\right);x\right)\right|}{1+x^2}+\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|f(x)\right|}{1+x^2}$$

$$\le\frac{1}{\sqrt{b_n}}\sup_{0\le x\le x_0}\frac{\left|T_n(f;x)-f(x)\right|}{1+x^2}+\frac{1}{\sqrt{b_n}}K_f\sup_{x>x_0}\frac{\left|T_n\left(\left(1+t^2\right);x\right)\right|}{1+x^2}+\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|f(x)\right|}{1+x^2}$$

$$\le\frac{1}{\sqrt{b_n}}\left\|T_n(f)-f\right\|_{C[0,x_0]}+\frac{1}{\sqrt{b_n}}\left\|f\right\|_2\sup_{x>x_0}\frac{\left|T_n\left(\left(1+t^2\right);x\right)\right|}{1+x^2}+\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|f(x)\right|}{1+x^2} \qquad (2.10)$$

eşitsizliği elde edilir. Şimdi $T_n(f;x)$ dizisinin tanımını dikkate alarak

$\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|T_n\left(\left(1+t^2\right);x\right)\right|}{1+x^2}$ ifadesini hesaplayalım. Açıktır ki $0\le\alpha_2\le\alpha_1\le\beta_2\le\beta_1$ 'den

$$\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|T_n\left(\left(1+t^2\right);x\right)\right|}{1+x^2}=\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|T_{n,\alpha,\beta}\left(\left(1+t^2\right);x\right)\right|}{1+x^2}$$

$$\le\frac{\alpha_3^2+2\alpha_3\beta_3}{\sqrt{b_n}}+2\frac{\alpha_3\beta_3}{\sqrt{b_n}}\frac{\alpha_1}{n+\beta_1}b_n+\frac{1}{\sqrt{b_n}}\left(\frac{\alpha_1\beta_3}{n+\beta_1}b_n\right)^2$$

$$+\frac{1}{\sqrt{b_n}}\frac{2\alpha_1\beta_3^2}{n+\beta_1}b_n+\frac{1}{\sqrt{b_n}}\frac{\beta_3^2(n-1)}{n}+\frac{1}{\sqrt{b_n}}\frac{\beta_3^2}{n+\beta_1}b_n$$

elde edilir. Yukarıdaki eşitsizliği (2.10) 'da yerine yazarsak

$$\frac{1}{\sqrt{b_n}}\sup_{0\le x<\infty}\frac{\left|T_n(f;x)-f(x)\right|}{1+x^2}$$

$$\le\frac{1}{\sqrt{b_n}}\left\|T_n(f)-f\right\|_{C[0,x_0]}+\frac{1}{\sqrt{b_n}}\|f\|_2\sup_{x>x_0}\frac{\left|T_n\left(\left(1+t^2\right);x\right)\right|}{1+x^2}+\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{|f(x)|}{1+x^2}$$

$$\le\frac{1}{\sqrt{b_n}}\left\|T_n(f)-f\right\|_{C[0,x_0]}$$

$$+\|f\|_2\left\{\frac{\alpha_3^2+2\alpha_3\beta_3}{\sqrt{b_n}}+2\frac{\alpha_3\beta_3}{\sqrt{b_n}}\frac{\alpha_1}{n+\beta_1}b_n+\frac{1}{\sqrt{b_n}}\left(\frac{\alpha_1\beta_3}{n+\beta_1}b_n\right)^2\right.$$

$$\left.+\frac{1}{\sqrt{b_n}}\frac{2\alpha_1\beta_3^2}{n+\beta_1}b_n+\frac{1}{\sqrt{b_n}}\frac{\beta_3^2(n-1)}{n}+\frac{1}{\sqrt{b_n}}\frac{\beta_3^2}{n+\beta_1}b_n\right\}$$

