Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 621469, 17 pages
doi:10.1155/2010/621469

Research Article

Some Fixed Point Theorems on Ordered Metric Spaces and Application

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Received 2 July 2009; Accepted 13 January 2010

Academic Editor: Juan Jose Nieto

We present some fixed point results for nondecreasing and weakly increasing operators in a partially ordered metric space using implicit relations. Also we give an existence theorem for common solution of two integral equations.

1. Introduction

Existence of fixed points in partially ordered sets has been considered recently in [1], and some generalizations of the result of [1] are given in [2–6]. Also, in [1] some applications to matrix equations are presented, in [3, 4] some applications to periodic boundary value problem and to some particular problems are, respectively, given. Later, in [6] O'Regan and Petruşel gave some existence results for Fredholm and Volterra type integral equations. In some of the above works, the fixed point results are given for nondecreasing mappings.

We can order the purposes of the paper as follows.

First, we give a slight generalization of some of the results of the above papers using an implicit relation in the following way.

In [1, 3], the authors used the following contractive condition in their result, there exists $k \in [0,1)$ such that

$$d(fx, fy) \le kd(x, y) \quad \text{for } y \preceq x. \tag{1.1}$$

Afterwards, in [2], the authors used the nonlinear contractive condition, that is,

$$d(fx, fy) \le \varphi(d(x, y)) \quad \text{for } y \preceq x, \tag{1.2}$$

where $\psi : [0,\infty) \to [0,\infty)$ is anondecreasing function with $\lim_{n\to\infty}\psi^n(t) = 0$ for $t > 0$, instead of (1.1). Also in [2], the authors proved a fixed point theorem using generalized nonlinear contractive condition, that is,

$$d(fx, fy) \leq \psi\left(\max\left\{d(x,y), d(x,fx), d(y,fy), \frac{1}{2}[d(x,fy)+d(y,fx)]\right\}\right) \tag{1.3}$$

for $y \preceq x$, where ψ is as above. In the Section 3, we generalized the above contractive conditions using the implicit relation technique in such a way that

$$T(d(Fx,Fy), d(x,y), d(x,Fx), d(y,Fy), d(x,Fy), d(y,Fx)) \leq 0 \tag{1.4}$$

for $y \preceq x$, where $T : \mathbb{R}_+^6 \to \mathbb{R}$ is a function as given in Section 2. We can obtain various contractive conditions from (1.4). For example, if we choose

$$T(t_1,\ldots,t_6) = t_1 - \psi\left(\max\left\{t_2, t_3, t_4, \frac{1}{2}[t_5+t_6]\right\}\right) \tag{1.5}$$

in (1.4), then, we have (1.3). Similarly we can have the contractive conditions in [7–9] from (1.4).

In some of the above mentioned theorems, the fixed point results are given for nondecreasing mappings. Also in these theorems the following condition is used:

$$\text{there exists } x_0 \in X \text{ such that } x_0 \preceq fx_0. \tag{1.6}$$

In Section 4, we give some examples such that two weakly increasing mappings need not be nondecreasing. Therefore, we give a common fixed point theorem for two weakly increasing operators in partially ordered metric spaces using implicit relation technique. Also we did not use the condition (1.6) in this theorem. At the end, to see the applicability of our result, we give an existence theorem for common solution of two integral equations using a result of the Section 4.

2. Implicit Relation

Implicit relations on metric spaces have been used in many articles. See for examples, [10–15].

Let $\mathbb{R}_+$ denote the nonnegative real numbers, and let $\mathcal{T}$ be the set of all continuous functions $T : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

T_1 : $T(t_1,\ldots,t_6)$ is nonincreasing in variables $t_2,\ldots,t_6$;

T_2 : there exists a right continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(0) = 0$, $f(t) < t$ for $t > 0$, such that for $u \geq 0$,

$$T(u, v, u, v, 0, u+v) \leq 0 \tag{2.1}$$

or

$$T(u,v,0,0,v,v) \le 0 \tag{2.2}$$

implies $u \le f(v)$;

$T_3 : T(u,0,u,0,0,u) > 0$, for all $u > 0$.

Example 2.1. $T(t_1,\ldots,t_6) = t_1 - \alpha \max\{t_2,t_3,t_4\} - (1-\alpha)[at_5 + bt_6]$, where $0 \le \alpha < 1$, $0 \le a < 1/2$, $0 \le b < 1/2$.

