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Research Article

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Approximation Properties of Ibragimov-Gadjiev-Durrmeyer Operators on $Lp\left(\mathbb{R}^{+}\right)$

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Abstract: We deal with the approximation properties of a new class of positive linear Durrmeyer type operators which offer a reconstruction of integral type operators including well known Durrmeyer operators. This reconstruction allows us to investigate approximation properties of the Durrmeyer operators at the same time. It is first shown that these operators are a positive approximation process in $Lp\left(\mathbb{R}^{+}\right)$. While we are showing this property of the operators we consider the Ditzian-Totik modulus of smoothness and corresponding K-functional. Then, weighted norm convergence, whose proof is based on Korovkin type theorem on $Lp\left(\mathbb{R}^{+}\right)$, is given. At the end of the paper we show several examples of classical sequences that can be obtained from the Ibragimov-Gadjiev-Durrmeyer operators.

Keywords: Durrmeyer operators, Ibragimov-Gadjiev operators, modulus of continuity, L_p approximation.

MSC: 41A36, 41A25, 41A35.

1 Introduction

One of the way to obtain an approximation process for spaces of integrable functions on unbounded interval is studying Durrmeyer type integral modification of the concerned operator. In recent years, this type of modification has been the subject of investigation of several mathematicians. For example, Ditzian and Ivanov [1] proved direct and converse results for linear combinations and derivatives of Bernstein type operators in $L_p\left(\mathbb{R}^{+}\right)$ spaces. And later in [2] Baskakov-Durrmeyer type operators in $L_p\left(\mathbb{R}^{+}\right)$ spaces were introduced by Heilmann. Then Agrawal [3] studied some direct results for a linear combination of a new sequence of linear positive operators in $L_p\left(\mathbb{R}^{+}\right)$. Among the others, we refer the readers to [4, 5] and the references therein.

On the other hand, in 1970, Ibragimov and Gadjiev [6] established a sequence of linear positive operators to collect the many well known operators. They contain Bernstein, Szasz, Bernstein-Cholodwsky and Baskakov operators and others. More recent results on these operators may be found in [7–11].

Later on, integral version of Ibragimov-Gadjiev operators, called Ibragimov-Gadjiev-Durrmeyer operators, were constructed and some approximation results were investigated in [12] by Aral and Acar. Authors presented Voronovskaya type theorem and its quantitative version, thus characterized the order of approximation of the operators in pointwise manner for function $f \in C\left(\mathbb{R}^{+}\right)$.

In this paper we investigate convergence of Ibragimov-Gadjiev-Durrmeyer operators in $L_p\left(\mathbb{R}^{+}\right)$ spaces. Our first aim is to show that the sequence of operators $M_n f$ which will be given below, is an approximation process in $L_p\left(\mathbb{R}^{+}\right)$ spaces. We give the proof with a quantitative estimate. This quantitative result is based

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on Ditzian-Totik modulus of smoothness and its equivalence to appropriate K-functional. As a second part of this work, we give weighted approximation formula using a weighted Korovkin type theorem in a weighted L_p space.

Now we recall these operators.

Definition 1. *Let $(\varphi_n(t))_{n\in\mathbb{N}}$ and $(\psi_n(t))_{n\in\mathbb{N}}$ be sequences of functions in $C(\mathbb{R}^+)$, which is the space of continuous function on $\mathbb{R}^+ := [0, \infty)$, such that $\varphi_n(0) = 0$, $\psi_n(t) > 0$, for all t and $\lim_{n\to\infty} 1/n^2\psi_n(0) = 0$. Also let $(\alpha_n)_{n\in\mathbb{N}}$ denote a sequence of positive numbers which satisfy the following conditions*

$$\lim_{n\to\infty}\frac{\alpha_n}{n} = 1 \text{ and } \lim_{n\to\infty}\alpha_n\psi_n(0) = l_1,\ l_1 > 0.$$

$$\begin{aligned} M_n(f;x) &= (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!} \\ &\quad\times\int_0^\infty f(y)\,K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy \end{aligned} \tag{1}$$

