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Research Article

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Performance and stochastic stability of the adaptive fading extended Kalman filter with the matrix forgetting factor

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Abstract: In this paper, the stability of the adaptive fading extended Kalman filter with the matrix forgetting factor when applied to the state estimation problem with noise terms in the non–linear discrete–time stochastic systems has been analysed. The analysis is conducted in a similar manner to the standard extended Kalman filter's stability analysis based on stochastic framework. The theoretical results show that under certain conditions on the initial estimation error and the noise terms, the estimation error remains bounded and the state estimation is stable.
The importance of the theoretical results and the contribution to estimation performance of the adaptation method are demonstrated interactively with the standard extended Kalman filter in the simulation part.

Keywords: Kalman filter, Non-linear systems, Stability, Adaptive filtering

MSC: 15A51, 37Hxx

1 Introduction

The Kalman filter (KF) and the standard extended Kalman filter (EKF) are two most popular methods used for the state estimation in linear and non-linear systems, respectively. They have maintained their popularity from their discovery to present day since they can be easily applied to the estimation problem in many diverse areas including natural and physical sciences, military and economics. The KF yields the optimum state estimation when the system dynamics is fully known and the system noise processes is Gaussian white noise [1–5]. On the other hand, both the KF and the EKF might give biased estimates and diverge when the initial estimates are not sufficiently good or the arbitrary noise matrices have not been chosen appropriately or any changes occur in the system dynamics [6, 7]. To overcome these problems, sevaral adaptive filtering techniques [8–18] are proposed. Among them is the adaptive fading extended Kalman filter with the matrix forgetting factor (AFEKF) [8]. The AFEKF is based on scalling the error covariance of the prediction with the diagonal matrix forgetting factor. The calculation of the diagonal entries are described in [8, 19]. The AFEKF compensates the effects of poor initial information or any changes in system parameters.

As the EKF, any adaptive EKF can be used for state estimation in non–linear systems. However, it is crutial to decide which filter to use because the filter estimates are desired to be close to the true values during the filtering processes, in other words, the estimation error should be the smallest and the estimates should be stable. To address the importance of this issue the stability and convergence analysis of the discrete–time EKF are studied [7, 8, 20–24].

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Hence, the stability analysis as well as the determination of the stability conditions of the AFEKF are very important. The convergence and stability properties of the AFEKF, without noise terms, can be found in [8] where it is shown that the AFEKF is exponentially stable for deterministic non–linear systems, namely, the estimation error is bounded.

With this study, we extend the results of the article [8] by eliminating the restriction on the noise terms. Then, using the direct method of Lyapunov, it has been proved that under certain conditions the AFEKF is still an exponential observer i.e., the dynamics of the estimation error is exponentially stable. It is an important result as the real–life systems are usually not noise free.

Troughout the manuscript, $\|\cdots\|$ denotes the Euclidean norm of a real vector or the spectral norm of a real matrix.

The rest of the manuscript is structured as follows. We review the state estimation problem for non–linear stochastic discrete–time systems and present some auxiliary results from the stochastic stability theory in Section 2. In Section 3, the AFEKF is introduced and its boundedness of the error is proved. The numerical simulation is given in Section 4. The conclusions are discussed in Section 5.

2 Review: state estimation and stochastic boundedness

This section overviews some definitions and fundamental results on the stochastic theory. Recall that a non–linear discrete time stochastic system is given by the equations:

$$x_{n+1} = f(x_n, u_n) + G_n w_n, \tag{1}$$

$$y_n = h(x_n) + D_n v_n, \tag{2}$$

where $n \in \mathbb{N}_0$ is the discrete time point, $x_n \in \mathbb{R}^q$ is the state vector, $u_n \in \mathbb{R}^q$ is the input vector and $y_n \in \mathbb{R}^m$ is the output vector. Moreover, $v_n \in \mathbb{R}^k, w_n \in \mathbb{R}^l$ are uncorrelated zero-mean white noise process with identity covarience and $D_n \in \mathbb{R}^{m\times k}, G_n \in \mathbb{R}^{q\times l}$ are time varying matrices. The functions f and h are assumed to be of class C^1 i.e. continuously differentiable functions.

The state estimator for the system is

$$\hat{x}_{n+1} = f(\hat{x}_n, u_n) + K_n (y_n - h(\hat{x}_n)) \tag{3}$$

where $K_n \in \mathbb{R}^{q\times m}$ changes in time, is called the observer gain. $\hat{x}_n$ represents the estimated states.

We define

$$A_n = \frac{\partial f}{\partial x}(\hat{x}_n, u_n), \tag{4}$$

$$C_n = \frac{\partial h}{\partial x}(\hat{x}_n). \tag{5}$$

We also define the estimate error vector as

$$\zeta_n = x_n - \hat{x}_n. \tag{6}$$

By subtracting (3) from (1) and taking equations (2), (4)-(5) into account we get

$$\zeta_{n+1} = (A_n - K_n C_n)\zeta_n + r_n + s_n, \tag{7}$$

where

$$r_n = \varphi_n(x_n, \hat{x}_n, u_n) - K_n \chi_n(x_n, \hat{x}_n), \tag{8}$$

$$s_n = G_n w_n - K_n D_n v_n. \tag{9}$$

To analyze the error dynamics given in the equation (7) we recall the following lemma on the boundedness of stochastic processes.