Let $u > 0$ and $T(u,v,u,v,0,u+v) = u - \alpha \max\{u,v\} - (1-\alpha)b(u+v) \le 0$. If $u \ge v$, then $(1-b)u \le bv$ which implies $b \ge 1/2$, a contradiction. Thus $u < v$ and $u \le ((\alpha + (1-\alpha)b)/(1-(1-\alpha)b))v = \beta v$. Similarly, let $u > 0$ and $T(u,v,0,0,v,v) = u - \alpha v - (1-\alpha)(a+b)v = u - (\alpha + (1-\alpha)(a+b))v \le 0$, then $u \le (\alpha + (1-\alpha)(a+b))v = \gamma v$. If $u = 0$, then $u \le \gamma v$. Thus T_2 is satisfied with $f(t) = \max\{\beta,\gamma\}t$. Also $T(u,0,u,0,0,u) = u - \alpha u - (1-\alpha)bu = (1-\alpha)(1-b)u > 0$, for all $u > 0$. Therefore, $T \in \mathcal{T}$.

Example 2.2. $T(t_1,\ldots,t_6) = t_1 - k \max\{t_2,t_3,t_4,(1/2)(t_5+t_6)\}$, where $k \in (0,1)$.

Let $u > 0$ and $T(u,v,u,v,0,u+v) = u - k\max\{u,v\} \le 0$. If $u \ge v$, then $u \le ku$, which is a contradiction. Thus $u < v$ and $u \le kv$. Similarly, let $u > 0$ and $T(u,v,0,0,v,v) = u - kv \le 0$, then we have $u \le kv$. If $u = 0$, then $u \le kv$. Thus T_2 is satisfied with $f(t) = kt$. Also $T(u,0,u,0,0,u) = u - ku > 0$, for all $u > 0$. Therefore, $T \in \mathcal{T}$.

Example 2.3. $T(t_1,\ldots,t_6) = t_1 - \phi(\max\{t_2,t_3,t_4,(1/2)(t_5+t_6)\})$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is right continuous and $\phi(0) = 0$, $\phi(t) < t$ for $t > 0$.

Let $u > 0$ and $T(u,v,u,v,0,u+v) = u - \phi(\max\{u,v\}) \le 0$. If $u \ge v$, then $u - \phi(u) \le 0$, which is a contradiction. Thus $u < v$ and $u \le \phi(v)$. Similarly, let $u > 0$ and $T(u,v,0,0,v,v) = u - \phi(v) \le 0$, then we have $u \le \phi(v)$. If $u = 0$, then $u \le \phi(v)$. Thus T_2 is satisfied with $f = \phi$. Also $T(u,0,u,0,0,u) = u - \phi(u) > 0$, for all $u > 0$. Therefore, $T \in \mathcal{T}$.

Example 2.4. $T(t_1,\ldots,t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6$, where $a > 0$, $b,c,d \ge 0$, $a+b+c < 1$ and $a + d < 1$.

Let $u > 0$ and $T(u,v,u,v,0,u+v) = u^2 - u(av + bu + cv) \le 0$. Then $u \le ((a+c)/(1-b))v = h_1 v$. Similarly, let $u > 0$ and $T(u,v,0,0,v,v) = u^2 - auv - dv^2 \le 0$, then we have $u \le ((a + \sqrt{4d + a^2})/2)v = h_2 v$. If $u = 0$, then $u \le h_2 v$. Thus T_2 is satisfied with $f(t) = \max\{h_1,h_2\}t$. Also $T(u,0,u,0,0,u) = (1-b)u^2 > 0$, for all $u > 0$. Therefore, $T \in \mathcal{T}$.

3. Fixed Point Theorem for Nondecreasing Mappings

We need the following lemma for the proof of our theorems.

Lemma 3.1 (see [16]). *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a right continuous function such that $f(t) < t$ for every $t > 0$, then $\lim_{n\to\infty} f^n(t) = 0$, where f^n denotes the n-times repeated composition of f with itself.*

Theorem 3.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \to X$ is a nondecreasing mapping such that for all $x, y \in X$ with $y \preceq x$,*

$$T(d(Fx, Fy), d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx)) \leq 0, \tag{3.1}$$

where $T \in \mathcal{T}$. Also

$$F \text{ is continuous}, \tag{3.2}$$

or

$$\begin{aligned} &\text{if } \{x_n\} \subset X \text{ is a nondecreasing sequence with } x_n \longrightarrow x \text{ in } X, \\ &\text{then } x_n \preceq x \quad \forall n \end{aligned} \tag{3.3}$$

hold. If there exists an $x_0 \in X$ with $x_0 \preceq F(x_0)$, then F has a fixed point.