$K_n^{(\nu)}(x,0,\alpha_n\psi_n(0)) = \frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{u=\alpha_n\psi_n(t),t=0}$ is a sequence of functions of three variable x, t, u, such that for each $x, t \in \mathbb{R}^+ = [0, \infty)$ and for each $n \in \mathbb{N}$, $K_n(x,t,u)$ is entire analytic function with respect to variable u, satisfying the following conditions:

1. *Every function of this sequence is an entire function with respect to u for fixed $x, t \in \mathbb{R}^+$ and $K_n(x,0,0) = 1$ for $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$,*
2. $\left[(-1)^\nu\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{u=u_1,t=0}\right] \geq 0$ *for $\nu = 0, 1, \ldots,$ any fixed $u = u_1$ and $x \in \mathbb{R}^+$,*
   (This notation means that the derivative with respect to u is taken ν times, then one set $u = u_1$ and $t = 0$)
3. $\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{u=u_1,t=0} = -nx\left[\frac{\partial^{\nu-1}}{\partial u^{\nu-1}}K_{m+n}(x,t,u)\Big|_{u=u_1,t=0}\right]$ *for all $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$, $\nu = 0, 1, \ldots, m$ is a number such that $m + n = 0$ or a natural number.*
4. *$K_n(0,0,u) = 1$ for any $u \in \mathbb{R}$ and*
   $$\lim_{x\to\infty}x^pK_n^{(\nu)}(x,0,u_1) = 0,$$
   for any $p \in \mathbb{N}$ and fixed $u = u_1$.
5. *For any fixed t and u the function $K_n(x,t,u)$ is continuously differentiable with respect to variable $x \in \mathbb{R}^+$ and satisfying the equality*
   $$\frac{d}{dx}K_n(x,0,u_1) = -nu_1K_{m+n}(x,t,u_1)$$
   for fixed $u = u_1$.
   We assume that the function $K_n(x,t,u)$ in addition to the condition (1)-(5) satisfies also the condition:
6. $\frac{n+\nu m}{1+u_1mx}K_n^{(\nu)}(x,0,u_1) = nK_{n+m}^{(\nu)}(x,0,u_1)$ *for all $x \in \mathbb{R}^+$, $n \in \mathbb{N}$, $\nu = 0, 1, \ldots,$ and fixed $u = u_1$.*

Since $K_n(x,t,u)$ is an entire function with respect to variable u by the assumption (1), we can write Taylor expansion for $K_n(x,t,u)$ at any point $u_1 \in \mathbb{R}$ as

$$K_n(x,t,u) = \sum_{\nu=0}^{\infty}\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{u=u_1}\frac{(u-u_1)^\nu}{\nu!}$$

and replacing $u = \varphi_n(t)$, $u_1 = \alpha_n\psi_n(t)$ and $t = 0$, where (α_n) is the sequence defined in (1),

$$K_n(x,0,0) = \sum_{\nu=0}^{\infty}\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{u=\alpha_{n\psi n}(t),t=0}\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}$$

is obtained by the condition $\varphi_n(0) = 0$. Taking into account that $K_n(x,0,0) = 1$ by the condition (1), we have

$$\sum_{\nu=0}^{\infty}\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{u=\alpha_{n\psi n}(t),t=0}\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!} = 1. \tag{2}$$

The family of operators $M_n(f;x)$ is linear and positive. Also, the operators $M_n(f;x)$ reduce to following well-known operators in a special case:

(i) if we choose $K_n(x,t,u) = [1+t+ux]^{-n}$, $\alpha_n = n$, $\psi_n(0) = 1/n$, the operators (1) reduce to Baskakov-Durrmeyer operators,

(ii) if we choose $K_n(x,t,u) = e^{-n(t+ux)}$, $\alpha_n = n$, $\psi_n(0) = 1/n$, the operators (1) reduce to Szasz-Durrmeyer operators,

(iii) If $K_n(z)$ is entire analytic function and $K_n(x,t,u) = K_n(t+ux)$, $\alpha_n = n$, $\psi_n(0) = 1/n$, the operators (1) reduce to generalized Baskakov-Durrmeyer operators.

2 Auxiliary Results

In this section we give some lemmas which will be required to prove the main results. The proofs of Lemma 1-Lemma 4 were given in [12], but for the readers' convenience we recall them here again.