Lemma 2.1. *Let $V_n(\zeta_n)$ be a stochastic process and $\underline{\upsilon}, \bar{\upsilon}, \mu > 0$ and $0 < \alpha < 1$ be real numbers such that the inequalities*

$$\underline{\upsilon}\,\|\zeta_n\|^2 \leq V_n(\zeta_n) \leq \bar{\upsilon}\,\|\zeta_n\|^2 \tag{10}$$

and

$$E\{V_{n+1}(\zeta_{n+1})|\,\zeta_n\} - V_n(\zeta_n) \leq \mu - \alpha V_n(\zeta_n) \tag{11}$$

are carried out by every solutions of the equation (7). Then the stochastic process is exponentially bounded in mean square, that is,

$$E\left\{\|\zeta_n\|^2\right\} \leq \frac{\bar{\upsilon}}{\underline{\upsilon}} E\left\{\|\zeta_0\|^2\right\}(1-\alpha)^n + \frac{\mu}{\underline{\upsilon}} \sum_{i=1}^{n-1} (1-\alpha)^i \tag{12}$$

for every $n \in \mathbb{N}_0$. Moreover, the stochastic process is bounded with probability one.

Proof. See [25]. □

3 Error bounds for the AFEKF

Definition 3.1. *A discrete- time adaptive fading extended Kalman filter with the matrix forgetting factor is given by the following coupled difference equations*

$$\hat{x}_{n+1} = f(\hat{x}_n, u_n) + K_n(y_n - h(\hat{x}_n)) \tag{13}$$

and Riccati difference equation:

$$P_{n+1} = A_n \Lambda_n P_n \Lambda_n^T A_n^T + \Lambda_n Q_n \Lambda_n^T - K_n\big(C_n \Lambda_n P_n \Lambda_n^T C_n^T + R_n\big) K_n^T, \tag{14}$$

where K_n is the Kalman gain given by

$$K_n = A_n \Lambda_n P_n \Lambda_n^T C_n^T \left(C_n \Lambda_n P_n \Lambda_n^T C_n^T + R_n\right)^{-1}. \tag{15}$$

Moreover, $\Lambda_n = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_q)$ is a time varying $q \times q$ dimensional diagonal matrix forgetting factor with $\lambda_i \geq 1$ $i = 1, 2, \cdots, q$; (see [8, 19] for the computation of Λ_n). Furthermore, Q_n and R_n are positive define, symmetric matrices with dimensions $q \times q$ and $m \times m$, respectively, and the covariances matrices for the currupting noise terms in (1)-(2).

Theorem 3.2. *Consider a nonlinear stochastic system given by (1)-(2) and an extended Kalman filter as stated in Definition 3.1. Let the following assumptions hold.*

1. *There are real numbers $\bar{a}, \bar{c}, \underline{p}, \bar{p} > 0$ and $\underline{\lambda}, \bar{\lambda} \geq 1$ such that the following bounds hold for every $n \in \mathbb{N}_0$*

$$\|A_n\| \leq \bar{a}, \tag{16a}$$

$$\|C_n\| \leq \bar{c}, \tag{16b}$$

$$\underline{p} I \leq P_n \leq \bar{p} I, \tag{16c}$$

$$\underline{q} I \leq Q_n, \tag{16d}$$

$$\underline{r} I \leq R_n, \tag{16e}$$

$$\underline{\lambda} I \leq \Lambda_n \leq \bar{\lambda} I, \tag{16f}$$

where $\underline{q}$ and $\underline{r}$ are the smallest eigenvalues of the matrices Q_n and R_n, respectively. Moreover, $\underline{\lambda}$ and $\bar{\lambda}$ are the smallest and the largest diagonal entries of Λ_n, respectively.

2. *A_n is nonsingular matrix for every $n \in \mathbb{N}_0$*
3. *There are positive real numbers $\varepsilon_\varphi, \varepsilon_\chi, \kappa_\varphi, \kappa_\chi > 0$ such that the nonlinear functions φ, χ in (8) are bounded via*

$$\|\varphi(x, \hat{x}, u)\| \leq \kappa_\varphi \|x - \hat{x}\|^2, \tag{17}$$

$$\|\chi(x, \hat{x})\| \leq \kappa_\chi \|x - \hat{x}\|^2. \tag{18}$$

Then the estimation error ζ_n given by (6) is exponentially bounded in mean square and bounded with probability one, provided that the initial estimation error satisfies

$$\|\zeta_0\| \leq \epsilon \tag{19}$$

and the covariance matrices of the noise terms are bounded via

$$G_n \Lambda_n \Lambda_n^T G_n^T \leq \delta I, \tag{20}$$

$$D_n D_n^T \leq \delta I \tag{21}$$

for some $\delta, \epsilon > 0$.

To prove Theorem 3.2 we need the following auxiliary results .