$$+\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{|f(x)|}{1+x^2}$$

elde ederiz. Bu durumda (2.9) 'dan

$$\frac{1}{\sqrt{b_n}}\sup_{0\le x<\infty}\frac{|T_n(f;x)-f(x)|}{1+x^2}$$

$$\le \frac{1}{\sqrt{b_n}}\|T_n(f)-f\|_{C[0,x_0]}$$

$$+\|f\|_2\left\{\frac{\alpha_3^2+2\alpha_3\beta_3}{\sqrt{b_n}}+2\frac{\alpha_3\beta_3}{\sqrt{b_n}}\frac{\alpha_1}{n+\beta_1}b_n+\frac{1}{\sqrt{b_n}}\left(\frac{\alpha_1\beta_3}{n+\beta_1}b_n\right)^2\right.$$

$$\left.+\frac{1}{\sqrt{b_n}}\frac{2\alpha_1\beta_3^2}{n+\beta_1}b_n+\frac{1}{\sqrt{b_n}}\frac{\beta_3^2(n-1)}{n}+\frac{1}{\sqrt{b_n}}\frac{\beta_3^2}{n+\beta_1}b_n\right\}$$

$$+\frac{1}{\sqrt{b_n}}\varepsilon$$

olarak bulunur. Yukarıdaki eşitsizliğin her iki tarafından $n\to\infty$ için limit alınırsa birinci terim Korovkin's Teoremi'nden olmak üzere

$$\lim_{n\to\infty}\frac{1}{\sqrt{b_n}}\sup_{0\le x<\infty}\frac{|T_n(f;x)-f(x)|}{1+x^2}=\lim_{n\to\infty}\frac{1}{\sqrt{b_n}}\sup_{x\in\left[\frac{\alpha_2}{n+\beta_2}b_n,\frac{n+\alpha_2}{n+\beta_2}b_n\right]}\frac{|T_{n,\alpha,\beta}(f;x)-f(x)|}{1+x^2}=0$$

elde edilir.

2.3. $T_{n,\alpha,\beta}(f;x)$ Operatörünün Türevindeki Yakınsaklık

Bu bölümde $\alpha_3=0$ ve $\beta_3=1$ alarak $T_{n,\alpha,\beta}(f;x)$ operatörünü klasik anlamdaki Bernstein-Chlodowsky operatörüne dönüştüreceğiz. Böylelikle Ibragimov-Gadjiev tipli operatörler için A.D.Gadjiev ve N.İspir tarafından 1999 yılında yayınlanan makaledeki verileri $[10]$ ve birinci bölümdeki Tanım 1.2.2'yi kullanarak $T_{n,\alpha,\beta}(f;x)$operatörünün türevindeki yakınsaklığını elde edeceğiz.

Teorem 2.3.1:

f fonksiyonu $[0,\infty)$ aralığında $(k-1)$-kez türevlenebilir ve k-yıncı türevi sürekli bir fonksiyon olsun. Aynı zamanda $0<\alpha\leq 1$ ve $k\geq 1$ tamsayısı için f fonksiyonun türevi Lip_M^{α} sınıfına ait olsun. Bu durumda,

$$\lim_{n\to\infty}\sup_{x\in\left[\frac{\alpha_2}{n+k+\beta_2}b_{n+k},\frac{n+k+\alpha_2}{n+k+\beta_2}b_{n+k}\right]}\frac{\left|T_{n+k,\alpha,\beta}^{(k)}(f;x)-f^{(k)}(x)\right|}{1+x^{\alpha}}=0$$

eşitliği gerçeklenir.

İspat:

Biz Ortalama Değer Teoremi'nden biliyoruz ki

$$\Delta_h^k f\left(\frac{r+\alpha_1}{n+k+\beta_1}b_{n+k}\right)=f^{(k)}(\xi_r)\frac{(b_{n+k})^k}{(n+k+\beta_1)^k} \tag{2.11}$$

$\frac{r+\alpha_1}{n+k+\beta_1}b_{n+k}<\xi_r<\frac{r+\alpha_1+k}{n+k+\beta_1}b_{n+k}$ olacak şekilde bir ξ_r sayısı vardır.