Proof. If $Fx_0 = x_0$, then the proof is finished; so suppose $x_0 \neq Fx_0$. Now let $x_n = Fx_{n-1}$ for $n \in \{1, 2, \ldots\}$. Notice that, since $x_0 \preceq Fx_0$ and F is nondecreasing, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots. \tag{3.4}$$

Now since $x_{n-1} \preceq x_n$, we can use the inequality (3.1) for these points, then we have

$$T(d(Fx_n, Fx_{n-1}), d(x_n, x_{n-1}), d(x_n, Fx_n), d(x_{n-1}, Fx_{n-1}), d(x_n, Fx_{n-1}), d(x_{n-1}, Fx_n)) \leq 0 \tag{3.5}$$

and so

$$T(d(x_{n+1}, x_n), d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), 0, d(x_{n-1}, x_{n+1})) \leq 0. \tag{3.6}$$

Now using T_1, we have

$$T(d(x_{n+1}, x_n), d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), 0, d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \leq 0, \tag{3.7}$$

and from T_2 there exists a right continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(0) = 0$, $f(t) < t$, for $t > 0$, such that for all $n \in \{1, 2, \ldots\}$,

$$d(x_{n+1}, x_n) \leq f(d(x_n, x_{n-1})). \tag{3.8}$$

If we continue this procedure, we can have

$$d(x_{n+1}, x_n) \leq f^n(d(x_1, x_0)), \tag{3.9}$$

and so from Lemma 3.1,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.10}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. Suppose it is not true. Then we can find a $\delta > 0$ and two sequence of integers $\{m(k)\}, \{n(k)\}, m(k) > n(k) \geq k$ with

$$r_k = d\left(x_{n(k)}, x_{m(k)}\right) \geq \delta \quad \text{for } k \in \{1, 2, \ldots\}. \tag{3.11}$$

We may also assume

$$d\left(x_{m(k)-1}, x_{n(k)}\right) < \delta \tag{3.12}$$

by choosing $m(k)$ to be the smallest number exceeding $n(k)$ for which (3.11) holds. Now (3.9), (3.11), and (3.12) imply

$$\delta \leq r_k \leq d\left(x_{m(k)}, x_{m(k)-1}\right) + d\left(x_{m(k)-1}, x_{n(k)}\right) \leq f^{m(k)-1}(d(x_0, x_1)) + \delta \tag{3.13}$$

and so

$$\lim_{k\to\infty} r_k = \delta. \tag{3.14}$$

Also since

$$\delta \leq r_k \leq d\left(x_{n(k)}, x_{n(k)+1}\right) + d\left(x_{m(k)}, x_{m(k)+1}\right) + d\left(x_{n(k)+1}, x_{m(k)+1}\right), \tag{3.15}$$

we have from (3.9) that

$$\delta \leq r_k \leq f^{n(k)}(d(x_0, x_1)) + f^{m(k)}(d(x_0, x_1)) + d\left(x_{m(k)+1}, x_{n(k)+1}\right). \tag{3.16}$$

On the other hand, since $x_{n(k)} \preceq x_{m(k)}$, we can use the condition (3.1) for these points. Therefore, we have

$$\begin{aligned} &T\left(d\left(Fx_{m(k)}, Fx_{n(k)}\right), d\left(x_{m(k)}, x_{n(k)}\right), d\left(x_{m(k)}, Fx_{m(k)}\right),\right. \\ &\quad \left. d\left(x_{n(k)}, Fx_{n(k)}\right), d\left(x_{m(k)}, Fx_{n(k)}\right), d\left(x_{n(k)}, Fx_{m(k)}\right)\right) \leq 0 \end{aligned} \tag{3.17}$$

and so

$$\begin{aligned} &T\Big(d\left(Fx_{m(k)}, Fx_{n(k)}\right), r_k, f^{m(k)}(d(x_0, x_1)), f^{n(k)}(d(x_0, x_1)), \\ &\quad r_k + f^{n(k)}(d(x_0, x_1)), r_k + f^{m(k)}(d(x_0, x_1))\Big) \leq 0. \end{aligned} \tag{3.18}$$

Now letting $k \to \infty$ and using (3.14), we have, by continuity of T, that

$$T\left(\lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right), \delta, 0, 0, \delta, \delta\right) \leq 0. \tag{3.19}$$

From T_2, we have $\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq f(\delta)$. Therefore, letting $k \to \infty$ in (3.16), we have $\delta \leq f(\delta)$. This is a contradiction since $f(t) < t$ for $t > 0$. Thus $\{x_n\}$ is a Cauchy sequence in X, so there exists an $x \in X$ with $\lim_{n\to\infty} x_n = x$.