Lemma 3.3. *Under the conditions of Theorem 3.2 there is a real number $0 < \alpha < 1$ such that*

$$1 - \alpha = \frac{1}{\underline{\lambda}^2 \left(1 + \dfrac{\underline{\lambda}^2 \underline{q}}{\bar{\lambda}^2 \bar{p} \left(\bar{a} + \bar{a} \bar{\lambda}^2 \bar{p} \bar{c} \dfrac{1}{\underline{r}} \right)^2} \right)}$$

and

$$\Pi_n = \left(\Lambda_n P_n \Lambda_n^T \right)^{-1}$$

satisfies the inequality

$$(A_n - K_n C_n)^T \, \Pi_{n+1} \, (A_n - K_n C_n) \leq (1 - \alpha) \, \Pi_n \tag{22}$$

for $n \geq 0$ with the Kalman gain K_n given in (15).

Proof. The proof mimics Lemma 3.1 in [25]. Substituting (15) in (14) and rearranging the resulting equation yields

$$P_{n+1} = (A_n - K_n C_n) \Lambda_n P_n \Lambda_n^T (A_n - K_n C_n)^T + \Lambda_n Q_n \Lambda_n^T + K_n C_n \Lambda_n P_n \Lambda_n^T (A_n - K_n C_n)^T. \tag{23}$$

Multiplying the factor $(A_n - K_n C_n) \Lambda_n P_n \Lambda_n^T$ by A_n^{-1} from left and using Equation (15) yields

$$A_n^{-1} (A_n - K_n C_n) \Lambda_n P_n \Lambda_n^T = \Lambda_n P_n \Lambda_n^T - \Lambda_n P_n \Lambda_n^T C_n^T \left(C_n \Lambda_n P_n \Lambda_n^T C^T + R_n \right)^{-1} C_n \Lambda_n P_n \Lambda_n^T. \tag{24}$$

Note that the right side of the equation (24) is a symmetric matrix. Thus, applying matrix inversion lemma in [26] we obtain

$$A_n^{-1} (A_n - K_n C_n) \Lambda_n P_n \Lambda_n^T = \left(\Lambda_n P_n \Lambda_n^T + C_n R_n^{-1} C_n^T \right)^{-1} \geq 0. \tag{25}$$

Furthermore,

$$A_n^{-1} K_n C_n = \Lambda_n P_n \Lambda_n^T C_n^T \left(C_n \Lambda_n P_n \Lambda_n^T C_n^T + R_n \right)^{-1} C_n \geq 0. \tag{26}$$

Due to above equations (25) and (26) along with $\Lambda_n P_n \Lambda_n^T = \left(\Lambda_n P_n \Lambda_n^T\right)^T$ we obtain

$$K_n C_n \Lambda_n P_n \Lambda_n^T (A_n - K_n C_n)^T = A_n \left[A_n^{-1} K_n C_n\right]\left[A_n^{-1}(A_n - K_n C_n)\Lambda_n P_n \Lambda_n^T\right]^T A_n^T \geq 0. \quad (27)$$

From the equations (23) and (26) we have

$$P_{n+1} \geq (A_n - K_n C_n)\Lambda_n P_n \Lambda_n^T (A_n - K_n C_n)^T + \Lambda_n Q_n \Lambda_n^T. \quad (28)$$

The inequality (25) implies that $(A_n - K_n C_n)^{-1}$ exists, so we obtain

$$P_{n+1} \geq (A_n - K_n C_n)\left[\Lambda_n P_n \Lambda_n^T + \left(A_n K_n C_n\right)^{-1} + \Lambda_n Q_n \Lambda_n^T \left((A_n - K_n C_n)^T\right)^{-1}\right](A_n - K_n C_n)^T. \quad (29)$$

From (15) and (16a)-(16f) we have

$$\|K_n\| \leq \|A_n\| \|\Lambda_n\| \|P_n\| \left\|\Lambda_n^T\right\| \left\|C_n^T\right\| \left\|\left(C_n \Lambda_n P_n \Lambda_n^T C^T + R_n\right)^{-1}\right\| \leq \bar{a}\bar{\lambda}^2 \bar{p}\bar{c}\frac{1}{\underline{r}}. \quad (30)$$

Substituting the inequalities (16a)-(16f) into (29) we obtain

$$P_{n+1} \geq (A_n - K_n C_n)\left[\Lambda_n P_n \Lambda_n^T + \frac{\underline{\lambda}^2 \underline{q}}{\left(\bar{a} + \bar{a}\bar{\lambda}^2 \bar{p}\bar{c}\frac{1}{\underline{r}}\right)^2} I\right](A_n - K_n C_n)^T. \quad (31)$$

Multiplying both sides of (31) from left and right with Λ_{n+1} and Λ_{n+1}^T, respectively, and using the inequalty (16f) gives

$$\Lambda_{n+1} P_{n+1} \Lambda_{n+1}^T \geq \underline{\lambda}^2 (A_n - K_n C_n)\left[\Lambda_n P_n \Lambda_n^T + \frac{\underline{\lambda}^2 \underline{q}}{\left(\bar{a} + \bar{a}\bar{\lambda}^2 \bar{p}\bar{c}\frac{1}{\underline{r}}\right)^2} I\right]\left(A_n - K_n C_n\right)^T. \quad (32)$$

