Al Arifi et al. *Journal of Inequalities and Applications* (2016) 2016:189
DOI 10.1186/s13660-016-1132-y

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Lyapunov-type inequalities for a fractional p-Laplacian equation

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Abstract

In this paper, we present new Lyapunov-type inequalities for a fractional boundary value problem that models a turbulent flow in a porous medium. The obtained inequalities are used to obtain a lower bound for the eigenvalues of corresponding equations.

MSC: 34A08; 15A42; 26D15; 76F70

Keywords: Lyapunov-type inequality; fractional derivative; eigenvalues; turbulent flow

1 Introduction

The p-Laplacian operator arises in different mathematical models that describe physical and natural phenomena (see, for example, [1–6]). In particular, it is used in some models related to turbulent flows (see, for example, [7–9]).

In this paper, we present some Lyapunov-type inequalities for a fractional-order model for turbulent flow in a porous medium. More precisely, we are interested with the nonlinear fractional boundary value problem

$$\begin{cases} D_{a^+}^{\beta}(\Phi_p(D_{a^+}^{\alpha}u(t))) + \chi(t)\Phi_p(u(t)) = 0, & a < t < b, \\ u(a) = u'(a) = u'(b) = 0, & D_{a^+}^{\alpha}u(a) = D_{a^+}^{\alpha}u(b) = 0, \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3$, $1 < \beta \leq 2$, $D_{a^+}^{\alpha}$, $D_{a^+}^{\beta}$ are the Riemann-Liouville fractional derivatives of orders α, β, $\Phi_p(s) = |s|^{p-2}s$, $p > 1$, and $\chi : [a,b] \to \mathbb{R}$ is a continuous function. Under certain assumptions imposed on the function q, we obtain necessary conditions for the existence of nontrivial solutions to (1.1). Some applications to eigenvalue problems are also presented.

For completeness, let us recall the standard Lyapunov inequality [10], which states that if u is a nontrivial solution of the problem

$$\begin{cases} u''(t) + \chi(t)u(t) = 0, & a < t < b, \\ u(a) = u(b) = 0, \end{cases}$$

where $a < b$ are two consecutive zeros of u, and $\chi : [a,b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |\chi(t)|\,\mathrm{d}t > \frac{4}{b-a}. \tag{1.2}$$

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Note that in order to obtain this inequality, it is supposed that a and b are two consecutive zeros of u. In our case, as it will be observed in the proof of our main result, we assume just that u is a nontrivial solution to (1.1).

Inequality (1.2) is useful in various applications, including oscillation theory, stability criteria for periodic differential equations, and estimates for intervals of disconjugacy.

Several generalizations and extensions of inequality (1.2) to different boundary value problems exist in the literature. As examples, we refer to [11–16] and the references therein.

Recently, some Lyapunov-type inequalities for fractional boundary value problems have been obtained. Ferreira [17] established a fractional version of inequality (1.2) for a fractional boundary value problem involving the Riemann-Liouville fractional derivative of order $1 < \alpha \leq 2$. More precisely, Ferreira [17] studied the fractional boundary value problem

$$\begin{cases} D_{a^+}^{\alpha} u(t) + \chi(t)u(t) = 0, & a < t < b, \\ u(a) = u(b) = 0, \end{cases} \tag{1.3}$$

where $D_{a^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order $1 < \alpha \leq 2$, and $\chi : [a,b] \to \mathbb{R}$ is a continuous function. In this case, it was proved that if (1.3) has a nontrivial solution, then

$$\int_a^b |\chi(t)| \, \mathrm{d}t > \Gamma(\alpha) \left(\frac{4}{b-a} \right)^{\alpha-1},$$

where Γ is the Euler gamma function. Observe that if we take $\alpha = 2$ in the last inequality, we obtain the standard Lyapunov inequality (1.2).

Ferreira [18] established a fractional version of inequality (1.2) for a fractional boundary value problem involving the Caputo fractional derivative of order $1 < \alpha \leq 2$. In both papers [17, 18], the author presented nice applications to obtain intervals where certain Mittag-Leffler functions have no real zeros.