Lemma 1. *The condition* (5) *is equivalent to the following equality*

$$\frac{d}{dx}K_n^{(\nu)}(x,0,u_1) = \frac{\nu}{x}K_n^{(\nu)}(x,0,u_1) - nu_1K_{n+m}^{(\nu)}(x,0,u_1).$$

Proof. By ν-multiple application of condition (3), we obtain

$$K_n^{(\nu)}(x,0,u_1) = (-1)^{\nu} n(n+m)\dots(n+(\nu-1)m)x^{\nu}K_{n+\nu m}(x,0,u_1). \tag{3}$$

Applying condition (5) we get

$$(-1)^{\nu}\frac{d}{dx}K_n^{(\nu)}(x,0,u_1) = n(n+m)\dots(n+(\nu-1)m)$$
$$\times\left\{\nu x^{\nu-1}K_{n+\nu m}(x,0,u_1) - x^{\nu}(n+\nu m)u_1K_{n+(\nu+1)m}(x,0,u_1)\right\}.$$

Using (3) we get desired result. □

Conclusion 1. *Using (3) and Lemma 1, we get*

$$\frac{d}{dx}K_n^{(\nu)}(x,0,u_1) = (-n)\left[\nu K_{n+m}^{(\nu-1)}(x,0,u_1) + u_1K_{n+m}^{(\nu)}(x,0,u_1)\right].$$

Lemma 2. *We have*

$$\int_0^{\infty} K_n^{(\nu)}(x,0,u_1)\,dx = (-1)^{\nu}\frac{\nu!}{(n-m)u_1^{\nu+1}}.$$

Proof. Using integration by parts and conditions (1) and (4) we have

$$\int_0^{\infty} K_n^{(\nu)}(x,0,u_1)\,dx = -\int_0^{\infty} x\frac{d}{dx}K_n^{(\nu)}(x,0,u_1)\,dx.$$

Using Lemma 1, we get

$$\int_0^{\infty} K_n^{(\nu)}(x,0,u_1)\,dx = -\nu\int_0^{\infty} K_n^{(\nu)}(x,0,u_1)\,dx + nu_1\int_0^{\infty} xK_{n+m}^{(\nu)}(x,0,u_1)\,dx.$$

Also by condition (3), we have

$$\int_0^{\infty} K_n^{(\nu)}(x,0,u_1)\,dx = -\nu\int_0^{\infty} K_n^{(\nu)}(x,0,u_1)\,dx - u_1\int_0^{\infty} K_n^{(\nu+1)}(x,0,u_1)\,dx.$$

Hence we can write

$$\int_0^{\infty} K_n^{(\nu)}(x,0,u_1)\,dx = \frac{-u_1}{\nu+1}\int_0^{\infty} K_n^{(\nu+1)}(x,0,u_1)\,dx.$$

By ν- times application of above equality and using condition (5) and (1), we get

$$\begin{aligned}\int_0^\infty K_n^{(\nu)}(x,0,u_1)\,dx &= -\frac{\nu}{u_1}\int_0^\infty K_n^{(\nu-1)}(x,0,u_1)\,dx\\ &\vdots\\ &= (-1)^\nu\frac{\nu!}{u_1^\nu}\int_0^\infty K_n(x,0,u_1)\,dx\\ &= \frac{(-1)^{\nu+1}\nu!}{(n-m)\,u_1^{\nu+1}}\int_0^\infty \frac{d}{dx}K_{n-m}(x,0,u_1)\,dx\\ &= (-1)^\nu\frac{\nu!}{(n-m)\,u_1^{\nu+1}}. \end{aligned} \tag{4}$$

□

Conclusion 2. *Using condition* (6) *and Lemma 1, we get*

$$x(1+u_1mx)\frac{d}{dx}K_n^{(\nu)}(x,0,u_1) = (\nu - xu_1n)\,K_n^{(\nu)}(x,0,u_1). \tag{5}$$

Lemma 3. *Let ν, $n\in\mathbb{N}$. For any natural number r we have*

$$\int_0^\infty x^rK_n^{(\nu)}(x,0,u_1)\,dx = \frac{(-1)^\nu(\nu+r)!}{(n-m)(n-2m)\dots(n-pm)(n-(r+1)m)\,u_1^{\nu+r+1}}. \tag{6}$$