Taking the inverse of both sides of (32) and multiplying from left and right with $(A_n - K_n C_n)^T$ and $(A_n - K_n C_n)$ we have,

$$\left(A_n - K_n C_n\right)^T \Pi_{n+1} (A_n - K_n C_n) \geq \frac{1}{\underline{\lambda}^2}\left[1 + \frac{\underline{\lambda}^2 \underline{q}}{\bar{\lambda}^2 \bar{p}\left(\bar{a} + \bar{a}\bar{\lambda}^2 \bar{p}\bar{c}\frac{1}{\underline{r}}\right)^2}\right]^{-1} \Pi_n. \quad (33)$$

Then the result follows.

$$(1-\alpha) = \frac{1}{\underline{\lambda}^2} \frac{1}{\left(1 + \frac{\underline{\lambda}^2 \underline{q}}{\bar{\lambda}^2 \bar{p}\left(\bar{a} + \bar{a}\bar{\lambda}^2 \bar{p}\bar{c}\frac{1}{\underline{r}}\right)^2}\right)}. \quad (34)$$

□

Lemma 3.4. *Let the conditions of Theorem 3.2 be fulfilled, let $\Pi_n = \left(\Lambda_n P_n \Lambda_n^T\right)^{-1}$ and K_n, r_n given in (15),(8). Then there are positive real numbers $\varepsilon', \kappa_{nonl}$ such that*

$$r_n^T \Pi_n \left[2(A_n - K_n C_n)(x_n - \hat{x}_n) + r_n\right] \leq \kappa_{nonl} \|x_n - \hat{x}_n\|^3 \quad (35)$$

holds for $\|x_n - \hat{x}_n\| \leq \varepsilon'$.

Proof. From (15), (16a)-(16f) and $C_n \Lambda_n P_n \Lambda_n' C_n' > 0$ we have

$$\|K_n\| \leq \bar{a}\bar{\lambda}^2 \bar{p}\bar{c}\frac{1}{\underline{r}} \quad (36)$$

and using in (8) gives

$$\|r_n\| \leq \|\varphi(x_n, \hat{x}_n, u_n)\| + \bar{a}\bar{\lambda}^2 \bar{p}\bar{c}\frac{1}{\underline{r}} \|\chi(x_n, \hat{x}_n)\|. \quad (37)$$

By choosing $\varepsilon' = \min(\varepsilon_\varphi, \varepsilon_\chi)$ and using (17), (18) we obtain

$$\|r_n\| \le \kappa_\varphi \|x_n - \hat{x}_n\|^2 + \bar{a}\bar{\lambda}^2\bar{p}\bar{c}\frac{1}{\underline{r}}\kappa_\chi \|(x_n, \hat{x}_n)\|^2 . \tag{38}$$

Since $\|x_n - \hat{x}_n\|^2 \le \varepsilon_n'$, we have

$$\|r_n\| \le \kappa_\varphi \|x_n - \hat{x}_n\|^2 . \tag{39}$$

Define

$$\kappa' = \kappa_\varphi + \left(\bar{a}\bar{\lambda}^2\bar{p}\bar{c}\frac{1}{\underline{r}}\right)\kappa_\chi . \tag{40}$$

Then, for $\|x_n - \hat{x}_n\|^2 \le \varepsilon'$, from (38) by taking $\Pi_n = \left(\Lambda_n P_n \Lambda_n^T\right)^{-1}$ and using (16a)-(16f) we obtain

$$r_n^T \Pi_n\Big[2\big(A_n - K_n C_n\big)\big(x_n - \hat{x}_n\big) + r_n\Big] \le \kappa' \|x_n - \hat{x}_n\|^2 \frac{1}{\underline{\lambda}^2\underline{p}}\left[2\left(\bar{a} + \bar{a}\bar{\lambda}^2\bar{p}\bar{c}^2\frac{1}{\underline{r}}\right)\right] \times \|x_n - \hat{x}_n\| + \kappa'\varepsilon' \|x_n - \hat{x}_n\| . \tag{41}$$

Rearranging (41) gives

$$r_n^T \Pi_n\left[2\big(A_n - K_n C_n\big)(x_n - \hat{x}_n) + r_n\right] \le \kappa' \frac{1}{\underline{\lambda}^2\underline{p}}\left[2\Big(\bar{a} + \bar{a}\bar{\lambda}^2\bar{p}\bar{c}^2\frac{1}{\underline{r}}\Big) + \kappa'\varepsilon'\right]\|x_n - \hat{x}_n\|^3 \tag{42}$$

$$= \kappa_{nonl} \|x_n - \hat{x}_n\|^3 \tag{43}$$

where

$$\kappa_{nonl} = \kappa' \frac{1}{\underline{\lambda}^2\underline{p}}\left[2\left(\bar{a} + \bar{a}\bar{\lambda}^2\bar{p}\bar{c}^2\frac{1}{\underline{r}}\right) + \kappa'\varepsilon'\right]. \qquad \square$$

Lemma 3.5. *Let the conditions of Theorem 3.2 be fulfilled, let* $\Pi_n = \left(\Lambda_n P_n \Lambda_n^T\right)^{-1}$ *and* K_n, s_n *given in (15), (9). Then there is a positive real number* κ_{noise} *independent of* δ *such that*

$$E\left\{s_n^T \Pi_{n+1} s_n\right\} \le \kappa_{noise}\delta \tag{44}$$

holds.