If (3.2) holds, then clearly $x = Fx$. Now suppose (3.3) holds. Suppose $d(x, Fx) > 0$. Now since $\lim_{n\to\infty} x_n = x$, then from (3.3), $x_n \preceq x$ for all n. Using the inequality (3.1), we have

$$T(d(Fx, Fx_n), d(x, x_n), d(x, Fx), d(x_n, Fx_n), d(x, Fx_n), d(x_n, Fx)) \leq 0, \tag{3.20}$$

so letting $n \to \infty$ from the last inequality, we have

$$T(d(Fx, x), 0, d(x, Fx), 0, 0, d(x, Fx)) \leq 0, \tag{3.21}$$

which is a contradiction to T_3. Thus $d(x, Fx) = 0$ and so $x = Fx$. □

Remark 3.3. Note that if we take that

T_4: there exists a nondecreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{n\to\infty} f^n(t) = 0$ for each $t > 0$, such that for $u \geq 0$,

$$T(u, v, u, v, 0, u + v) \leq 0 \tag{3.22}$$

or

$$T(u, v, 0, 0, v, v) \leq 0 \tag{3.23}$$

implies $u \leq f(v)$,

Instead of T_2 in Theorem 3.2, again we can have the same result.

If we combine Theorem 3.2 with Example 2.1, we obtain the following result.

Corollary 3.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \to X$ is a nondecreasing mapping such that for all $x, y \in X$ with $y \preceq x$,*

$$d(Fx, Fy) \leq \alpha \max\{d(x, y), d(x, Fx), d(y, Fy)\} + (1 - \alpha)\left[a d(x, Fy) + b d(y, Fx)\right], \tag{3.24}$$

where $0 \leq \alpha < 1$, $0 \leq a < 1/2$, $0 \leq b < 1/2$. Also

$$F \text{ is continuous} \tag{3.25}$$

or

$$\text{if } \{x_n\} \subset X \text{ is a nondecreasing sequence with } x_n \longrightarrow x \text{ in } X, \\ \text{then } x_n \preceq x \quad \forall n \tag{3.26}$$

hold. If there exists an $x_0 \in X$ *with* $x_0 \preceq F(x_0)$, *then* F *has a fixed point.*

Remark 3.5. Theorem 2.2 of [2] follows from Example 2.3, Remark 3.3, and Theorem 3.2.

Remark 3.6. We can have some new results from other examples and Theorem 3.2.

Remark 3.7. In Theorem 1 [1], it is proved that if

$$\text{every pair of elements has a lower bound and an upper bound,} \tag{3.27}$$

then for every $x \in X$,

$$\lim_{n\to\infty} F^n(x) = y, \tag{3.28}$$

where y is the fixed point of F such that

$$y = \lim_{n\to\infty} F^n(x_0) \tag{3.29}$$

and hence F has a unique fixed point. If condition (3.27) fails, it is possible to find examples of functions F with more than one fixed point. There exist some examples to illustrate this fact in [3].

4. Fixed Point Theorem for Weakly Increasing Mappings

Now we give a fixed point theorem for two weakly increasing mappings in ordered metric spaces using an implicit relation. Before this, we will define an implicit relation for the contractive condition of the theorem.

Let $\mathcal{T}'$ be the set of all continuous functions $T : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying T_1 and the following conditions:

T_2': there exists a right continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(0) = 0$, $f(t) < t$ for $t > 0$, such that for $u \geq 0$,

$$T(u, v, u, v, 0, u + v) \leq 0 \tag{4.1}$$

or

$$T(u, v, v, u, u + v, 0) \leq 0 \tag{4.2}$$

or

$$T(u, v, 0, 0, v, v) \leq 0 \tag{4.3}$$

implies $u \le f(v)$;

T_3': $T(u,0,u,0,0,u) > 0$ and $T(u,0,0,u,u,0) > 0$, for all $u > 0$.

We can easily show that, all functions in the Examples in Section 2 are in $\mathcal{T}'$.

Definition 4.1 (see [17, 18]). Let $(X,\preceq)$ be a partially ordered set. Two mappings $F,G : X \to X$ are said to be weakly increasing if $Fx \preceq GFx$ and $Gx \preceq FGx$ for all $x \in X$.

Note that, two weakly increasing mappings need not be nondecreasing.

Example 4.2. Let $X = \mathbb{R}_+$ endowed with usual ordering. Let $F,G : X \to X$ defined by

$$Fx = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 1 < x < \infty, \end{cases} \qquad Gx = \begin{cases} \sqrt{x} & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 1 < x < \infty, \end{cases} \tag{4.4}$$

then it is obvious that $Fx \le GFx$ and $Gx \le FGx$ for all $x \in X$. Thus F and G are weakly increasing mappings. Note that both F and G are not nondecreasing.