Bu durumda

$$\xi_r=\frac{r+\alpha_1+\theta_r k}{n+k+\beta_1}b_{n+k}\quad ,0<\theta_r<1$$

olarak seçelim. O halde birinci bölüm Lemma 1.3.1 ve (2.11)'den herhangi bir $k\geq 0$ tamsayısı için,

$$T_{n+k,\alpha,\beta}^{(k)}(f;x)$$

$$=\frac{(n+k)!}{n!}\left(\frac{1}{b_{n+k}}\right)^k\left(\frac{n+k+\beta_2}{n+k}\right)^{n+k}\sum_{r=0}^{n}\Delta_h^k f\left(\frac{r+\alpha_1}{n+k+\beta_1}b_{n+k}\right)$$

$$\times\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$=\frac{(n+k)!}{n!(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^{n+k}\sum_{r=0}^{n} f^{(k)}\left(\frac{r+\alpha_1+\theta_r k}{n+k+\beta_1}b_{n+k}\right)$$

$$\times\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

eşitliğini elde ederiz. Kolaylıkla görülebilir ki

$$\lim_{n\to\infty}\frac{(n+k)!}{n!(n+k+\beta_1)^k}=1$$

dir. Bu durumda

$$T^{(k)}_{n+k,\alpha,\beta}(f;x)-f^{(k)}(x)$$

$$=\frac{(n+k)!}{n!(n+k+\beta_1)^k}\left\{\left(\frac{n+k+\beta_2}{n+k}\right)^{n+k}\sum_{r=0}^{n}\left(f^{(k)}\left(\frac{r+\alpha_1+\theta_r k}{n+k+\beta_1}b_{n+k}\right)-f^{(k)}(x)\right)\right.$$

$$\left.\times\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}\right\}$$

$$+f^{(k)}(x)\left(\frac{(n+k)!}{n!(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k-1\right)$$

elde ederiz. $f^{(k)}\in Lip_M^{\alpha}$ olduğundan

$$\left|T^{(k)}_{n+k,\alpha,\beta}(f;x)-f^{(k)}(x)\right|$$

$$\leq M\frac{(n+k)!}{n!(n+k+\beta_1)^k}\left\{\left(\frac{n+k+\beta_2}{n+k}\right)^{n+k}\sum_{r=0}^{n}\left(\left|\frac{r+\alpha_1+\theta_r k}{n+k+\beta_1}b_{n+k}-x\right|^{\alpha}\right)\right.$$

$$\left.\times\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}\right\}$$

$$+\left|f^{(k)}(x)\right|\left|\left(\frac{(n+k)!}{n!(n+k+\beta_1)^k}\right)\left(\frac{n+k+\beta_2}{n+k}\right)^k-1\right|$$

gerçeklenir. Şimdi Hölder's Eşitsizliği'ni ve aşağıdaki eşitsizlikte

$M_f = \max\{M_1, M_2\}$ olmak üzere

$$\left|f^{(k)}(x)\right| \le \left|f^{(k)}(0)\right| M_1 + M_2 x^{\alpha}$$

$$\le M_f\left(1 + x^{\alpha}\right)$$

eşitsizliğini kullanalım. Bu durumda

$$\left|T_{n+k,\alpha,\beta}^{(k)}(f;x) - f^{(k)}(x)\right|$$

$$\le M \frac{(n+k)!}{n!(n+k+\beta_1)^k}\left\{\left(\frac{n+k+\beta_2}{n+k}\right)^{n+k}\sum_{r=0}^{n}\left(1.\left|\frac{r+\alpha_1+\theta_r k}{n+k+\beta_1}b_{n+k} - x\right|^{\alpha}\right)\right.$$

$$\left.\times \binom{n}{r}\left(\frac{x}{b_{n+k}} - \frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2} - \frac{x}{b_{n+k}}\right)^{n-r}\right\}$$

$$+ M_f\left(1 + x^{\alpha}\right)\left|\left(\frac{(n+k)!}{n!(n+k+\beta_1)^k}\right)\left(\frac{n+k+\beta_2}{n+k}\right)^k - 1\right|$$