Jleli and Samet [19] studied a fractional differential equation involving the Caputo fractional derivative under mixed boundary conditions. More precisely, they considered the fractional differential equation

$${}^{\mathrm{C}}D_{a^+}^{\alpha} u(t) + \chi(t)u(t) = 0, \quad a < t < b, \tag{1.4}$$

under the mixed boundary conditions

$$u(a) = u'(b) = 0 \tag{1.5}$$

or

$$u'(a) = u(b) = 0, \tag{1.6}$$

where ${}^{\mathrm{C}}D_{a^+}^{\alpha}$ is the Caputo fractional derivative of order $1 < \alpha \leq 2$. For the boundary conditions (1.5) and (1.6), the following two Lyapunov-type inequalities were derived respectively:

$$\int_a^b (b-s)^{\alpha-2}\left|q(s)\right| \mathrm{d}s \geq \frac{\Gamma(\alpha)}{\max\{\alpha-1, 2-\alpha\}(b-a)}$$

and

$$\int_a^b (b-s)^{\alpha-1}\left|q(s)\right| \mathrm{d}s \geq \Gamma(\alpha).$$

The same equation (1.4) was considered by Rong and Bai [20] with the fractional boundary condition

$$u(a) = {}^{\mathrm{C}}D_{a^+}^{\beta} u(b) = 0,$$

where $0 < \beta \leq 1$.

For other related results, we refer to [21–23] and the references therein.

The paper is organized as follows. In Section 2, we recall some basic concepts on fractional calculus and establish some preliminary results that will be used in Section 3, where we state and prove our main result. In Section 4, we present some applications of the obtained Lyapunov-type inequalities to eigenvalue problems.

2 Preliminaries

For the convenience of the reader, we recall some basic concepts on fractional calculus to make easy the analysis of (1.1). For more details, we refer to [24].

Let $C[a,b]$ be the set of real-valued and continuous functions in $[a,b]$. Let $f \in C[a,b]$. Let $\alpha \geq 0$. The Riemann-Liouville fractional integral of order α of f is defined by $I_a^0 f \equiv f$ and

$$\left(I_{a^+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad \alpha > 0, t \in [a,b],$$

where Γ is the gamma function.

The Riemann-Liouville fractional derivative of order $\alpha > 0$ of f is defined by

$$\left(D_{a^+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s, \quad t \in [a,b],$$

where $n = [\alpha] + 1$.

Lemma 2.1 (see [24]) *Let $\alpha > 0$. If $D_{a^+}^{\alpha} u \in C[a,b]$, then*

$$I_{a^+}^{\alpha} D_{a^+}^{\alpha} u(t) = u(t) + \sum_{k=1}^{n} c_k (t-a)^{\alpha-k},$$

where $n = [\alpha] + 1$.

Now, in order to obtain an integral formulation of (1.1), we need the following results.

Lemma 2.2 *Let* $2 < \alpha \le 3$ *and* $y \in C[a,b]$. *Then the problem*

$$\begin{cases} D_{a^+}^{\alpha} u(t) + y(t) = 0, & a < t < b, \\ u(a) = u'(a) = u'(b) = 0 \end{cases}$$

has a unique solution

$$u(t) = \int_a^b G(t,s) y(s)\,\mathrm{d}s,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\frac{b-s}{b-a})^{\alpha-2}(t-a)^{\alpha-1} - (t-s)^{\alpha-1}, & a \le s \le t \le b, \\ (\frac{b-s}{b-a})^{\alpha-2}(t-a)^{\alpha-1}, & a \le t \le s \le b. \end{cases}$$

Proof From Lemma 2.1 we have

$$u(t) = -\big(I_{a^+}^{\alpha} y\big)(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + c_3(t-a)^{\alpha-3}$$

for some real constants c_i, $i = 1,2,3$, that is,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)\,\mathrm{d}s + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + c_3(t-a)^{\alpha-3}.$$

The condition $u(a) = 0$ yields $c_3 = 0$. Therefore,

$$u'(t) = -\frac{(\alpha-1)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-2} y(s)\,\mathrm{d}s + c_1(\alpha-1)(t-a)^{\alpha-2} + c_2(\alpha-2)(t-a)^{\alpha-3}.$$

The condition $u'(a) = 0$ implies that $c_2 = 0$. Then

$$u'(b) = -\frac{(\alpha-1)}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-2} y(s)\,\mathrm{d}s + c_1(\alpha-1)(b-a)^{\alpha-2}.$$

Since $u'(b) = 0$, we get

$$c_1 = \frac{1}{(b-a)^{\alpha-2}\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-2} y(s)\,\mathrm{d}s.$$