Proof. Using the equation (9) and after matrix distribution we obtain

$$s_n^T \Pi_{n+1} s_n = \left\{\big(G_n w_n - K_n D_n v_n\big)^T \Pi_{n+1}\big(G_n w_n - K_n D_n v_n\big)\right\} \tag{45}$$

$$= \Big\{(G_n w_n)^T \Pi_{n+1}(G_n w_n) - (G_n w_n)^T \Pi_{n+1}(K_n D_n v_n) - (K_n D_n v_n)^T \Pi_{n+1}(G_n w_n) + (K_n D_n v_n)^T \Pi_{n+1}(K_n D_n v_n)\Big\}. \tag{46}$$

Recall that the vectors w_n and v_n are uncorrelated, the terms containing both vanish so we have

$$s_n^T \Pi_{n+1} s_n = \{(G_n w_n)^T \Pi_{n+1}(G_n w_n) + (K_n D_n v_n)^T \Pi_{n+1}(K_n D_n v_n)\}. \tag{47}$$

By the group equations (16) and the inequality $C_n \Lambda_n P_n \Lambda_n^T C_n^T > 0$ we have

$$\|K_n\| < \bar{a}\bar{\lambda}^2\bar{p}\bar{c}\frac{1}{\underline{r}}. \tag{48}$$

This inequality yields

$$s_n^T \Pi_{n+1} s_n \le \frac{1}{\underline{\lambda}^2\underline{p}} w_n^T G_n^T G_n w_n + \frac{\bar{a}^2\bar{p}^2\bar{c}^2\bar{\lambda}^2}{\underline{p}\underline{r}^2} v_n^T D_n^T D_n v_n . \tag{49}$$

Taking the trace of the above inequality we get

$$s_n^T \Pi_{n+1} s_n \leq \frac{1}{\underline{\lambda}^2 \underline{p}} \operatorname{tr}\left(w_n^T G_n^T G_n w_n\right) + \frac{\bar{a}^2 \bar{p}^2 \bar{c}^2 \bar{\lambda}^2}{\underline{p}\underline{r}^2} \operatorname{tr}\left(v_n^T D_n^T D_n v_n\right). \tag{50}$$

Since $\operatorname{tr}(\Gamma\Delta) = \operatorname{tr}(\Delta\Gamma)$, using (50) we obtain

$$s_n^T \Pi_{n+1} s_n \leq \frac{1}{\underline{\lambda}^2 \underline{p}} \operatorname{tr}\left(G_n w_n w_n^T G_n^T\right) + \frac{\bar{a}^2 \bar{p}^2 \bar{c}^2 \bar{\lambda}^2}{\underline{p}\underline{r}^2} \operatorname{tr}\left(D_n v_n v_n^T D_n^T\right), \tag{51}$$

where D_n and G_n are deterministic matrices. Remember that w_n and v_n are vector valued white noise process, thus,

$$E\left\{v_n v_n^T\right\} = I \tag{52}$$

and

$$E\left\{w_n w_n^T\right\} = I \tag{53}$$

hold. Thus we have

$$E\left\{s_n^T \Pi_{n+1} s_n\right\} \leq \frac{1}{\underline{\lambda}^2 \operatorname{tr}\left(G_n \Lambda_n \Lambda_n^T G_n^T\right) \underline{p}} + \frac{\bar{a}^2 \bar{p}^2 \bar{c}^2 \bar{\lambda}^2}{\underline{p}\underline{r}^2} \operatorname{tr}\left(D_n v_n v_n^T D_n^T\right). \tag{54}$$

From the equations (20) and (21) we have

$$\operatorname{tr}\left(G_n \Lambda_n \Lambda_n^T G_n^T\right) \leq \delta \operatorname{tr}(I) = q\delta \tag{55}$$

and

$$\operatorname{tr}\left(D_n D_n^T\right) \leq \delta \operatorname{tr}(I) = m\delta, \tag{56}$$

where q and m are the number of rows G_n and D_n, respectively. Defining

$$\kappa_{noise} = \frac{q}{\underline{\lambda}^2 \underline{p}} + \frac{\bar{a}^2 \bar{c}^2 \bar{p}^2 \bar{\lambda}^2 m}{\underline{p}\underline{r}^2} \tag{57}$$

yields

$$E\left\{s_n^T \Pi_{n+1} s_n\right\} \leq \kappa_{noise}\delta. \tag{58}$$

This completes the proof. □

We are now ready to prove the main result stated in Theorem 3.2 of the paper.

