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Some generalizations of Caristi type fixed point theorem on partial metric spaces

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Abstract. In the persent paper, we give Bae and Suzuki type generalizations of Caristi's fixed point theorem on partial metric space.

1. Introduction

The concept of partial metric p on a nonempty set X was introduced by Matthews [10]. One of the most interesting properties of a partial metric is that $p(x,x)$ may not be zero for $x \in X$. Also, each partial metric p on a nonempty set X generates a T_0 topology on X. After the definition of partial metric space, Matthews proved the partial metric version of Banach fixed point theorem. Then, Valero [14], Oltra and Valero [11] and Altun et al [2], [3] gave some generalizations of the result of Matthews. Again, Romaguera [12] and Acar, Altun and Romaguera [1] proved the Caristi type fixed point theorems on this space. In this paper, we give partial metric version of some generalizations of Caristi's fixed point theorems given by Bae [4] and Suzuki [13].

First, we recall some definitions of partial metric space and some properties of theirs. See [7, 8, 10–12, 14] for details.

A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ (nonnegative real numbers) such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$ (T_0-separation axiom),

(p_2) $p(x,x) \leq p(x,y)$ (small self-distance axiom),

(p_3) $p(x,y) = p(y,x)$ (symmetry),

(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$ (modified triangular inequality).

A partial metric space (for short PMS) is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then, from p_1 and p_2, $x = y$. But if $x = y$, $p(x,y)$ may not be 0. A basic example of a PMS is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$. For another example, let I

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denote the set of all intervals $[a, b]$ for any real numbers $a \le b$. Let $p : I \times I \to \mathbb{R}^+$ be the function such that $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then (I, p) is a PMS. Other examples of PMS which are interesting from a computational point of view may be found in [6, 10].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family open p-balls

$$\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\},$$

where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on X.

Definition 1.1. (i) A sequence $\{x_n\}$ in a PMS (X, p) converges to a point $x \in X$ if and only if $p(x, x) = lim_{n\to\infty} p(x, x_n)$.

(ii) A sequence $\{x_n\}$ in a PMS (X, p) is called a *Cauchy sequence* if there exists (and is finite) $lim_{n,m\to\infty} p(x_n, x_m)$.

(iii) A PMS (X, p) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = lim_{n,m\to\infty} p(x_n, x_m)$.

The following lemma plays an important role to give fixed point results on a PMS.

Lemma 1.2. ([10, 11]) *(X, p) be a PMS.*

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).

(b) (X, p) is complete if and only if (X, p^s) is complete. Furthermore,

$lim_{n\to\infty} p^s(x_n, x) = 0$ *if and only if* $p(x, x) = lim_{n\to\infty} p(x_n, x) = lim_{n,m\to\infty} p(x_n, x_m)$.

Let (X, d) be a metric space and $T : X \to X$ be a mapping. If there exists a lower semicontinuous function $\phi : X \to [0, \infty)$ satisfying $d(x, Tx) \le \phi(x) - \phi(Tx)$, then T is called Caristi mapping for (X, d). Caristi proved in [5] that every Caristi mapping on a complete metric space has a fixed point. Then, Kirk [9] proved that the metric space (X, d) is complete if and only if every Caristi mapping for (X, d) has a fixed point. Also, as in Bae [4] and Suzuki [13], there are a lot of generalizations of Caristi's fixed point theorem in the literature.

After the definition of partial metric space, Kirk type characterization of this interesting space want to be given. For this, in [12], Romaguera proposed the following two alternatives to give an appropriate notion of a Caristi mapping in partial metric spaces.

(i) A self mapping T of a partial metric space (X, p) is called a p-Caristi mapping on X if there is a function $\phi : X \to [0, \infty)$ which is lower semicontinuous for (X, p) and satisfies

$$p(x, Tx) \le \phi(x) - \phi(Tx) \tag{1}$$

for all $x \in X$.

(ii) A self mapping T of a partial metric space (X, p) is called a p^s-Caristi mapping on X if there is a function $\phi : X \to [0, \infty)$ which is lower semicontinuous for (X, p^s) and satisfies (1).