Thus,

$$u(t) = -\int_a^t \frac{(t-s)^{\alpha-1} y(s)}{\Gamma(\alpha)}\,\mathrm{d}s + \int_a^b \frac{1}{\Gamma(\alpha)} \left(\frac{b-s}{b-a}\right)^{\alpha-2} (t-a)^{\alpha-1} y(s)\,\mathrm{d}s.$$

For the uniqueness, suppose that u_1 and u_2 are two solutions of the considered problem. Define $u = u_1 - u_2$. By linearity, u solves the boundary value problem

$$\begin{cases} D_{a^+}^{\alpha} u(t) = 0, & a < t < b, \\ u(a) = u'(a) = u'(b) = 0, \end{cases}$$

which has as a unique solution $u = 0$. Therefore, $u_1 = u_2$, and the uniqueness follows. □

Lemma 2.3 *Let $y \in C[a,b]$, $2 < \alpha \leq 3$, $1 < \beta \leq 2$, $p > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then the problem*

$$\begin{cases} D_{a^+}^{\beta}(\Phi_p(D_{a^+}^{\alpha} u(t))) + y(t) = 0, & a < t < b, \\ u(a) = u'(a) = u'(b) = 0, & D_{a^+}^{\alpha} u(a) = D_{a^+}^{\alpha} u(b) = 0 \end{cases}$$

has a unique solution

$$u(t) = -\int_a^b G(t,s)\Phi_q\left(\int_a^b H(s,\tau)y(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s,$$

where

$$H(t,s) = \frac{1}{\Gamma(\beta)}\begin{cases} (\frac{b-s}{b-a})^{\beta-1}(t-a)^{\beta-1} - (t-s)^{\beta-1}, & a \leq s \leq t \leq b, \\ (\frac{b-s}{b-a})^{\beta-1}(t-a)^{\beta-1}, & a \leq t \leq s \leq b. \end{cases}$$

Proof From Lemma 2.1 we have

$$\Phi_p\bigl(D_{a^+}^{\alpha} u\bigr)(t) = -\int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s)\,\mathrm{d}s + c_1(t-a)^{\beta-1} + c_2(t-a)^{\beta-2},$$

where c_i, $i = 1, 2$, are real constants. The condition $D_{a^+}^{\alpha} u(a) = 0$ implies that $\Phi_p(D_{a^+}^{\alpha} u)(a) = 0$, which yields $c_2 = 0$. The condition $D_{a^+}^{\alpha} u(b) = 0$ implies that $\Phi_p(D_{a^+}^{\alpha} u)(b) = 0$, which yields

$$c_1 = \frac{1}{(b-a)^{\beta-1}}\int_a^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} y(s)\,\mathrm{d}s.$$

Therefore,

$$\Phi_p\bigl(D_{a^+}^{\alpha} u\bigr)(t) = -\int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s)\,\mathrm{d}s + \frac{1}{(b-a)^{\beta-1}}\int_a^b \frac{(b-s)^{\beta-1}(t-a)^{\beta-1}}{\Gamma(\beta)} y(s)\,\mathrm{d}s,$$

that is,

$$\Phi_p\bigl(D_{a^+}^{\alpha} u\bigr)(t) = \int_a^b H(t,\tau)y(\tau)\,\mathrm{d}\tau.$$

Then we have

$$D_{a^+}^{\alpha} u(t) - \Phi_q\left(\int_a^b H(t,\tau)y(\tau)\,\mathrm{d}s\right) = 0.$$

Setting

$$\tilde{y}(t) = -\Phi_q\left(\int_a^b H(t,\tau)y(\tau)\,\mathrm{d}\tau\right),$$

we obtain

$$\begin{cases} D_{a^+}^{\alpha} u(t) + \tilde{y}(t) = 0, & a < t < b, \\ u(a) = u'(a) = u'(b) = 0. \end{cases}$$

Finally, applying Lemma 2.2, we obtain the desired result. □

The following estimates will be useful later.