Proof. Using the condition (3) recursively ν- times we get

$$\begin{aligned}\int_0^\infty x^rK_n^{(\nu)}(x,0,u_1)\,dx &= -\frac{1}{n-m}\int_0^\infty x^{r-1}K_{n-m}^{(\nu+1)}(x,0,u_1)\,dx.\\ &= \frac{1}{(n-m)(n-2m)}\int_0^\infty x^{r-2}K_{n-2m}^{(\nu+2)}(x,0,u_1)\,dx\\ &\vdots\\ &= \frac{(-1)^r}{(n-m)(n-2m)\dots(n-rm)}\int_0^\infty K_{n-rm}^{(\nu+r)}(x,0,u_1)\,dx.\end{aligned}$$

Using (4) it follows

$$\int_0^\infty x^rK_n^{(\nu)}(x,0,u_1)\,dx = \frac{(-1)^\nu(\nu+r)!}{(n-m)(n-2m)\dots(n-rm)(n-(r+1)m)\,u_1^{\nu+r+1}}.$$

□

Lemma 4. *Let ν, $n\in\mathbb{N}$. For any natural r we have*

$$\begin{aligned}M_n\left(t^r;x\right) &= \frac{n^{2r}}{(n-2m)\dots(n-pm)(n-(r+1)m)(\alpha_n)^r\left(n^2\psi_n(0)\right)^r}\\ &\quad\times\sum_{j=0}^{r} n(n+m)\dots(n+(j-1)m)\,C_{j,r}\,[\alpha_n\psi_n(0)]^j\,x^j,\end{aligned}$$

where $C_{j,r}=\frac{r!}{j!}\binom{r}{j}$. Also,

$$\begin{aligned}M_n(1;x) &= 1,\;\; M_n(t;x)=\frac{n^2}{(n-2m)\alpha_n}\left(\frac{\alpha_n}{n}x+\frac{1}{n^2\psi_n(0)}\right)\\ M_n\left(t^2;x\right) &= \frac{n^4}{(n-2m)(n-3m)\alpha_n^2}\left(\left(\frac{\alpha_n}{n}x\right)^2\frac{(m+n)}{n}+\frac{\alpha_n}{n}\frac{4}{n^2\psi_n(0)}x+\frac{2}{(n^2\psi_n(0))^2}\right)\end{aligned}$$

Proof. Directly from the definition of operator (1) we can write

$$\begin{aligned} M_n\left(t^r;x\right) &= (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!} \\ &\times\int_0^\infty t^r K_n^{(\nu)}(t,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dt. \end{aligned}$$

Using (6) with $u_1=\alpha_n\psi_n(0)$ we conclude that

$$\begin{aligned} M_n\left(t^r;x\right) &= (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!} \\ &\times\frac{(-1)^\nu(\nu+r)!}{(n-m)(n-2m)\dots(n-rm)(n-(r+1)m)(\alpha_n\psi_n(0))^{\nu+r+1}}\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}. \\ &= \sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!} \\ &\times\frac{1}{(n-2m)\dots(n-rm)(n-(r+1)m)(\alpha_n\psi_n(0))^r}(\nu+r)\dots(\nu+1). \end{aligned}$$