$p = \dfrac{2}{\alpha}$ ve $q = \dfrac{2}{2-\alpha}$ için

$$\left|T_{n+k,\alpha,\beta}^{(k)}(f;x) - f^{(k)}(x)\right|$$

$$\le M \frac{(n+k)!}{n!(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k$$

$$\times\left\{\left\{\left(\frac{n+k+\beta_2}{n+k}\right)^n \sum_{r=0}^{n}\left(\left|\frac{r+\alpha_1+\theta_r k}{n+k+\beta_1}b_{n+k} - x\right|^{\alpha\frac{2}{\alpha}}\right)\right.\right.$$

$$\left.\times \binom{n}{r}\left(\frac{x}{b_{n+k}} - \frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2} - \frac{x}{b_{n+k}}\right)^{n-r}\right\}^{\frac{\alpha}{2}}$$

$$\times\left\{\left(\frac{n+k+\beta_2}{n+k}\right)^n \sum_{r=0}^{n}|1|^{\frac{2}{2-\alpha}}\right.$$

$$\times \binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}\Bigg\}^{1-\frac{\alpha}{2}}\Bigg\}$$

$$+M_f\left(1+x^\alpha\right)\left|\left(\frac{(n+k)!}{n!(n+k+\beta_1)^k}\right)\left(\frac{n+k+\beta_2}{n+k}\right)^k-1\right|$$

$$\leq M\frac{(n+k)!}{n!(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k$$

$$\times\left\{\left(\frac{n+k+\beta_2}{n+k}\right)^n\sum_{r=0}^{n}\left(\frac{r+\alpha_1+\theta_r k}{n+k+\beta_1}b_{n+k}-x\right)^2\right.$$

$$\left.\times\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}\right\}^{\frac{\alpha}{2}}$$

$$+M_f\left(1+x^\alpha\right)\left|\left(\frac{(n+k)!}{n!(n+k+\beta_1)^k}\right)\left(\frac{n+k+\beta_2}{n+k}\right)^k-1\right|$$

elde edilir.

$$\widetilde{T}_{n,\alpha,\beta}(f;x)$$

$$=\left(\frac{n+k+\beta_2}{n+k}\right)^n\sum_{r=0}^{n}f\left(\frac{r+\alpha_1+k}{n+k+\beta_1}b_{n+k}\right)\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

olacak şekilde bir $\widetilde{T}_{n,\alpha,\beta}(f;x)$ operatörü tanımlayalım. Bu operatörü kullanarak açıktır ki

$$t=\frac{r+\alpha_1+k}{n+k+\beta_1}b_{n+k}$$

olmak üzere

$$\left|T^{(k)}_{n+k,\alpha,\beta}(f;x)-f^{(k)}(x)\right|\leq M\frac{(n+k)!}{n!(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k\left[\widetilde{T}_{n,\alpha,\beta}\left((t-x)^2;x\right)\right]^{\frac{\alpha}{2}}$$

$$+M_f\left(1+x^{\alpha}\right)\left|\left(\frac{(n+k)!}{n!(n+k+\beta_1)^k}\right)\left(\frac{n+k+\beta_2}{n+k}\right)^k-1\right| \tag{2.12}$$

dir. Şimdi $\widetilde{T}_{n,\alpha,\beta}\left((t-x)^2;x\right)$ operatörünü hesaplayalım. Bu durumda,

$$\widetilde{T}_{n,\alpha,\beta}\left((t-x)^2;x\right)$$

$$=\left(\frac{n+k+\beta_2}{n+k}\right)^n\sum_{r=0}^{n}\left(\frac{r+\alpha_1+k}{n+k+\beta_1}b_{n+k}-x\right)^2$$

$$\times\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$=\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{1}{n+k+\beta_1}\right)^2\sum_{r=0}^{n}\left((r+\alpha_1+k)b_{n+k}-x(n+k+\beta_1)\right)^2$$

$$\times\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$=\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=0}^{n}(r+\alpha_1+k)^2$$

$$\times\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$-\left(\frac{n+k+\beta_2}{n+k}\right)^n 2x(n+k+\beta_1)b_{n+k}\left(\frac{1}{n+k+\beta_1}\right)^2$$