Lemma 2.4 *We have*

$$0 \leq G(t,s) \leq G(b,s), \quad (t,s) \in [a,b] \times [a,b].$$

Proof Differentiating with respect to t, we obtain

$$G_t(t,s) = \frac{(\alpha-1)}{\Gamma(\alpha)} \begin{cases} (\frac{b-s}{b-a})^{\alpha-2}(t-a)^{\alpha-2} - (t-s)^{\alpha-2}, & a \leq s \leq t \leq b, \\ (\frac{b-s}{b-a})^{\alpha-2}(t-a)^{\alpha-2}, & a \leq t \leq s \leq b. \end{cases}$$

Set

$$g_1(t,s) = \left(\frac{b-s}{b-a}\right)^{\alpha-2}(t-a)^{\alpha-2} - (t-s)^{\alpha-2}, \quad a \leq s \leq t \leq b$$

and

$$g_2(t,s) = \left(\frac{b-s}{b-a}\right)^{\alpha-2}(t-a)^{\alpha-2}, \quad a \leq t \leq s \leq b.$$

Clearly,

$$g_2(t,s) \geq 0, \quad a \leq t \leq s \leq b.$$

On the other hand, using the inequality

$$(b-s)(t-a) \geq (t-s)(b-a), \quad a \leq s \leq t \leq b,$$

and the fact that $\alpha > 2$, we obtain

$$(b-s)^{\alpha-2}(t-a)^{\alpha-2} \geq (t-s)^{\alpha-2}(b-a)^{\alpha-2}, \quad a \leq s \leq t \leq b,$$

which yields

$$g_1(t,s) \geq 0, \quad a \leq s \leq t \leq b.$$

As consequence, we have

$$G_t(t,s) \geq 0, \quad (t,s) \in [a,b] \times [a,b].$$

Then $G(\cdot,s)$ is a nondecreasing function for all $s \in [a,b]$, which yields

$$0 = G(a,s) \leq G(t,s) \leq G(b,s), \quad (t,s) \in [a,b] \times [a,b].$$

The proof is complete. $\square$

Lemma 2.5 *We have*

$$0 \leq H(t,s) \leq H(s,s), \quad (t,s) \in [a,b] \times [a,b].$$

Proof Observe that $H(t,s) = G_t(t,s)$ for $\alpha = \beta + 1$. Then, from the proof of Lemma 2.4 we have

$$H(t,s) \geq 0, \quad (t,s) \in [a,b] \times [a,b].$$

On the other hand, for all $s \in [a,b]$, we have

$$\Gamma(\beta)H(s,s) = \left(\frac{b-s}{b-a}\right)^{\beta-1}(s-a)^{\beta-1}.$$

For $a \leq t \leq s \leq b$, we have

$$\begin{aligned} \Gamma(\beta)H(t,s) &= \left(\frac{b-s}{b-a}\right)^{\beta-1}(t-a)^{\beta-1} \\ &\leq \left(\frac{b-s}{b-a}\right)^{\beta-1}(s-a)^{\beta-1} \\ &= \Gamma(\beta)H(s,s). \end{aligned}$$

For $a \leq s < t \leq b$, we have

$$\Gamma(\beta)H(t,s) = \left(\frac{b-s}{b-a}\right)^{\beta-1}(t-a)^{\beta-1} - (t-s)^{\beta-1}.$$

Let $s \in [a,b)$ be fixed. Define the function $\psi : (s,b] \to \mathbb{R}$ by

$$\psi(t) = \Gamma(\beta)H(t,s), \quad t \in (s,b].$$

We have

$$\psi'(t) = (\beta-1)\left[\left(\frac{b-s}{b-a}\right)^{\beta-1}(t-a)^{\beta-2} - (t-s)^{\beta-2}\right], \quad t \in (s,b].$$

Using the inequalities

$$\left(\frac{b-s}{b-a}\right)^{\beta-1} \leq 1, \qquad \beta-2 \leq 0, \qquad (t-a)^{\beta-2} \leq (t-s)^{\beta-2},$$

we get

$$\psi'(t) \leq 0, \quad t \in (s,b].$$

Thus, for all $t \in (s,b]$, we have

$$\psi(t) \leq \psi(s),$$

that is,

$$\Gamma(\beta)H(t,s) \le \Gamma(\beta)H(s,s), \quad t \in (s,b].$$

The proof is complete. □

Now, we are ready to state and prove our main result.

3 Main result

Our main result is the following Lyapunov-type inequality.