Using the equality

$$(\nu+r)\dots(\nu+1)=\sum_{j=1}^{r}C_{j,r}\prod_{l=0}^{j-1}(\nu-l),$$

where $C_{j,r}=\frac{r!}{j!}\binom{r}{j}$ and (4) we have

$$\begin{aligned} M_n\left(t^r;x\right) &= \sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!} \\ &\times\frac{1}{(n-2m)\dots(n-rm)(n-(r+1)m)(\alpha_n\psi_n(0))^r}\sum_{j=1}^{r}C_{j,r}\prod_{l=0}^{j-1}(\nu-l) \\ &= \frac{1}{(n-2m)\dots(n-rm)(n-(r+1)m)(\alpha_n\psi_n(0))^r} \\ &\times\sum_{j=0}^{r}C_{j,r}\sum_{\nu=0}^{\infty}\prod_{l=0}^{j-1}(\nu-l)K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!} \\ &= \frac{1}{(n-2m)\dots(n-rm)(n-(r+1)m)(\alpha_n\psi_n(0))^r} \\ &\times\sum_{j=0}^{r}C_{j,r}\sum_{\nu=j}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu-j)!} \\ &= \frac{1}{(n-2m)\dots(n-rm)(n-(r+1)m)u_1^{r+1}} \\ &\times\sum_{j=1}^{r}C_{j,r}x^j\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-1)^j[-\alpha_n\psi_n(0)]^{\nu+j}}{(\nu)!} \\ &= \frac{n^{2r}}{(n-2m)\dots(n-rm)(n-(r+1)m)(\alpha_n)^r\left(n^2\psi_n(0)\right)^r} \\ &\times\sum_{j=1}^{r}n(n+m)\dots(n+(j-1)m)\,C_{j,r}[\alpha_n\psi_n(0)]^j x^j. \end{aligned}$$

□

Lemma 5. *Let $r \in \mathbb{N}_0$, $n \in \mathbb{N}$, $n > (2r+1)m$ and $T_{n,r}(x) = [M_n(x-\cdot)^r](x)$, $x \in \mathbb{R}^+$. We have the recursion formula:*

$$T_{n,0}(x) = 1 \quad , \quad T_{n,1}(x) = -\frac{(1+2xu_1m)}{u_1(n-2m)}, \tag{7}$$

$$[u_1n - u_1m(r+2)]T_{n,r+1}(x) = x(1+u_1mx)[-T'_{n,r}(x) + 2rT_{n,r-1}(x)] - (1+2xu_1m)\,(r+1)\,T_{n,r}(x), \;\; r \in \mathbb{N}.$$

We also have

$$T_{n,2r}(x) = \sum_{i=0}^{r} q_{i,2r}\left[\frac{\varphi^2(x)}{u_1}\right]^{r-i} \frac{1}{(n-2m)\ldots(n-(2r+1)m)} u_1^{-2i}, \tag{8}$$

where the real numbers $q_{i,2r}$ are independent of x and bounded uniformly in n and $\varphi(x) := \sqrt{x\,(1+xm\alpha_n\psi_n(0))}$.

Proof. We can easily calculate $T_{n,0}(x)$ and $T_{n,1}(x)$. Using the equality

$$\begin{aligned} T'_{n,r}(x) &= (n-m)\alpha_n\psi_n(0)\sum_{\upsilon=0}^{\infty}\frac{d}{dx}K_n^{(\upsilon)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!}\int_0^{\infty}(x-y)^rK_n^{(\upsilon)}(y,0,u_1)\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!}dy \\ &\quad + rT_{n,r-1}(x), \end{aligned}$$

and (5) we have

$$\begin{aligned} &\varphi^2(x)[T'_{n,r}(x) - rT_{n,r-1}(x)] \\ = \;& (n-m)\alpha_n\psi_n(0)\sum_{\upsilon=0}^{\infty}(\upsilon - xu_1n)K_n^{(\upsilon)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!}\int_0^{\infty}(x-y)^rK_n^{(\upsilon)}(y,0,u_1)\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!}dy \\ = \;& (n-m)\alpha_n\psi_n(0)\sum_{\upsilon=0}^{\infty}K^{(\upsilon)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!} \\ & \times\int_0^{\infty}(\upsilon - yu_1n)K_n^{(\upsilon)}(y,0,u_1)(x-y)^r\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!}dy - nu_1T_{n,r+1}(x) \\ = \;& (n-m)\alpha_n\psi_n(0)\sum_{\upsilon=0}^{\infty}K^{(\upsilon)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!} \\ & \times\int_0^{\infty}\varphi^2(y)\frac{d}{dy}K_n^{(\upsilon)}(y,0,u_1)(x-y)^r\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!}dy - nu_1T_{n,r+1}(x). \end{aligned}$$

Using integration by parts we deduce

$$\begin{aligned} &\varphi^2(x)[T'_{n,r}(x) - rT_{n,r-1}(x)] \\ = \;& (n-m)\alpha_n\psi_n(0)\sum_{\upsilon=0}^{\infty}K^{(\upsilon)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!} \\ & \times\int_0^{\infty}K_n^{(\upsilon)}(y,0,u_1)\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!}\left[-(1+2u_1my)(x-y)^r + r\varphi^2(y)(x-y)^{r-1}\right]dy \\ & -nu_1T_{n,r+1}(x). \end{aligned} \tag{9}$$