$$\times\sum_{r=0}^{n}(r+\alpha_1+k)\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$+\left(\frac{1}{n+k+\beta_1}\right)^2\left(x(n+k+\beta_1)\right)^2$$

$$=\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=0}^{n}r^2\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=0}^{n}2r(\alpha_1+k)$$

$$\times\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=0}^{n}(\alpha_1+k)^2$$

$$\times\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$-\left(\frac{n+k+\beta_2}{n+k}\right)^n 2x(n+k+\beta_1)b_{n+k}\left(\frac{1}{n+k+\beta_1}\right)^2$$

$$\times\sum_{r=0}^{n}r\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$-\left(\frac{n+k+\beta_2}{n+k}\right)^n 2x(n+k+\beta_1)b_{n+k}\left(\frac{1}{n+k+\beta_1}\right)^2$$

$$\times\sum_{r=0}^{n}(\alpha_1+k)\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$+\left(\frac{1}{n+k+\beta_1}\right)^2(x(n+k+\beta_1))^2$$

yazılabilir. Son eşitliğin sağ tarafındaki ilk toplamda r ekleyip çıkaralım.

O halde

$$\widetilde{T}_{n,\alpha,\beta}\left((t-x)^2;x\right)$$

$$=\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=0}^{n}(r^2-r)\binom{n}{r}$$

$$\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=0}^{n} r\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=0}^{n} 2r(\alpha_1+k)\binom{n}{r}$$

$$\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=0}^{n} (\alpha_1+k)^2\binom{n}{r}$$

$$\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$-\left(\frac{n+k+\beta_2}{n+k}\right)^n 2x(n+k+\beta_1)b_{n+k}\left(\frac{1}{n+k+\beta_1}\right)^2$$

$$\times\sum_{r=0}^{n} r\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$-\left(\frac{n+k+\beta_2}{n+k}\right)^n 2x(n+k+\beta_1)b_{n+k}\left(\frac{1}{n+k+\beta_1}\right)^2$$

$$\times\sum_{r=0}^{n} (\alpha_1+k)\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$+\left(\frac{1}{n+k+\beta_1}\right)^2\left(x(n+k+\beta_1)\right)^2$$

elde edilir. Gerekli sadeleştirmeler yapılırsa

$$\widetilde{T}_{n,\alpha,\beta}\left((t-x)^2;x\right)$$

$$=\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=2}^{n} n(n-1)\binom{n-2}{r-2}$$

$$\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=1}^{n}n\binom{n-1}{r-1}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=1}^{n}2n(\alpha_1+k)\binom{n-1}{r-1}$$

$$\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=0}^{n}(\alpha_1+k)^2\binom{n}{r}$$

$$\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$-\left(\frac{n+k+\beta_2}{n+k}\right)^n 2x(n+k+\beta_1)b_{n+k}\left(\frac{1}{n+k+\beta_1}\right)^2$$

$$\times\sum_{r=1}^{n}n\binom{n-1}{r-1}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$-\left(\frac{n+k+\beta_2}{n+k}\right)^n 2x(n+k+\beta_1)b_{n+k}\left(\frac{1}{n+k+\beta_1}\right)^2$$

$$\times\sum_{r=0}^{n}(\alpha_1+k)\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$+\left(\frac{1}{n+k+\beta_1}\right)^2\left(x(n+k+\beta_1)\right)^2$$

elde edilir. Yukarıdaki eşitliğin sağ tarafındaki birinci toplamda r yerine $r+2$, ikinci, üçüncü ve dördüncü toplamda r yerine $r+1$ yazarsak

$$\widetilde{T}_{n,\alpha,\beta}\left((t-x)^2;x\right)$$

$$=\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=0}^{n-2}n(n-1)\binom{n-2}{r}$$

$$\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r+2}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r-2}$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=0}^{n-1}n\binom{n-1}{r}$$

$$\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r+1}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r-1}$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=0}^{n-1}2n(\alpha_1+k)\binom{n-1}{r}$$