Theorem 3.1 *Suppose that* $2 < \alpha \le 3$, $1 < \beta \le 2$, $p > 1$, *and* $\chi : [a,b] \to \mathbb{R}$ *is a continuous function. If* (1.1) *has a nontrivial continuous solution, then*

$$\begin{aligned} &\int_a^b (b-s)^{\beta-1}(s-a)^{\beta-1}\big|\chi(s)\big|\,\mathrm{d}s \\ &\quad \ge \big[\Gamma(\alpha)\big]^{p-1}\Gamma(\beta)(b-a)^{\beta-1}\left(\int_a^b (b-s)^{\alpha-2}(s-a)\,\mathrm{d}s\right)^{1-p}. \end{aligned} \tag{3.1}$$

Proof We endow the set $C[a,b]$ with the Chebyshev norm $\|\cdot\|_\infty$ given by

$$\|u\|_\infty = \max\big\{\big|u(t)\big| : a \le t \le b\big\}, \quad u \in C[a,b].$$

Suppose that $u \in C[a,b]$ is a nontrivial solution of (1.1). From Lemma 2.3 we have

$$u(t) = -\int_a^b G(t,s)\Phi_q\left(\int_a^b H(s,\tau)\chi(\tau)\Phi_p\big(u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s, \quad t \in [a,b].$$

Let $t \in [a,b]$ be fixed. We have

$$\begin{aligned} \big|u(t)\big| &\le \int_a^b \big|G(t,s)\big|\left|\Phi_q\left(\int_a^b H(s,\tau)\chi(\tau)\phi_p\big(u(\tau)\big)\,\mathrm{d}\tau\right)\right|\mathrm{d}s \\ &= \int_a^b \big|G(t,s)\big|\left|\int_a^b H(s,\tau)\chi(\tau)\Phi_p\big(u(\tau)\big)\,\mathrm{d}\tau\right|^{q-1}\mathrm{d}s \\ &\le \int_a^b \big|G(t,s)\big|\theta(s)\,\mathrm{d}s, \end{aligned}$$

where

$$\theta(s) = \left(\int_a^b \big|H(s,\tau)\big|\big|\chi(\tau)\big|\big|u(\tau)\big|^{p-1}\,\mathrm{d}\tau\right)^{q-1}, \quad s \in [a,b].$$

Using Lemma 2.4 and Lemma 2.5, we obtain

$$\big|u(t)\big| \le \|u\|_\infty^{(p-1)(q-1)}\left(\int_a^b G(b,s)\,\mathrm{d}s\right)\left(\int_a^b H(s,s)\big|\chi(s)\big|\,\mathrm{d}s\right)^{q-1}.$$

Since the last inequality holds for every $t \in [a,b]$, we obtain

$$1 \leq \left(\int_a^b G(b,s)\,\mathrm{d}s\right)\left(\int_a^b H(s,s)\left|\chi(s)\right|\mathrm{d}s\right)^{q-1},$$

which yields the desired result. □

Corollary 3.2 *Suppose that* $2 < \alpha \leq 3$, $1 < \beta \leq 2$, $p > 1$, *and* $\chi : [a,b] \to \mathbb{R}$ *is a continuous function. If* (1.1) *has a nontrivial continuous solution, then*

$$\int_a^b \left|\chi(s)\right|\mathrm{d}s \geq \frac{4^{\beta-1}[\Gamma(\alpha)]^{p-1}\Gamma(\beta)}{(b-a)^{\beta-1}}\left(\int_a^b (b-s)^{\alpha-2}(s-a)\,\mathrm{d}s\right)^{1-p}. \tag{3.2}$$

Proof Let

$$\psi(s) = (b-s)(s-a), \quad s \in [a,b].$$

Observe that the function ψ has a maximum at the point $s^* = \frac{a+b}{2}$, that is,

$$\|\psi\|_\infty = \psi\left(s^*\right) = \frac{(b-a)^2}{4}.$$

The desired result follows immediately from the last equality and inequality (3.1). □

For $p = 2$, problem (1.1) becomes

$$\begin{cases} D_{a^+}^{\beta}(D_{a^+}^{\alpha}u(t)) + \chi(t)u(t) = 0, & a < t < b, \\ u(a) = u'(a) = u'(b) = 0, & D_{a^+}^{\alpha}u(a) = D_{a^+}^{\alpha}u(b) = 0, \end{cases} \tag{3.3}$$

where $2 < \alpha \leq 3$, $1 < \beta \leq 2$, and $\chi : [a,b] \to \mathbb{R}$ is a continuous function. In this case, taking $p = 2$ in Theorem 3.1, we obtain the following result.