Example 4.3. Let $X = [1,\infty) \times [1,\infty)$ be endowed with the coordinate ordering, that is, $(x,y) \preceq (z,w) \Leftrightarrow x \le z$ and $y \le w$. Let $F,G : X \to X$ be defined by $F(x,y) = (2x,3y)$ and $G(x,y) = (x^2,y^2)$, then $F(x,y) = (2x,3y) \preceq GF(x,y) = G(2x,3y) = (4x^2,9y^2)$ and $G(x,y) = (x^2,y^2) \preceq FG(x,y) = F(x^2,y^2) = (2x^2,3y^2)$. Thus F and G are weakly increasing mappings.

Example 4.4. Let $X = \mathbb{R}^2$ be endowed with the lexicographical ordering, that is, $(x,y) \preceq (z,w) \Leftrightarrow (x < z)$ or (if $x = z$, then $y \le w$). Let $F,G : X \to X$ be defined by

$$\begin{aligned} F(x,y) &= (\max\{x,y\},\min\{x,y\}), \\ G(x,y) &= \left(\max\{x,y\},\frac{x+y}{2}\right), \end{aligned} \tag{4.5}$$

then

$$\begin{aligned} F(x,y) &= (\max\{x,y\},\min\{x,y\}) \\ &\preceq GF(x,y) \\ &= G(\max\{x,y\},\min\{x,y\}) \\ &= \left(\max\{\max\{x,y\},\min\{x,y\}\},\frac{\max\{x,y\}+\min\{x,y\}}{2}\right) \\ &= \left(\max\{x,y\},\frac{x+y}{2}\right), \end{aligned}$$

$$\begin{aligned}G(x,y) &= \left(\max\{x,y\}, \frac{x+y}{2}\right)\\ &\preceq FG(x,y)\\ &= F\left(\max\{x,y\}, \frac{x+y}{2}\right)\\ &= \left(\max\left\{\max\{x,y\}, \frac{x+y}{2}\right\}, \min\left\{\max\{x,y\}, \frac{x+y}{2}\right\}\right)\\ &= \left(\max\{x,y\}, \frac{x+y}{2}\right).\end{aligned} \tag{4.6}$$

Thus F and G are weakly increasing mappings. Note that $(1,4) \preceq (2,3)$ but $F(1,4) = (4,1)(3,2) = F(2,3)$, then F is not nondecreasing. Similarly, G is not nondecreasing.

Theorem 4.5. *Let $(X,\preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X,d) is a complete metric space. Suppose $F,G : X \to X$ are two weakly increasing mappings such that for all comparable $x,y \in X$,*

$$T(d(Fx,Gy), d(x,y), d(x,Fx), d(y,Gy), d(x,Gy), d(y,Fx)) \le 0, \tag{4.7}$$

where $T \in \mathcal{T}'$. Also

$$F \text{ is continuous} \tag{4.8}$$

or

$$G \text{ is continuous} \tag{4.9}$$

or

$$\begin{aligned}&\text{if } \{x_n\} \subset X \text{ is a nondecreasing sequence with } x_n \longrightarrow x \text{ in } X,\\ &\text{then } x_n \preceq x \quad \forall n\end{aligned} \tag{4.10}$$

hold, then F and G have a common fixed point.

Remark 4.6. Note that, in this theorem we remove the condition "there exists an $x_0 \in X$ with $x_0 \preceq F(x_0)$" of Theorem 3.2. Again we can consider the result of Remark 3.7 for this theorem.

Proof of Theorem 4.5. First of all we show that if F or G has a fixed point, then it is a common fixed point of F and G. Indeed, let z be a fixed point of F. Now assume $d(z,Gz) > 0$. If we use the inequality (4.7), for $x = y = z$, we have

$$T(d(z,Gz), 0, 0, d(z,Gz), d(z,Gz), 0) \le 0, \tag{4.11}$$

which is a contradiction to T_3'. Thus $d(z, Gz) = 0$ and so z is a common fixed point of F and G. Similarly, if z is a fixed point of G, then it is also a fixed point of F. Now let x_0 be an arbitrary point of X. If $x_0 = Fx_0$, the proof is finished, so assume $x_0 \neq Fx_0$. We can define a sequence $\{x_n\}$ in X as follows:

$$x_{2n+1} = Fx_{2n}, \qquad x_{2n+2} = Gx_{2n+1} \quad \text{for } n \in \{0, 1, \ldots\}. \tag{4.12}$$