Since

$$r\varphi^2(y) - (1+2u_1my)(x-y) = r\varphi^2(x) - (r+1)(1+2u_1mx)(x-y) + u_1m(r+2)(x-y)^2,$$

we get from (9)

$$\begin{aligned} &\varphi^2(x)[T'_{n,r}(x) - rT_{n,r-1}(x)] \\ = \;& r\varphi^2(x)T_{n,r-1}(x) - (r+1)(1+2u_1mx)T_{n,r}(x) + u_1m(r+2)T_{n,r+1}(x) - nu_1T_{n,r+1}(x), \end{aligned}$$

which is formula (7). Using the recurrence above it is easy to obtain

$$T_{n,2}(x) = \varphi^2(x)\frac{(6m+2n)}{u_1(n-2m)(n-3m)} + \frac{2}{u_1^2(n-2m)(n-3m)},$$

$$\begin{aligned} T_{n,4}(x) &= \varphi^4(x)\left[\frac{252u_1^2mn+120u_1^2m^2+12u_1^2n^2}{u_1^4(n-2m)(n-3m)(n-4m)(n-5m)}\right] \\ &+\varphi^2(x)\left[\frac{72u_1n+120u_1m}{u_1^4(n-2m)(n-3m)(n-4m)(n-5m)}\right] + \frac{24}{u_1^4(n-2m)(n-3m)(n-4m)(n-5m)} \end{aligned}$$

and, recursively, we get the following general form

$$T_{n,2r}(x) = \sum_{i=0}^{r} q_{i,2r}\left[\frac{\varphi^2(x)}{u_1}\right]^{r-i}\frac{1}{(n-2m)\dots(n-(2r+1)m)}u_1^{-2i}.$$

□

Corollary 1. *For all $r \in \mathbb{N}_0$ and $x \in \mathbb{R}^+$ we have*

$$\left|T_{n,2r}(x)\right| \le C\left[(n-(2r+1)m)u_1\right]^{-r}\left(\varphi^2(x) + ((n-(2r+1)m)u_1)^{-1}\right)^r,$$

where C is a constant independent of n.

Proof. For $x \in \left[0, \frac{1}{u_1(n-(2r+1)m)}\right]$ we have $\varphi^2(x) \le \frac{(n-(2r+1)m)+m}{(n-(2r+1)m)^2u_1}$, thus by (8) we get

$$\begin{aligned} \left|T_{n,2r}(x)\right| &\le \sum_{i=0}^{r}\left|q_{i,2r}\right|\left[\frac{(n-(2r+1)m)+m}{(n-(2r+1)m)^2u_1^2}\right]^{r-i}\frac{1}{(n-2m)\dots(n-(2r+1)m)}u_1^{-2i} \\ &\le C(u_1(n-(2r+1)m))^{-2r}. \end{aligned}$$

For $x \in \left[\frac{1}{u_1(n-(2r+1)m)}, \infty\right)$ we have $\left[(n-(2r+1)m)u_1\varphi(x)^2\right]^{-1} \le \frac{n-rm}{n-rm+m} \le 1$, thus by (8) we get

$$\begin{aligned} \left|T_{n,2r}(x)\right| &\le ((n-(2r+1)m)u_1)^{-r}\varphi^{2r}(x)\sum_{i=0}^{r}\left|q_{i,2r}\right|\left[(n-(2r+1)m)u_1\varphi(x)^2\right]^{-i} \\ &\le C((n-(2r+1)m)u_1)^{-r}\varphi^{2r}(x). \end{aligned}$$

□

Let $n, r \in \mathbb{N}$, $n > m$ and $s \in \mathbb{R}^+$ we consider the notation

$$\begin{aligned} H_{n,r}(s) &= r(n-m)u_1\left\{\int_s^{\infty}\int_0^{s} - \int_0^{s}\int_s^{\infty}\right\} \\ &\times(s-y)^{r-1}\sum_{\nu=0}^{\infty}K_n^{(\nu)}(y,0,u_1)\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}K_n^{(\nu)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}dydx \end{aligned} \tag{10}$$

and we shall obtain a recurrence relation for $H_{n,r}(s)$ in the following Lemma.

