Research Article Open Access

Emrah Kılıç, Ilker Akkus*, and Gonca Kızılaslan

A variant of the reciprocal super Catalan matrix

DOI 10.1515/spma-2015-0014
Received June 23, 2015; accepted July 12, 2015

Abstract: Recently Prodinger [8] considered the reciprocal super Catalan matrix and gave explicit formulæ for its LU-decomposition, the LU-decomposition of its inverse, and obtained some related matrices. For all results, q-analogues were also presented. In this paper, we define and study a variant of the reciprocal super Catalan matrix with two additional parameters. Explicit formulæ for its LU-decomposition, LU-decomposition of its inverse and the Cholesky decomposition are obtained. For all results, q-analogues are also presented.

Keywords: Super Catalan numbers; LU-decomposition; Cholesky decomposition, q-analogues

MSC: 15B05, 05A05.

1 Introduction

For a given sequence $a_0,\ a_1,\ldots$, the Hankel matrix is defined as

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

see [6].

Taking certain special number sequences instead of $\{a_n\}$, some authors have defined and studied various combinatorial matrices, see [2, 4, 5, 9]. There are also papers that concentrated on the reciprocals of a given sequence: The Hilbert matrix is defined by

$$\mathcal{H}_n = \left[\frac{1}{i+j+1}\right],\ \ 0 \le i,j \le n,\ \ n = 0,1,\ldots,$$

see [1–3]. As a second example, the Filbert matrix is given by

$$\mathcal{F}_n = \left[\frac{1}{F_{i+j+1}}\right],\ \ 0 \le i,j \le n,\ \ n = 0,1,\ldots,$$

where F_n is the nth Fibonacci number, see [5, 9]. The last one is the reciprocal Pascal matrix with entries

$$\mathcal{R}_n = \left[\binom{i+j}{i}^{-1}\right],\ \ 0 \le i,j,$$

Emrah Kılıç: TOBB University of Economics and Technology, Mathematics Department, 06560 Ankara Turkey,
E-mail: ekilic@etu.edu.tr
***Corresponding Author: Ilker Akkus:** Kırıkkale University, Faculty of Science and Arts, Department of Mathematics, 71450 Kırıkkale Turkey, E-mail: iakkus.tr@gmail.com
Gonca Kızılaslan: Kırıkkale University, Faculty of Science and Arts, Department of Mathematics, 71450 Kırıkkale Turkey,
E-mail: goncakizilaslan@gmail.com

© 2015 Emrah Kılıç et al., licensee De Gruyter Open.
This work is licensed under the Creative Commons Attribution-NonCommercial-NoDerivs 3.0 License.

see [10].

Recently Prodinger [8] studied the matrix $\mathcal{A}$ whose entries are the super Catalan numbers, $\frac{(2i)!(2j)!}{i!j!(i+j)!}$. He derived analogues results for the matrix A with entries $\frac{i!j!(i+j)!}{(2i)!(2j)!}$. He also studied q-analogues of these matrices.

The Gaussian q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}},$$

where $(x;q)_n$ is the q-Pochhammer symbol defined by

$$(x;q)_n = (1-x)(1-xq)\ldots\left(1-xq^{n-1}\right).$$

Note that

$$\lim_{q\to 1}\begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k},$$

where $\binom{n}{k}$ is the usual binomial coefficient.

We can rewrite the entries of A via the usual binomial coefficients as follows

$$A_{i,j} = \binom{2i}{i}^{-1}\binom{2j}{j}^{-1}\binom{i+j}{i}.$$

In the present paper, we define the matrix $M = [m_{i,j}]$ with

$$m_{i,j} = \binom{2i+r}{i}^{-1}\binom{2j+s}{j}^{-1}\binom{i+j}{i}^{-1}$$

for $0 \le i, j < n$ and nonnegative integers r and s. By the definitions of A and M, note that the matrix M is a variant of the matrix A with two additional parameters. First we derive explicit expressions for the LU-decomposition of M which leads to a formula for the determinant via $\prod_{0\le i<n} U_{i,i}$. Further, we have expressions for the matrices $L^{-1}$and U^{-1}. Similarly we have explicit expressions for the LU-decomposition of $M^{-1} = AB$, and, for A^{-1}, B^{-1}. The latter expressions depend on the size N of the matrix M^{-1}. Via this decomposition, we find that the entries of M^{-1} are integers. We denote the q-analogue of the matrix M by $\mathcal{M}$. We also give formulæ for the q-analogues of all these results. Finally we give expressions for the Cholesky decomposition of M when the matrix is symmetric.