$$\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r+1}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r-1}$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)^n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\sum_{r=0}^{n}(\alpha_1+k)^2\binom{n}{r}$$

$$\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$-\left(\frac{n+k+\beta_2}{n+k}\right)^n 2x(n+k+\beta_1)b_{n+k}\left(\frac{1}{n+k+\beta_1}\right)^2$$

$$\times\sum_{r=0}^{n-1}n\binom{n-1}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r+1}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r-1}$$

$$-\left(\frac{n+k+\beta_2}{n+k}\right)^n 2x(n+k+\beta_1)b_{n+k}\left(\frac{1}{n+k+\beta_1}\right)^2$$

$$\times\sum_{r=0}^{n}(\alpha_1+k)\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^{r}\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$+\left(\frac{1}{n+k+\beta_1}\right)^2 \left(x(n+k+\beta_1)\right)^2$$

elde ederiz. Gerekli düzenlemeler yapılırsa

$$\widetilde{T}_{n,\alpha,\beta}\left((t-x)^2;x\right)$$

$$=\left(\frac{n+k+\beta_2}{n+k}\right)^n \left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2 n(n-1)\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^2$$

$$\times\sum_{r=0}^{n-2}\binom{n-2}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r \left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r-2}$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)^n \left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2 n\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)$$

$$\times\sum_{r=0}^{n-1}\binom{n-1}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r \left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r-1}$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)^n \left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2 2n(\alpha_1+k)\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)$$

$$\times\sum_{r=0}^{n-1}\binom{n-1}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r \left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r-1}$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)^n \left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2 (\alpha_1+k)^2$$

$$\times\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r \left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$-\left(\frac{n+k+\beta_2}{n+k}\right)^n 2xn(n+k+\beta_1)b_{n+k}\left(\frac{1}{n+k+\beta_1}\right)^2 \left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)$$

$$\times\sum_{r=0}^{n-1}\binom{n-1}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r \left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r-1}$$

$$-\left(\frac{n+k+\beta_2}{n+k}\right)^n 2x(n+k+\beta_1)b_{n+k}\left(\frac{1}{n+k+\beta_1}\right)^2(\alpha_1+k)$$

$$\times\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}$$

$$+\left(\frac{1}{n+k+\beta_1}\right)^2(x(n+k+\beta_1))^2$$

$$=\left(\frac{n+k+\beta_2}{n+k}\right)^2\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2 n^2\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^2$$

$$-\left(\frac{n+k+\beta_2}{n+k}\right)^2 n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^2$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)n\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)$$

$$+\left(\frac{n+k+\beta_2}{n+k}\right)\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2 2n(\alpha_1+k)\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)+\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2(\alpha_1+k)^2$$

$$-\left(\frac{n+k+\beta_2}{n+k}\right)2xn\left(\frac{b_{n+k}}{n+k+\beta_1}\right)\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)-2x\left(\frac{b_{n+k}}{n+k+\beta_1}\right)(\alpha_1+k)+x^2$$

$$=\left(\frac{n}{n+k}\right)^2\left(\frac{1}{n+k+\beta_1}\right)^2(x(n+k+\beta_2)-\alpha_2 b_{n+k})^2$$

$$-n\left(\frac{1}{n+k}\right)^2\left(\frac{1}{n+k+\beta_1}\right)^2(x(n+k+\beta_2)-\alpha_2 b_{n+k})^2$$

$$+\left(\frac{b_{n+k}}{n+k}\right)n\left(\frac{1}{n+k+\beta_1}\right)^2(x(n+k+\beta_2)-\alpha_2 b_{n+k})$$

$$+2n(\alpha_1+k)\left(\frac{b_{n+k}}{n+k}\right)\left(\frac{1}{n+k+\beta_1}\right)^2(x(n+k+\beta_2)-\alpha_2 b_{n+k})$$

$$+b_{n+k}^2\left(\frac{1}{n+k+\beta_1}\right)^2(\alpha_1+k)^2-\left(\frac{2xn}{n+k}\right)\left(\frac{1}{n+k+\beta_1}\right)(x(n+k+\beta_2)-\alpha_2 b_{n+k})$$