Corollary 3.3 *Suppose that* $2 < \alpha \leq 3$, $1 < \beta \leq 2$, *and* $\chi : [a,b] \to \mathbb{R}$ *is a continuous function. If* (3.3) *has a nontrivial continuous solution, then*

$$\int_a^b (b-s)^{\beta-1}(s-a)^{\beta-1}\left|\chi(s)\right|\mathrm{d}s \geq \Gamma(\alpha)\Gamma(\beta)(b-a)^{\beta-1}\left(\int_a^b (b-s)^{\alpha-2}(s-a)\,\mathrm{d}s\right)^{-1}.$$

Taking $p = 2$ in Corollary 3.2, we obtain the following result.

Corollary 3.4 *Suppose that* $2 < \alpha \leq 3$, $1 < \beta \leq 2$, *and* $\chi : [a,b] \to \mathbb{R}$ *is a continuous function. If* (3.3) *has a nontrivial continuous solution, then*

$$\int_a^b \left|\chi(s)\right|\mathrm{d}s \geq \frac{4^{\beta-1}\Gamma(\alpha)\Gamma(\beta)}{(b-a)^{\beta-1}}\left(\int_a^b (b-s)^{\alpha-2}(s-a)\,\mathrm{d}s\right)^{-1}.$$

4 Applications to eigenvalue problems

In this section, we present some applications of the obtained results to eigenvalue problems.

Corollary 4.1 *Let λ be an eigenvalue of the problem*

$$\begin{cases} D_{0^+}^{\beta}(\Phi_p(D_{0^+}^{\alpha}u(t))) + \lambda\Phi_p(u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, & D_{0^+}^{\alpha}u(0) = D_{0^+}^{\alpha}u(1) = 0, \end{cases} \tag{4.1}$$

where $2 < \alpha \leq 3$, $1 < \beta \leq 2$, *and* $p > 1$. *Then*

$$|\lambda| \geq \frac{\Gamma(2\beta)}{\Gamma(\beta)}\left(\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{\Gamma(\alpha-1)}\right)^{p-1}. \tag{4.2}$$

Proof Let λ be an eigenvalue of (4.1). Then there exists a nontrivial solution $u = u_\lambda$ to (4.1). Using Theorem 3.1 with $(a,b) = (0,1)$ and $\chi(s) = \lambda$, we obtain

$$|\lambda|\int_0^1 (1-s)^{\beta-1}s^{\beta-1}\,\mathrm{d}s \geq \left[\Gamma(\alpha)\right]^{p-1}\Gamma(\beta)\left(\int_0^1 (1-s)^{\alpha-2}s\,\mathrm{d}s\right)^{1-p}.$$

Observe that

$$\int_0^1 (1-s)^{\beta-1}s^{\beta-1}\,\mathrm{d}s = B(\beta,\beta)$$

and

$$\int_0^1 (1-s)^{\alpha-2}s\,\mathrm{d}s = \int_0^1 s^{2-1}(1-s)^{(\alpha-1)-1}\,\mathrm{d}s = B(2,\alpha-1),$$

where B is the beta function defined by

$$B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1}\,\mathrm{d}s, \quad x,y > 0.$$

Using the identity

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

we get the desired result. □

Corollary 4.2 *Let λ be an eigenvalue of the problem*

$$\begin{cases} D_{0^+}^{\beta}(D_{0^+}^{\alpha}u(t)) + \lambda u(t) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, & D_{0^+}^{\alpha}u(0) = D_{0^+}^{\alpha}u(1) = 0, \end{cases}$$

where $2 < \alpha \leq 3$ *and* $1 < \beta \leq 2$. *Then*

$$|\lambda| \geq \frac{\Gamma(\alpha)\Gamma(\alpha+1)\Gamma(2\beta)}{\Gamma(\alpha-1)\Gamma(\beta)}. \tag{4.3}$$

Proof It follows from inequality (4.2) by taking $p = 2$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors carried out the proof. All authors conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

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Acknowledgements
The authors extend their appreciation to Distinguished Scientist Fellowship Program (DSFP) at King Saud University (Saudi Arabia).

Received: 21 January 2016 Accepted: 20 July 2016 Published online: 02 August 2016

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