To prove the claimed result, we firstly guess relevant quantities and then use the q-Zeilberger algorithm (for more details, see [7, 11, 12]) to justify relevant equalities.

2 Decomposition of *M*

For the matrix M, we list here the formulæ that were found; all indices start at (0, 0).

$$L_{i,j} = \binom{i}{j}\binom{2j}{j}\binom{2j+r}{j}\binom{i+j}{j}^{-1}\binom{2i+r}{i}^{-1},$$

$$L^{-1}_{i,j} = (-1)^{i+j}\binom{i}{j}\binom{i+j-1}{j}\binom{2j+r}{j}\binom{2i-1}{i}^{-1}\binom{2i+r}{i}^{-1},$$

$$U_{i,j} = (-1)^{i}\binom{j}{i}\binom{2i-1}{i}^{-1}\binom{i+j}{i}^{-1}\binom{2i+r}{i}^{-1}\binom{2j+s}{j}^{-1},$$

$$U^{-1}_{i,j} = (-1)^i \binom{j}{i}\binom{i+j-1}{i}\binom{2j}{j}\binom{2i+s}{i}\binom{2j+r}{j},$$

$$A_{i,j} = (-1)^{i+j}\binom{i}{j}\binom{N}{i}\binom{N+i}{i}\binom{2i+s}{i}\binom{N}{j}^{-1}\binom{N+j}{j}^{-1}\binom{2j+s}{j}^{-1},$$

$$A^{-1}_{i,j} = \binom{i}{j}\binom{N}{i}\binom{N+i}{i}\binom{2i+s}{i}\binom{N}{j}^{-1}\binom{N+j}{j}^{-1}\binom{2j+s}{j}^{-1},$$

$$B_{i,j} = (-1)^{N+j}\binom{j}{i}\binom{N}{j}\binom{N+j}{j}\binom{2j+r}{j}\binom{2i+s}{i},$$

$$B^{-1}_{i,j} = (-1)^{N+j}\binom{j}{i}\binom{N}{i}^{-1}\binom{N+i}{i}^{-1}\binom{2i+r}{i}^{-1}\binom{2j+s}{j}^{-1}.$$

Since $\det M_N = \prod_{0\le i<N} U_{i,i}$, we have the following formula

$$\det M_N = (-1)^{\binom{N}{2}} \prod_{0\le i<N} \binom{2i-1}{i}^{-1}\binom{2i}{i}^{-1}\binom{2i+r}{i}^{-1}\binom{2i+s}{i}^{-1}.$$

Now we give related proofs in some detail. Note that instead of proving that $AB = M^{-1}$ it is more convenient to prove the equivalent $B^{-1}A^{-1} = M$.

$$\begin{aligned}\sum_k L_{i,k}L^{-1}_{k,j} &= \frac{i!i!i!(i+r)!(2j+r)!}{j!j!j!(j+r)!(2i+r)!}\sum_k (-1)^{k+j} 2k \frac{(k+j-1)!}{(i+k)!(i-k)!(k-j)!}\\ &= \frac{i!i!i!(i+r)!(2j+r)!(2j-1)!}{j!j!j!(j+r)!(2i+r)!(2i)!}\sum_k (-1)^{k+j} 2k\binom{2i}{i+k}\binom{k+j-1}{k-j}\\ &= \frac{i!i!i!(i+r)!(2j+r)!(2j-1)!}{j!j!j!(j+r)!(2i+r)!(2i)!}(2j\,[i=j])\\ &= [i=j]\,.\end{aligned}$$

$$\begin{aligned}\sum_k U_{i,k}U^{-1}_{k,j} &= \frac{i!i!(i-1)!(i+r)!(2j)!(2j+r)!}{j!j!(j-1)!(j+r)!(2i-1)!(2i+r)!}\sum_k (-1)^{i+k}\frac{(k+j-1)!}{(i+k)!(k-i)!(j-k)!}\\ &= \frac{i!i!(i-1)!(i+r)!(2j)!(2j+r)!}{j!j!(j-1)!(j+r)!(2i-1)!(2i+r)!}\sum_k (-1)^{i+k}\frac{1}{2k}\binom{2k}{i+k}\binom{k+j-1}{j-k}\\ &= \frac{i!i!(i-1)!(i+r)!(2j)!(2j+r)!}{j!j!(j-1)!(j+r)!(2i-1)!(2i+r)!}(\frac{1}{2j}\,[i=j])\\ &= [i=j]\,.\end{aligned}$$