$$-2x(\alpha_1+k)\left(\frac{b_{n+k}}{n+k+\beta_1}\right)+x^2$$

$$=n(n-1)\left(\frac{1}{n+k}\right)^2\left(\frac{1}{n+k+\beta_1}\right)^2 x^2(n+k+\beta_2)^2$$

$$-2\left(\frac{n}{n+k}\right)^2\left(\frac{1}{n+k+\beta_1}\right)^2 x(n+k+\beta_2)\alpha_2 b_{n+k}$$

$$+\left(\frac{n}{n+k}\right)^2\left(\frac{1}{n+k+\beta_1}\right)^2\alpha_2^2 b_{n+k}^2+\left(\frac{b_{n+k}}{n+k}\right)\left(\frac{1}{n+k+\beta_1}\right)^2 nx(n+k+\beta_2)$$

$$-\left(\frac{b_{n+k}}{n+k}\right)\left(\frac{1}{n+k+\beta_1}\right)^2 n\alpha_2 b_{n+k}+2(\alpha_1+k)n\left(\frac{b_{n+k}}{n+k}\right)\left(\frac{1}{n+k+\beta_1}\right)^2 x(n+k+\beta_2)$$

$$-2(\alpha_1+k)n\left(\frac{b_{n+k}}{n+k}\right)\left(\frac{1}{n+k+\beta_1}\right)^2\alpha_2 b_{n+k}+b_{n+k}^2\left(\frac{1}{n+k+\beta_1}\right)^2(\alpha_1+k)^2$$

$$-\left(\frac{2xn}{n+k}\right)\left(\frac{1}{n+k+\beta_1}\right)\left(x(n+k+\beta_2)-\alpha_2 b_{n+k}\right)-2x(\alpha_1+k)\left(\frac{b_{n+k}}{n+k+\beta_1}\right)+x^2$$

$$=x^2\left[\left(\left(\frac{n}{n+k}\right)\left(\frac{n+k+\beta_2}{n+k+\beta_1}\right)-1\right)^2-n\left(\frac{1}{n+k}\right)^2\left(\frac{n+k+\beta_2}{n+k+\beta_1}\right)\right]$$

$$+x\left\{\left(\frac{b_{n+k}}{n+k+\beta_1}\right)\left\{\left(\frac{2n}{n+k}\right)\alpha_2-2(\alpha_1+k)-2\left(\frac{n}{n+k}\right)^2\left(\frac{n+k+\beta_2}{n+k+\beta_1}\right)\alpha_2\right.\right.$$

$$\left.\left.+2(\alpha_1+k+1)\left(\frac{n}{n+k}\right)\left(\frac{n+k+\beta_2}{n+k+\beta_1}\right)\right\}\right\}$$

$$+\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\left\{\left[\left(\frac{n}{n+k}\right)\alpha_2-(\alpha_1+k)\left(\frac{b_{n+k}}{n+k+\beta_1}\right)\right]^2+2\alpha_2\left(\frac{n}{n+k}\right)\right\}$$

olarak bulunur. Bu durumda

$$\gamma_n=\left[\left(\left(\frac{n}{n+k}\right)\left(\frac{n+k+\beta_2}{n+k+\beta_1}\right)-1\right)^2-n\left(\frac{1}{n+k}\right)^2\left(\frac{n+k+\beta_2}{n+k+\beta_1}\right)\right]$$

$$\sigma_n = \left\{\left(\frac{b_{n+k}}{n+k+\beta_1}\right)\left\{\left(\frac{2n}{n+k}\right)\alpha_2 - 2(\alpha_1+k) - 2\left(\frac{n}{n+k}\right)^2\left(\frac{n+k+\beta_2}{n+k+\beta_1}\right)\alpha_2\right.\right.$$

$$\left.\left. + 2(\alpha_1+k+1)\left(\frac{n}{n+k}\right)\left(\frac{n+k+\beta_2}{n+k+\beta_1}\right)\right\}\right\}$$