Without loss of generality, we can suppose that the successive terms of $\{x_n\}$ are different. Otherwise, we are again finished. Note that since F and G are weakly increasing, we have

$$\begin{aligned} x_1 &= Fx_0 \preceq GFx_0 = Gx_1 = x_2, \\ x_2 &= Gx_1 \preceq FGx_1 = Fx_2 = x_3 \end{aligned} \tag{4.13}$$

and continuing this process, we have

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots . \tag{4.14}$$

Now since x_{2n-1} and x_{2n} are comparable then, we can use the inequality (4.7) for these points then we have

$$\begin{aligned} &T(d(Fx_{2n}, Gx_{2n-1}), d(x_{2n}, x_{2n-1}), d(x_{2n}, Fx_{2n}), d(x_{2n-1}, Gx_{2n-1}), \\ &\quad d(x_{2n}, Gx_{2n-1}), d(x_{2n-1}, Fx_{2n})) \leq 0 \end{aligned} \tag{4.15}$$

and so

$$T(d(x_{2n+1}, x_{2n}), d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), 0, d(x_{2n-1}, x_{2n+1})) \leq 0. \tag{4.16}$$

Now using T_1, we have

$$T(d(x_{2n+1}, x_{2n}), d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), 0, d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})) \leq 0, \tag{4.17}$$

and form T_2' there exists a right continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(0) = 0$, $f(t) < t$, for $t > 0$, we have for all $n \in \{1, 2, \ldots\}$

$$d(x_{2n+1}, x_{2n}) \leq f(d(x_{2n}, x_{2n-1})). \tag{4.18}$$

Similarly, since x_{2n} and x_{2n+1} are comparable, then we can use the inequality (4.7) for these points then we have

$$\begin{aligned} &T(d(Fx_{2n}, Gx_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n}, Fx_{2n}), d(x_{2n+1}, Gx_{2n+1}), \\ &\quad d(x_{2n}, Gx_{2n+1}), d(x_{2n+1}, Fx_{2n})) \leq 0 \end{aligned} \tag{4.19}$$

and so

$$T(d(x_{2n+1},x_{2n+2}),d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),d(x_{2n},x_{2n+2}),0)\leq 0. \quad (4.20)$$

Now again using T_1, we have

$$\begin{aligned}&T(d(x_{2n+1},x_{2n+2}),d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),\\&\quad d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2}),0)\leq 0,\end{aligned} \quad (4.21)$$

and form T_2', we have for all $n\in\{1,2,\ldots\}$,

$$d(x_{2n+1},x_{2n+2})\leq f(d(x_{2n},x_{2n+1})). \quad (4.22)$$

Therefore, from (4.18) and (4.22), we can have, for all $n\in\{2,3,\ldots\}$

$$d(x_{n+1},x_n)\leq f(d(x_n,x_{n-1})) \quad (4.23)$$

and so

$$d(x_{n+1},x_n)\leq f^{n-1}(d(x_2,x_1)). \quad (4.24)$$

Thus from Lemma 3.1, we have, since $d(x_2,x_1)>0$,

$$\lim_{n\to\infty}d(x_{n+1},x_n)=0. \quad (4.25)$$

Next we show that $\{x_n\}$ is a Cauchy sequence. For this it is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose it is not true. Then we can find an $\delta>0$ such that for each even integer $2k$, there exist even integers $2m(k)>2n(k)>2k$ such that

$$d\left(x_{2n(k)},x_{2m(k)}\right)\geq\delta \quad \text{for } k\in\{1,2,\ldots\}. \quad (4.26)$$

We may also assumethat

$$d\left(x_{2m(k)-2},x_{2n(k)}\right)<\delta \quad (4.27)$$

by choosing $2m(k)$ to be the smallest number exceeding $2n(k)$ for which (4.26) holds. Now (4.24), (4.26), and (4.27) imply

$$\begin{aligned}0<\delta&\leq d\left(x_{2n(k)},x_{2m(k)}\right)\\&\leq d\left(x_{2n(k)},x_{2m(k)-2}\right)+d\left(x_{2m(k)-2},x_{2m(k)-1}\right)+d\left(x_{2m(k)-1},x_{2m(k)}\right)\\&\leq\delta+f^{2m(k)-3}(d(x_2,x_1))+f^{2m(k)-2}(d(x_2,x_1))\end{aligned} \quad (4.28)$$

and so

$$\lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)}) = \delta. \tag{4.29}$$

Also, by the triangular inequality,

$$\begin{aligned} \left|d(x_{2n(k)}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})\right| &\le d(x_{2m(k)-1}, x_{2m(k)}) \le f^{2m(k)-2}(d(x_2, x_1)), \\ \left|d(x_{2n(k)+1}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})\right| &\le d(x_{2m(k)-1}, x_{2m(k)}) + d(x_{2n(k)}, x_{2n(k)+1}) \\ &\le f^{2m(k)-2}(d(x_2, x_1)) + f^{2n(k)-1}(d(x_2, x_1)). \end{aligned} \tag{4.30}$$