$$\sum_k A_{i,k}A^{-1}_{k,j} = \frac{j!j!(N-j)!(j+s)!(N+i)!(2i+s)!}{i!i!(N-i)!(i+s)!(N+j)!(2j+s)!}\sum_k (-1)^{i+k}\frac{1}{(i-k)!(k-j)!}$$

$$
\begin{aligned}
&= \frac{j!j!(N-j)!(j+s)!(N+i)!(2i+s)!}{i!i!(i-j)!(N-i)!(i+s)!(N+j)!(2j+s)!} \sum_k (-1)^{i+k} \binom{i-j}{k-j} \\
&= \frac{j!j!(N-j)!(j+s)!(N+i)!(2i+s)!}{i!i!(i-j)!(N-i)!(i+s)!(N+j)!(2j+s)!} [i=j] \\
&= [i=j] .
\end{aligned}
$$

$$
\begin{aligned}
\sum_k B_{i,k} B^{-1}_{k,j} &= \frac{j!j!(j+s)!(2i+s)!}{i!i!(i+s)!(2j+s)!} \sum_k (-1)^{j+k} \frac{1}{(k-i)!(j-k)!} \\
&= \frac{j!j!(j+s)!(2i+s)!}{i!i!(j-i)!(i+s)!(2j+s)!} \sum_k (-1)^{j+k} \binom{j-i}{k-i} \\
&= \frac{j!j!(j+s)!(2i+s)!}{i!i!(j-i)!(i+s)!(2j+s)!} [i=j] \\
&= [i=j] .
\end{aligned}
$$

$$
\begin{aligned}
\sum_k B^{-1}_{i,k} A^{-1}_{k,j} &= \frac{i!i!j!j!(i+r)!(j+s)!}{(2i+r)!(2j+s)!} \sum_k \frac{(-1)^{N+k}(N-i)!(N+k)!(N-j)!}{(N-k)!(k-i)!(N+j)!(k-j)!(N+i)!} \\
&= \frac{i!i!j!j!(i+r)!(j+s)!}{(2i+r)!(2j+s)!(i+j)!} \sum_k (-1)^{N+k} \binom{N-i}{k-i} \binom{N+k}{k-j} \binom{N+i}{i+j}^{-1} \\
&= \frac{i!i!j!j!(i+r)!(j+s)!}{(2i+r)!(2j+s)!(i+j)!} \\
&= M.
\end{aligned}
$$

Now we compute an arbitrary entry of AB :

$$
\begin{aligned}
\sum_k A_{i,k} B_{k,j} &= \frac{(2i+s)!(2j+r)!}{i!i!j!j!(i+s)!(j+r)!} \sum_k (-1)^{N+i+j+k} \frac{(N+i)!(N-k)!(N+j)!}{(i-k)!(N+k)!(j-k)!(N-j)!(N-i)!} \\
&= \frac{(2i+s)!(2j+r)!(i+j)!}{i!i!j!j!(i+s)!(j+r)!} \sum_k (-1)^{N+i+j+k} \binom{N+i}{i-k} \binom{N-k}{j-k} \binom{N+j}{i+j} \\
&= \binom{2i+s}{i} \binom{2j+r}{j} \binom{i+j}{i} \sum_k (-1)^{N+i+j+k} \binom{N+i}{i-k} \binom{N-k}{j-k} \binom{N+j}{i+j} .
\end{aligned}
$$

3 Decomposition of the matrix $\mathcal{M}$

First we recall that the matrix $\mathcal{M}$ has entries $\begin{bmatrix} 2i+r \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} i+j \\ i \end{bmatrix}_q^{-1}$. Then we have the following formulæ without proofs:

$$
L_{i,j} = \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} 2j \\ j \end{bmatrix}_q \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} i+j \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} 2i+r \\ i \end{bmatrix}_q^{-1},
$$

$$
L^{-1}_{i,j} = (-1)^{i+j} q^{\binom{i-j}{2}} \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} 2i-1 \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} 2i+r \\ i \end{bmatrix}_q^{-1},
$$

$$
U_{i,j} = (-1)^i q^{i(3i-1)/2} \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} 2i-1 \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} i+j \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} 2i+r \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1},
$$

$$U^{-1}_{i,j} = (-1)^i q^{\binom{i+1}{2}-j(j+i)} \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_q \begin{bmatrix} 2j \\ j \end{bmatrix}_q \begin{bmatrix} 2i+s \\ i \end{bmatrix}_q \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q,$$

$$A_{i,j} = (-1)^{i+j} q^{N(j-i)+\binom{i-j+1}{2}} \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} N \\ i \end{bmatrix}_q \begin{bmatrix} N+i \\ i \end{bmatrix}_q \begin{bmatrix} 2i+s \\ i \end{bmatrix}_q \begin{bmatrix} N \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} N+j \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1},$$