$$\tau_n = \left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\left\{\left[\left(\frac{n}{n+k}\right)\alpha_2 - (\alpha_1+k)\left(\frac{b_{n+k}}{n+k+\beta_1}\right)\right]^2 + 2\alpha_2\left(\frac{n}{n+k}\right)\right\}$$

alınırsa

$$\widetilde{T}_{n,\alpha,\beta}\left((t-x)^2;x\right) = x^2\gamma_n + x\sigma_n + \tau_n$$

elde edilir. Yukarıdaki eşitliği (2.12)'de yerine yazarsak

$$\left|T^{(k)}_{n+k,\alpha,\beta}(f;x) - f^{(k)}(x)\right| \le M\frac{(n+k)!}{n!(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k\left[\widetilde{T}_{n,\alpha,\beta}\left((t-x)^2;x\right)\right]^{\frac{\alpha}{2}}$$

$$+ M_f\left(1+x^\alpha\right)\left|\left(\frac{(n+k)!}{n!(n+k+\beta_1)^k}\right)\left(\frac{n+k+\beta_2}{n+k}\right)^k - 1\right|$$

$$\le M\frac{(n+k)!}{n!(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k\left[x^2\gamma_n + x\sigma_n + \tau_n\right]^{\frac{\alpha}{2}}$$

$$+ M_f\left(1+x^\alpha\right)\left|\left(\frac{(n+k)!}{n!(n+k+\beta_1)^k}\right)\left(\frac{n+k+\beta_2}{n+k}\right)^k - 1\right|$$

elde ederiz. Yukarıdaki eşitsizliğin her iki tarafında önce $\frac{1}{\left(1+x^\alpha\right)}$ ile çarpıp

sonrada $x \in \left[\frac{\alpha_2}{n+k+\beta_2}b_{n+k}, \frac{n+k+\alpha_2}{n+k+\beta_2}b_{n+k}\right]$ üzerinden supremum alalım.

Bu durumda,

$$\frac{\left|T^{(k)}_{n+k,\alpha,\beta}(f;x) - f^{(k)}(x)\right|}{\left(1+x^\alpha\right)} \le M\frac{(n+k)!}{n!(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k\frac{1}{\left(1+x^\alpha\right)}\left[x^2\gamma_n + x\sigma_n + \tau_n\right]^{\frac{\alpha}{2}}$$

$$+M_f\left|\left(\frac{(n+k)!}{n!(n+k+\beta_1)^k}\right)\left(\frac{n+k+\beta_2}{n+k}\right)^k-1\right|$$

$$\sup_{x\in\left[\frac{\alpha_2}{n+k+\beta_2}b_{n+k},\frac{n+k+\alpha_2}{n+k+\beta_2}b_{n+k}\right]}\frac{\left|T^{(k)}_{n+k,\alpha,\beta}(f;x)-f^{(k)}(x)\right|}{(1+x^{\alpha})}$$

$$\leq M\frac{(n+k)!}{n!(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k\sup_{x\in\left[\frac{\alpha_2}{n+k+\beta_2}b_{n+k},\frac{n+k+\alpha_2}{n+k+\beta_2}b_{n+k}\right]}\frac{1}{(1+x^{\alpha})}\left[x^2\gamma_n+x\sigma_n+\tau_n\right]^{\frac{\alpha}{2}}$$

$$+M_f\left|\left(\frac{(n+k)!}{n!(n+k+\beta_1)^k}\right)\left(\frac{n+k+\beta_2}{n+k}\right)^k-1\right|$$

elde edilir. Yukarıdaki eşitsizlikte $n\to\infty$ için limit alınırsa

$$\lim_{n\to\infty}\sup_{x\in\left[\frac{\alpha_2}{n+k+\beta_2}b_{n+k},\frac{n+k+\alpha_2}{n+k+\beta_2}b_{n+k}\right]}\frac{\left|T^{(k)}_{n+k,\alpha,\beta}(f;x)-f^{(k)}(x)\right|}{1+x^{\alpha}}=0$$

elde edilerek ispat tamamlanır.

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