Therefore, we get

$$\begin{aligned} \lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-1}) &= \delta, \\ \lim_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)-1}) &= \delta. \end{aligned} \tag{4.31}$$

Also we have

$$\begin{aligned} \delta &\le d(x_{2n(k)}, x_{2m(k)}) \\ &\le d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2m(k)}) \\ &\le f^{2n(k)-2}(d(x_2, x_1)) + d(Fx_{2n(k)}, Gx_{2m(k)-1}). \end{aligned} \tag{4.32}$$

On the other hand, since $x_{2n(k)}$ and $x_{2m(k)-1}$ are comparable, we can use the condition (4.7) for these points. Therefore, we have

$$\begin{aligned} &T(d(Fx_{2n(k)}, Gx_{2m(k)-1}), d(x_{2n(k)}, x_{2m(k)-1}), d(x_{2n(k)}, Fx_{2n(k)}), \\ &\quad d(x_{2m(k)-1}, Gx_{2m(k)-1}), d(x_{2n(k)}, Gx_{2m(k)-1}), d(x_{2m(k)-1}, Fx_{2n(k)})) \le 0 \end{aligned} \tag{4.33}$$

and so

$$\begin{aligned} &T(d(x_{2n(k)+1}, x_{2m(k)}), d(x_{2n(k)}, x_{2m(k)-1}), d(x_{2n(k)}, x_{2n(k)+1}), \\ &\quad d(x_{2m(k)-1}, x_{2m(k)}), d(x_{2n(k)}, x_{2m(k)}), d(x_{2m(k)-1}, x_{2n(k)+1})) \le 0. \end{aligned} \tag{4.34}$$

Now, considering (4.29) and (4.31) and letting $k \to \infty$ in the last inequality, we have, by continuity of T, that

$$T\left(\lim_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)}), \delta, 0, 0, \delta, \delta\right) \le 0. \tag{4.35}$$

From T_2', we have $\lim_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)}) \leq f(\delta)$. Therefore, letting $k \to \infty$ in (4.32), we have $\delta \leq f(\delta)$. This is a contradiction since $f(t) < t$ for $t > 0$. Thus $\{x_{2n}\}$ is a Cauchy sequence in X, so $\{x_n\}$ is a Cauchy sequence. Therefore, there exists an $x \in X$ with $\lim_{n\to\infty} x_n = x$.

If (4.8) or (4.9) hold then clearly $x = Fx = Gx$. Now suppose (4.10) holds. Suppose $d(x, Fx) > 0$. Now since $\lim_{n\to\infty} x_n = x$, then from (4.10), $x_{2n-1} \preceq x$ for all n. Using the inequality (4.7), we have

$$T(d(Fx, Gx_{2n-1}), d(x, x_{2n-1}), d(x, Fx), d(x_{2n-1}, Gx_{2n-1}), d(x, Gx_{2n-1}), d(x_{2n-1}, Fx)) \leq 0. \tag{4.36}$$

So letting $n \to \infty$ from the last inequality, we have

$$T(d(Fx, x), 0, d(x, Fx), 0, 0, d(x, Fx)) \leq 0 \tag{4.37}$$

which is a contradiction to T_3'. Thus $d(x, Fx) = 0$ and so $x = Fx = Gx$. □

Remark 4.7. We can have some new results from Theorem 4.5 with some examples for T.

For example, we can have the following corollary.

Corollary 4.8. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Suppose $F, G : X \to X$ are two weakly increasing mappings such that for all comparable $x, y \in X$,*

$$d(Fx, Gy) \leq \phi(d(x, y)), \tag{4.38}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a right continuous function such that $\phi(0) = 0$, $\phi(t) < t$ for $t > 0$. Also

$$F \text{ or } G \text{ is continuous} \tag{4.39}$$

or

$$\begin{aligned} &\text{if } \{x_n\} \subset X \text{ is a nondecreasing sequence with } x_n \longrightarrow x \text{ in } X, \\ &\text{then } x_n \preceq x \quad \forall n \end{aligned} \tag{4.40}$$

hold, then F and G have a common fixed point.

Proof. Let $T(t_1, \ldots, t_6) = t_1 - \phi(t_2)$, then it is obvious that $T \in \mathcal{T}'$. Therefore, the proof is complete from Theorem 4.5. □

5. Application

Consider the integral equations

$$x(t) = \int_a^b K_1(t,s,x(s))ds + g(t), \quad t \in [a,b],$$
$$x(t) = \int_a^b K_2(t,s,x(s))ds + g(t), \quad t \in [a,b]. \tag{5.1}$$

The purpose of this section is to give an existence theorem for common solution of (5.1) using Corollary 4.8. This section is related to those [19–22].