$$A^{-1}_{i,j} = q^{(j-i)(N-i)} \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} N \\ i \end{bmatrix}_q \begin{bmatrix} N+i \\ i \end{bmatrix}_q \begin{bmatrix} 2i+s \\ i \end{bmatrix}_q \begin{bmatrix} N \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} N+j \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1},$$

$$B_{i,j} = (-1)^{N+j} q^{\binom{j+1}{2}-\binom{N+1}{2}-Nj+i^2} \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} N \\ j \end{bmatrix}_q \begin{bmatrix} N+j \\ j \end{bmatrix}_q \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} 2i+s \\ i \end{bmatrix}_q,$$

$$B^{-1}_{i,j} = (-1)^{N+j} q^{\binom{N+1}{2}-\binom{j+1}{2}+i(N-j)} \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} N \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} N+i \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} 2i+r \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1}.$$

4 Cholesky Decomposition of M

Related with the Cholesky decomposition of matrix $M = CC^T$, we have that for $r = s$

$$C_{i,j} = \mathbf{i}^j \sqrt{2} \binom{i}{j} \binom{2i+r}{i}^{-1} \binom{i+j}{j}^{-1}$$

and

$$C^{-1}_{i,j} = (-1)^j \mathbf{i}^i \sqrt{2} \binom{i}{j} \binom{i+j-1}{j} \binom{2j+r}{j}.$$

Thus we write

$$\begin{aligned}
\sum_k C_{i,k} C^{-1}_{k,j} &= \frac{i!i!i!(i+r)!(2j+r)!}{j!j!j!(j+r)!(2i+r)!} \sum_k (-1)^{j-k} 2k \frac{(k+j-1)!}{(i-k)!(i+k)!(k-j)!} \\
&= \frac{i!i!i!(i+r)!(2j+r)!(2j-1)!}{j!j!j!(j+r)!(2i+r)!(2i)!} \sum_k (-1)^{j-k} 2k \binom{2i}{i+k} \binom{k+j-1}{k-j} \\
&= \frac{i!i!i!(i+r)!(2j+r)!(2j-1)!}{j!j!j!(j+r)!(2i+r)!(2i)!} (2j[i=j]) \\
&= [i=j].
\end{aligned}$$

The q-analogues of matrix C and its inverse are

$$\mathfrak{C}_{i,j} = \mathbf{i}^j q^{j(3j-1)/4} (1+q^j)^{1/2} \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} 2i+r \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} i+j \\ j \end{bmatrix}_q^{-1}$$

and

$$\mathfrak{C}^{-1}_{i,j} = (-1)^j \mathbf{i}^i q^{\binom{i-j}{2}-i(3i-1)/4} (1+q^i)^{1/2} \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q.$$

References

[1] J. E. Andersen and C. Berg, Quantum Hilbert matrices and orthogonal polynomials, math.CA:arXiv:math/0703546v1.
[2] M. D. Choi, Tricks or treats with the Hilbert matrix, *Amer. Math. Monthly* 90 (5) (1983), 301–312.
[3] D. Hilbert, Ein Beitrag zur Theorie des Legendreschen Polynoms, *Acta Math.* 18 (1894), 155–159. (367–370 in "Gesammelte Abhandlungen II", Berlin 1933.)
[4] E. Kılıç and H. Prodinger, The q-Pilbert matrix, *Inter. J. Computer Math.*, 89 (10) (2012), 1370–1377.
[5] E. Kılıç and H. Prodinger, Variants of the Filbert matrix, *The Fibonacci Quarterly* 51(2) (2013), 153–162.
[6] V. Y. Pan, Structured matrices and polynomials, Birkhauser Boston, Inc., Boston, MA, Springer-Verlag, New York, 2001.
[7] M. Petkovsek, H. Wilf, and D. Zeilberger, $A = B$, A.K. Peters, Ltd., 1996.
[8] H. Prodinger, The reciprocal super Catalan matrix, *Special Matrices* 3 (2015), 111–117
[9] T. M. Richardson, The Filbert matrix, *The Fibonacci Quarterly* 39 (3) (2001), 268–275.
[10] T. M. Richardson. The reciprocal Pascal matrix, ArXiv:1405.6315, 2014.
[11] A. Riese, A Mathematica q-analogue of Zeilberger's algorithm for proving q-hypergeometric identities, Diploma Thesis, RISC, J. Kepler University, Linz, Austria, 1995.
[12] A. Riese, http://www.risc.uni-linz.ac.at/research/combinat/software/qZeil/index.php.