Also in the same paper, Romaguera defined the 0-complete PMS as follows:

A sequence x_n in a PMS (X,p) is called 0-Cauchy if $lim_{m,n\to\infty} p(x_n, x_m) = 0$ and (X, p) is called 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $z \in X$ such that $p(z, z) = 0$. It is clear that every complete PMS is 0-complete. After then, Romaguera proved that, a partial metric space (X, p) is 0-complete if and only if every p^s-Caristi mapping T on X has a fixed point.

Note that in the above proposed two alternatives, the identity mapping on X is neither p-Caristi nor p^s-Caristi mapping, although it is a Caristi mapping for metric space. For this reason, in [1], a new notation of Caristi mapping, which avoids this disadvantage, has been introduced as follows:

A self mapping T of a partial metric space (X, p) is called a Caristi mapping on X if there is a function $\phi : X \to [0,\infty)$ which is a lower semicontinuous function for (X, p^s) and satisfies $p(x, Tx) \le p(x, x) + \phi(x) - \phi(Tx)$ for all $x \in X$. Then the following theorem, which characterize of completeness of partial metric space, has been given.

Theorem 1.3. *A partial metric space (X, p) is complete if and only if every Caristi mapping on X has a fixed point.*

2. The main result

In this section, we give Bae [4] and Suzuki [13] type generalized versions of Theorem 1.3 in partial metric spaces.

Theorem 2.1. *Let (X, p) be a complete partial metric space, $\phi : X \to [0,\infty)$ be a lower semicontinuous for (X, p^s) satisfying*

$$p(x, x) = p(x, y) \quad \textit{implies} \quad \phi(y) \le \phi(x), \tag{2}$$

and $\psi : X \to [0,\infty)$ be a function satisfying

$$sup\{\psi(x) : x \in X, \phi(x) \le inf_{w\in X}\phi(w) + \mu\} < \infty$$

for some $\mu > 0$. If $T : X \to X$ be a mapping satisfying

$$p(x, Tx) \le p(x, x) + \psi(x)\{\phi(x) - \phi(Tx)\} \tag{3}$$

for all $x \in X$, then T has a fixed point in X.

Proof. In the case of $\psi(x) > 0$, from (3) we have $\phi(Tx) \le \phi(x)$. In the case of $\psi(x) = 0$, we have $p(x, x) = p(x, Tx)$ and hence from (2) $\phi(Tx) \le \phi(x)$. Therefore $\phi(Tx) \le \phi(x)$ for all $x \in X$. Put

$$Y = \{x \in X : \phi(x) \le inf_{w\in X}\phi(w) + \mu\}$$

and

$$\gamma = sup_{w\in Y}\psi(w) < \infty.$$

Since (X, p) is complete, then from Lemma 1.2, (X, p^s) is complete and also ϕ is a lower semicontinuous function for (X, p^s), then Y is closed in (X, p^s). Hence (Y, p^s) is complete and then from Lemma 1.2, (Y, p) is complete. It is clear that Y is nonempty and $TY \subseteq Y$ because of $\phi(Tx) \le \phi(x)$ for $x \in X$. We also have

$$p(x, Tx) \le p(x, x) + \gamma\{\phi(x) - \phi(Tx)\}$$

for all $x \in Y$ and the function $\varphi : Y \to [0,\infty)$ defined by $\varphi(x) = \gamma\phi(x)$ is a lower semicontinuous for (Y, p^s). Therefore, from Theorem 1.3, T has a fixed point $z \in Y \subseteq X$. □

Remark 2.2. If p is an ordinary metric in Theorem 2.1, then the condition (2) is clearly satisfied. Therefore we obtain Theorem 2 of [13].

Theorem 2.3. *Let (X, p) be a complete partial metric space, $\phi : X \to [0, \infty)$ be a lower semicontinuous for (X, p^s) satisfying (2) and $c : [0, \infty) \to [0, \infty)$ be an upper semicontinuous function. If $T : X \to X$ be a mapping satisfying*

$$p(x, Tx) \leq p(x, x) + \max\{c(\phi(x)), c(\phi(Tx))\}\{\phi(x) - \phi(Tx)\} \tag{4}$$

for all $x \in X$. Then T has a fixed point in X.