Let $\ll$ be a partial order relation on $\mathbb{R}^n$.

Theorem 5.1. *Consider the integral equations* (5.1).

(i) $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : \mathbb{R}^n \to \mathbb{R}^n$ *are continuous;*

(ii) *for each* $t, s \in [a,b]$,

$$K_1(t,s,x(s)) \ll K_2\left(t, s, \int_a^b K_1(s,\tau,x(\tau))d\tau + g(s)\right),$$
$$K_2(t,s,x(s)) \ll K_1\left(t, s, \int_a^b K_2(s,\tau,x(\tau))d\tau + g(s)\right); \tag{5.2}$$

(iii) *there exist a continuous function* $p : [a,b] \times [a,b] \to \mathbb{R}_+$ *and a right continuous and nondecreasing function* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\phi(0) = 0$ *and* $\phi(t) < t$ *for* $t > 0$, *such that*

$$|K_1(t,s,u) - K_2(t,s,v)| \leq p(t,s)\phi(|u-v|) \tag{5.3}$$

for each $t, s \in [a,b]$ *and comparable* $u, v \in \mathbb{R}^n$;

(iv) $\sup_{t\in[a,b]} \int_a^b p(t,s)ds \leq 1$.

Then the integral equations (5.1) *have a unique common solution* x^* *in* $C([a,b],\mathbb{R}^n)$.

Proof. Let $X := C([a,b],\mathbb{R}^n)$ with the usual supremum norm, that is, $\|x\| = \max_{t\in[a,b]}|x(t)|$, for $x \in C([a,b],\mathbb{R}^n)$. Consider on X the partial order defined by

$$x, y \in C([a,b],\mathbb{R}^n), \quad x \preceq y \quad \text{iff } x(t) \ll y(t) \quad \text{for any } t \in [a,b]. \tag{5.4}$$

Then $(X, \preceq)$ is a partially ordered set. Also $(X, \|\cdot\|)$ is a complete metric space. Moreover, for any increasing sequence $\{x_n\}$ in X converging to $x^* \in X$, we have $x_n(t) \ll x^*(t)$ for any $t \in [a,b]$. Also for every $x, y \in X$, there exists $c(x,y) \in X$ which is comparable to x and y [6].

Define $F, G : X \to X$, by

$$\begin{aligned} Fx(t) &= \int_a^b K_1(t,s,x(s))ds + g(t), \quad t \in [a,b], \\ Gx(t) &= \int_a^b K_2(t,s,x(s))ds + g(t), \quad t \in [a,b]. \end{aligned} \tag{5.5}$$

Now from (ii), we have, for all $t \in [a,b]$,

$$\begin{aligned} Fx(t) &= \int_a^b K_1(t,s,x(s))ds + g(t) \\ &\ll \int_a^b K_2\left(t,s,\int_a^b K_1(s,\tau,x(\tau))d\tau + g(s)\right)ds + g(t) \\ &= \int_a^b K_2(t,s,Fx(s))ds + g(t) \\ &= GFx(t), \\ Gx(t) &= \int_a^b K_2(t,s,x(s))ds + g(t) \\ &\ll \int_a^b K_1\left(t,s,\int_a^b K_2(s,\tau,x(\tau))d\tau + g(s)\right)ds + g(t) \\ &= \int_a^b K_1(t,s,Gx(s))ds + g(t) \\ &= FGx(t). \end{aligned} \tag{5.6}$$

Thus, we have $Fx \preceq GFx$ and $Gx \preceq FGx$ for all $x \in X$. This shows that F and G are weakly increasing. Also for each comparable $x, y \in X$, we have

$$\begin{aligned} |Fx(t) - Gy(t)| &= \left|\int_a^b K_1(t,s,x(s))ds - \int_a^b K_2(t,s,y(s))ds\right| \\ &\leq \int_a^b |K_1(t,s,x(s)) - K_2(t,s,y(s))|ds \\ &\leq \int_a^b p(t,s)\phi(|x(s) - y(s)|)ds \\ &\leq \phi(\|x-y\|)\int_a^b p(t,s)ds \\ &\leq \phi(\|x-y\|), \quad \text{for any } t \in [a,b]. \end{aligned} \tag{5.7}$$

Hence $\|Fx - Gy\| \leq \phi(\|x - y\|)$ for each comparable $x, y \in X$. Therefore, all conditions of Corollary 4.8 are satisfied. Thus the conclusion follows. □

Acknowledgment

The authors thank the referees for their appreciation, valuable comments, and suggestions.

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