Proof of Theorem 3.2. There exists a function depending on error estimate

$$V_n(\zeta_n) = \zeta_n^T \Pi_n \zeta_n \tag{59}$$

with $\Pi_n = \left(\Lambda_n P_n \Lambda_n^T\right)^{-1}$ since P_n is positive definite. From the inequalities (16c)-(16f) we have

$$\frac{1}{\bar{p}\bar{\lambda}^2} \|\zeta_n\|^2 \leq V_n(\zeta_n) \leq \frac{1}{\underline{p}\underline{\lambda}^2} \|\zeta_n\|^2, \tag{60}$$

which is similar to (10) with $\underline{v} = \frac{1}{\bar{p}\bar{\lambda}^2}$ and $\bar{v} = \frac{1}{\underline{p}\underline{\lambda}^2}$. We need an upper bound on $E\{V_{n+1}(\zeta_{n+1}) | \zeta_n\}$ as stated in (11) to meet the requirements of Lemma 2.1. From (7) we obtain

$$V_n(\zeta_{n+1}) = \left[(A_n - K_n C_n)\zeta_n + r_n + s_n\right]^T \Pi_{n+1} \left[(A_n - K_n C_n)\zeta_n + r_n + s_n\right]. \tag{61}$$

Using Lemma 3.3 we obtain

$$V_n(\zeta_{n+1}) \leq (1-\alpha) V_n(\zeta_n) + r_n^T \Pi_{n+1}\big(2(A_n - K_n C_n)\zeta_n + r_n\big) + 2 s_n^T \Pi_{n+1}\big((A_n - K_n C_n)\zeta_n + r_n\big) + s_n^T \Pi_{n+1} s_n. \tag{62}$$

Taking the conditional expectation $E\left\{V_{n+1}\left(\zeta_{n+1}\right)|\zeta_n\right\}$ and considering the white noise property it can be seen that the term $E\left\{s_n^T \Pi_{n+1}\left(\left(A_n - K_n C_n\right)\zeta_n + r_n\right)|\zeta_n\right\}$ vanishes since neither Π_{n+1} nor $A_n, K_n, C_n, r_n, s_n, \zeta_n$ depend on v_n or w_n. The remaining terms are estimated by Lemma 3.4 and Lemma 3.5 as

$$E\left\{V_{n+1}\left(\zeta_{n+1}\right)|\zeta_n\right\} - V_n\left(\zeta_n\right) \le -\alpha V_n\left(\zeta_n\right) + \kappa_{nonl}\left\|\zeta_n\right\|^3 + \kappa_{noise}\delta \tag{63}$$

for $\|\zeta_n\| \le \varepsilon'$. We define

$$\varepsilon = \min\left(\varepsilon', \frac{\alpha}{2\bar{p}\bar{\lambda}^2\kappa_{noise}}\right). \tag{64}$$

Then from (59), (60) under condition $\|\zeta_n\| \le \varepsilon$ we obtain

$$\kappa_{nonl}\left\|\zeta_n\right\|\left\|\zeta_n\right\|^2 \le \frac{\alpha}{2\bar{p}\bar{\lambda}^2}\left\|\zeta_n\right\|^2 \le \frac{\alpha}{2}V_n\left(\zeta_n\right). \tag{65}$$

Substituting into (63) yields

$$E\left\{V_{n+1}\left(\zeta_{n+1}\right)|\zeta_n\right\} - V_n\left(\zeta_n\right) \le -\alpha V_n\left(\zeta_n\right) + \underbrace{\kappa_{nonl}\left\|\zeta_n\right\|^3}_{\le\frac{\alpha}{2}V_n(\zeta_n)} + \kappa_{noise}\delta \le -\frac{\alpha}{2}V_n\left(\zeta_n\right) + \kappa_{noise}\delta \tag{66}$$

for $\|\zeta_n\| \le \varepsilon$. Therefore we are able to apply Lemma 2.1 with $\|\zeta_0\| \le \varepsilon$, $\underline{v} = \frac{1}{\bar{p}\bar{\lambda}^2}$, $\bar{v} = \frac{1}{\underline{p\lambda}^2}$ and $\mu = \kappa_{noise}\delta$. However, with some $\tilde{\varepsilon} \le \varepsilon$ for $\tilde{\varepsilon} \le \|\zeta_n\| \le \varepsilon$ we have to guarantee the inequality

$$E\left\{V_{n+1}\left(\zeta_{n+1}\right)|\zeta_n\right\} - V_n\left(\zeta_n\right) \le -\frac{\alpha}{2}V_n\left(\zeta_n\right) + \kappa_{noise}\delta \le 0. \tag{67}$$

Choosing with the aid of (64)

$$\delta = \frac{\alpha\tilde{\varepsilon}^2}{2\bar{p}\bar{\lambda}^2\kappa_{noise}} \tag{68}$$

with some $\tilde{\varepsilon} \le \varepsilon$ we have for $\|\zeta_n\| \ge \tilde{\varepsilon}$

$$\kappa_{noise} \le \frac{\alpha}{2\bar{p}\bar{\lambda}^2}\left\|\zeta_n\right\|^2 \le \frac{\alpha}{2}V_n\left(\zeta_n\right), \tag{69}$$

which says that (67) holds. In result we conclude that the estimation error remains bounded if the initial error and noise terms are bounded as stated in (19)-(21). □