Proof. Fix $\gamma > c(inf_{w\in X}\phi(w))$. Then since c is upper semicontinuous, there exist $\mu > 0$ such that $c(t) \leq inf_{w\in X}\phi(w) + \mu$ for $t \in [inf_{w\in X}\phi(w), inf_{w\in X}\phi(w) + \mu]$. Define a function ψ from X into $[0, \infty)$ by

$$\psi(x) = \max\{c(\phi(x)), c(\phi(Tx))\}$$

for all $x \in X$. As in the proof of Theorem 2.1, we can prove that $\phi(Tx) \leq \phi(x)$ for all $x \in X$. Then for $x \in X$ with $\phi(x) \leq inf_{w\in X}\phi(w) + \mu$, we have $\phi(Tx) \leq inf_{w\in X}\phi(w) + \mu$ and hence $\psi(x) \leq \gamma$. Therefore, we obtain

$$\sup\{\psi(x) : x \in X, \phi(x) \leq inf_{w\in X}\phi(w) + \mu\} \leq \gamma < \infty$$

So by Theorem 2.1 we obtain the desired result. □

Theorem 2.4. *Let (X, p) be a complete partial metric space, $\phi : X \to [0, \infty)$ be a lower semicontinuous for (X, p^s) satisfying (2) and $c : [0, \infty) \to [0, \infty)$ be a nondecreasing function. If $T : X \to X$ be a mapping satisfying either*

$$p(x, Tx) \leq p(x, x) + c(\phi(x))\{\phi(x) - \phi(Tx)\}, \tag{5}$$

or

$$p(x, Tx) \leq p(x, x) + c(\phi(Tx))\{\phi(x) - \phi(Tx)\}. \tag{6}$$

Then T has a fixed point in X.

Proof. As in the proof of Theorem 2.1, we can prove that $\phi(Tx) \leq \phi(x)$ for all $x \in X$. Hence, we have $c(\phi(Tx)) \leq c(\phi(x))$. So (6) implies (5). Therefore we only prove this theorem in the case of (5). Define a function ψ from X into $[0, \infty)$ by

$$\psi(x) = c(\phi(x))$$

for $x \in X$. Then we have

$$\sup\{\psi(x) : x \in X, \phi(x) \leq inf_{w\in X}\phi(w) + 1\} \leq c(inf_{w\in X}\phi(w) + 1) < \infty.$$

By Theorem 2.1, T has a fixed point in X. □

Theorem 2.5. *Let (X, p) be a complete partial metric space, $\phi : X \to [0, \infty)$ be a lower semicontinuous for (X, p^s) satisfying (2) and $c : [0, \infty) \to [0, \infty)$ be an upper semicontinuous function. If $T : X \to X$ be a mapping satisfying $p(x, Tx) \leq \phi(x)$ for all $x \in X$ and*

$$p(x, Tx) \leq p(x, x) + c(p(x, Tx))\{\phi(x) - \phi(Tx)\}$$

for all $x \in X$. Then T has a fixed point in X.

Proof. Define a function $\psi : X \to [0, \infty)$ by

$$\psi(x) = c(p(x, Tx)).$$

For $x \in X$ with $\phi(x) \leq inf_{w\in X}\phi(w) + 1$, we have

$$\begin{aligned} \psi(x) &\leq \sup\{c(t) : 0 \leq t \leq p(x, Tx)\} \\ &\leq \sup\{c(t) : 0 \leq t \leq \phi(x)\} \\ &\leq \sup\{c(t) : 0 \leq t \leq inf_{w\in X}\phi(w) + 1\}. \end{aligned}$$

Hence

$$\sup\{\psi(x) : x \in X, \phi(x) \leq inf_{w\in X}\phi(w) + 1\} \leq \max\{c(t) : 0 \leq t \leq inf_{w\in X}\phi(w) + 1\} < \infty$$

because c is upper semicontinuous. So, by Theorem 2.1, we obtain the desired result. □

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