4 Simulation study

In the previous section it is shown that the estimation error of the discrete–time AFEKF is bounded under two conditions: (1) sufficiently small initial estimation error (2) sufficiently small noise assumptions. Here, we run simulations to illustrate numerically the significance of these assumptions and to show the numerical behaviour of the theory we obtained. For this purpose we consider the Lotka-Volterra (prey-predator) model in which the population growth of two interactive species is described. The model consists of a pair of non–linear differential equations

$$\frac{dx_1(t)}{dt} = ax_1(t) - bx_1(t)x_2(t), \tag{70}$$

$$\frac{dx_2(t)}{dt} = -mx_2(t) - rx_1(t)x_2(t), \tag{71}$$

where $x_1(t)$ is the number of the first species (prey) in time t, $x_2(t)$ is the number of the second species (predator) in time t, a is the reproduction rate of preys, m is the death rate of predators and parameters b and r describe the interaction of the two species.

The state–space notation with perturbed gaussian white noise for the differential system is

$$x_{t+1} = \begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \begin{bmatrix} 1 + \left(a - bx_{2,t}\right)\Delta t & 0 \\ 0 & 1 + \left(-m + rx_{1,t}\right)\Delta t \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + G_n w_t,$$

$$y_t = \begin{bmatrix} 0 & 1 \end{bmatrix} x_t + D_t v_t, \tag{72}$$

where y_t is the measurements in time t, Δt is integration time interval subdivider. Also, w_t, v_t are uncorrelated system and measurement noise terms with a mean of zero and Q, R covariance matrices, respectively [27].

We compare the EKF and the AFEKF with the initial estimates and the noise terms given in Table 1 over 250 replicated samples. The exact values of the parameters used in the simulations are given in Table 2.

Table 1. The initial state estimates and noise terms used in the simulation

	Stability conditions are met		Stability conditions are violated	
Initial state – $\hat{x}_0$	$[5\ 2]'$		$[3\ 3]'$	
Process noise – G_t	$0.1 \times I_2$		$0.000001 \times I_2$	
Measurement noise – D_t	0.1		0.5	
Unknown parameters	$a = 0.2$	$b = 0.06$	$a = 0.2$	$b = 0.16$
	$m = 0.1$	$r = 0.01$	$m = 0.12$	$r = 0.01$

Table 2. True values of the unknown parameters in Simulation

Parameter name	Exact value
a	0.2
b	0.06
m	0.10
r	$\begin{cases} 0.01 & \text{if } t < 50 \\ 0.03 & \text{if } t \geq 50 \end{cases}$

The simulations results are displayed in Figures 1-5. Figure 1 describes the estimation error during the simulation process. It is obvious that if the conditions in (18) to (20) are satisfied, then the estimation error remains bounded for both the EKF and the AFEKF. In Figure 2, the sum of the squared estimation errors are shown. The estimation error in the AFEKF at time t is smaller than that of the EKF, thus, the AFEKF converges to true value faster than the EKF. On the other hand, when the conditions defined by (18) to (20) are violated, the state estimates of the EKF diverge from true states as seen in Figure 3. Hence, the estimation error of the EKF grows without bound. However, under the same conditions, the AFEKF's state estimates by using forgetting factors in Figure 4 converge to the true state values and the estimation errors remain bounded. Finally, Figure 5 demonstrates the performance improvement in the sum of the squared estimation errors when the stability conditions are violated.

5 Conclusion

In this study, we have analyzed the error behavior of the AFEKF when it is applied to the general estimation problems for non–linear stochastic discrete–time systems. The results show that the estimation error remains bounded in the mean square sense under certain conditions. This includes small initial estimation error, small disturbing noise terms, positive definite and bounded Ricatti difference equations. We have presented some numerical simulations to prove the importance of the stability conditions as well as to evaluate the performance of the AFEKF compared to the standart the EKF. The simulations presented state that small initial estimation error results in bounded estimation error in both the EKF and the AFEKF. However, when the initial estimation error is not small enough, the estimation error in the EKF is much bigger than the AFEKF which shows that the forgetting factors prevent from the filtering estimation to diverge.

Fig. 1. Estimation error for State 1 and State 2 (Stability conditions are met)

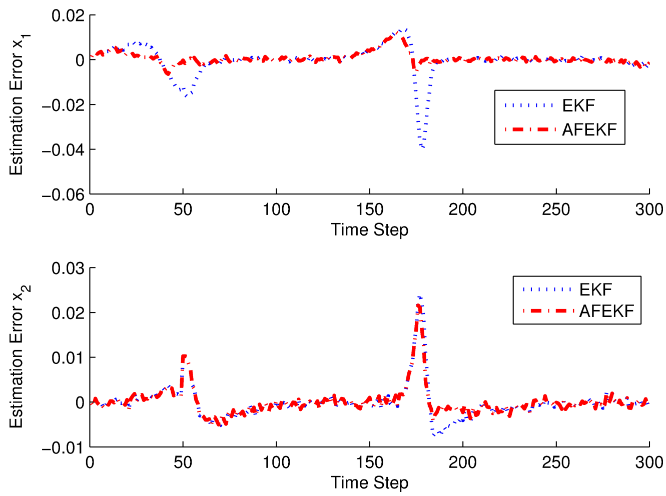


Fig. 2. Sum of the squared estimation errors (Stability conditions are met)

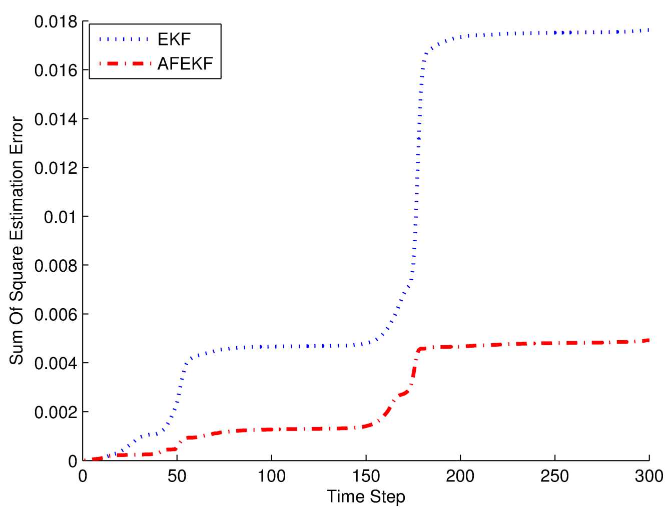


Fig. 3. State 1 and State 2 estimations (Stability conditions are violated)

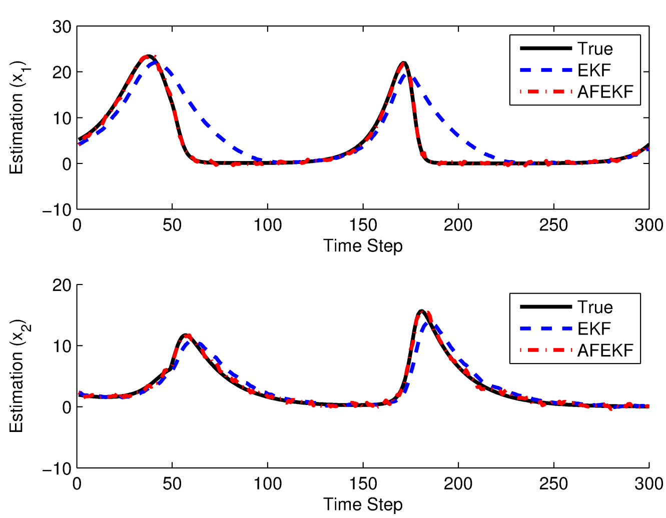


Fig. 4. Forgetting factors

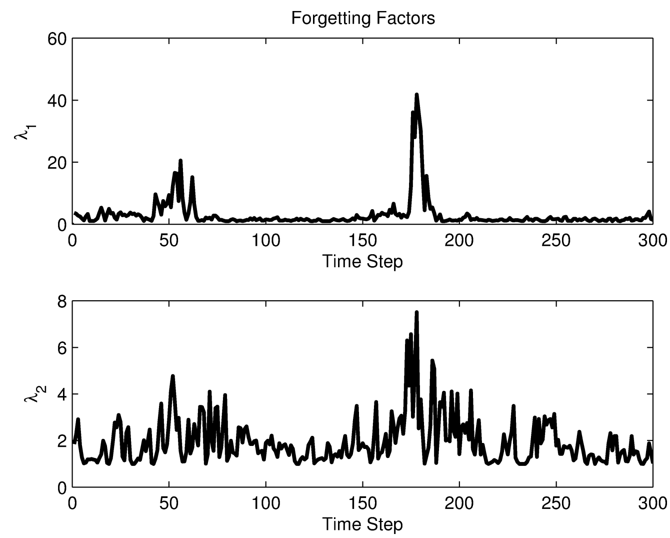
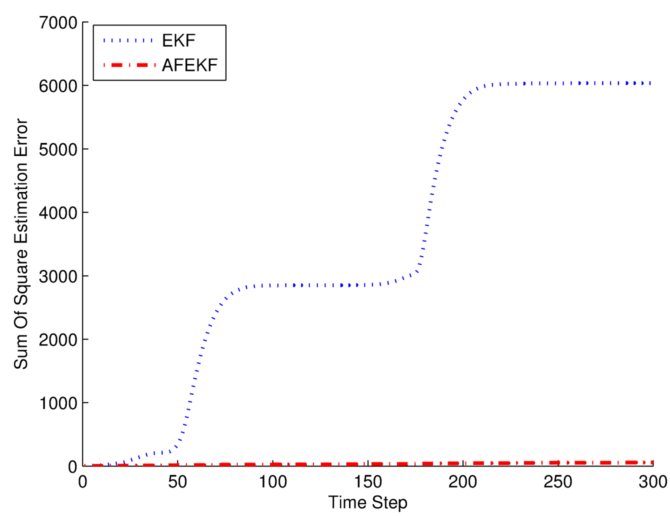


Fig. 5. Sum of the squared estimation errors (Stability conditions are violated)



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