Lemma 6. *For $n, r \in \mathbb{N}$, $n > 2rm$ we have*

$$\begin{aligned} H_{n,r}(s) &= s^rK_{n-m}(s,0,u_1) - n\frac{u_1^2}{\nu+1}\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!} \\ &\times\sum_{k=0}^{\infty}K_{n-m}^{(\nu+1)}(s,0,u_1)\int_0^{\infty}K_{n+m}^{(\nu)}(y,0,u_1)(s-y)^rdy, \end{aligned} \tag{11}$$

$$H_{n,0}(s)=1,\qquad H_{n,1}(s)=0\ ,\quad H_{n,2}(s)=\frac{2\varphi^2(s)}{u_1(n-2m)},$$

$$[u_1 n-u_1 m(r+1)]H_{n,r+1}(s)=\varphi^2(s)[-H^{'}_{n,r}(s)+2rH_{n,r-1}(s)]-r(1+2su_1 m)H_{n,r}(s),\ \ r\geq 1, \tag{12}$$

and

$$H_{n,2r}(s)=\sum_{i=0}^{r-1} q_{i,2r}\left[\frac{\varphi^2(s)}{u_1}\right]^{r-i}\frac{1}{(n-2m)\ldots(n-2rm)}u_1^{-2i},\ \ r>0, \tag{13}$$

where the real numbers $q_{i,2r}$ are independent of x and bounded uniformly in n.

Proof. We first prove (11).

$$\begin{aligned}
H_{n,r}(s) &= u_1 r(n-m)\left\{\int_0^\infty\int_0^s-\int_0^s\int_0^s-\int_0^s\int_0^\infty+\int_0^s\int_0^s\right\}(s-y)^{r-1}\\
&\quad\times\sum_{\nu=0}^{\infty}K_n^{(\nu)}(y,0,u_1)\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}K_n^{(\nu)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dydx\\
&= u_1 r(n-m)\int_0^s(s-y)^{r-1}\sum_{\nu=0}^{\infty}K_n^{(\nu)}(y,0,u_1)\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\int_0^\infty K_n^{(\nu)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dxdy\\
&\quad -u_1 r(n-m)\int_0^s\int_0^\infty(s-y)^{r-1}\sum_{\nu=0}^{\infty}K_n^{(\nu)}(y,0,u_1)\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}K_n^{(\nu)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dydx
\end{aligned}$$

Using (2) and Lemma 2 we have

$$H_{n,r}(s)=s^r-u_1 r(n-m)\sum_{\nu=0}^{\infty}\int_0^s K_n^{(\nu)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\int_0^\infty(s-y)^{r-1}K_n^{(\nu)}(y,0,u_1)\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dydx. \tag{14}$$

Since $n>mr$, using integration by parts we deduce

$$r\int_0^\infty(s-y)^{r-1}K_n^{(\nu)}(y,0,u_1)dy=\int_0^\infty(s-y)^r\frac{d}{dy}(K_n^{(\nu)}(y,0,u_1))dy+\begin{cases}s^r & \nu=0,t=0\\ 0 & \nu\neq 0\end{cases}. \tag{15}$$

Using Conclusion 1 and the fact $K_{n-m}(0,0,u)=1$,we get

$$-(n-m)u_1\int_0^s K_n^{(0)}(x,0,u_1)s^r dx=s^r\int_0^s\frac{d}{dx}K_{n-m}^{(0)}(x,0,u_1)dx. \tag{16}$$

If we replace (15) and (16) in (14), we obtain

$$\begin{aligned}
H_{n,r}(s) &= s^r K_{n-m}(s,0,u_1) + u_1(n-m)\sum_{k=0}^{\infty}\int_0^s K_n^{(\upsilon)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!} \\
&\quad \times \int_0^{\infty}(s-y)^r n[\upsilon K_{n+m}^{(\upsilon-1)}(y,0,u_1) + u_1 K_{n+m}^{(\upsilon)}(y,0,u_1)]\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!}dydx \\
&= s^r K_{n-m}(s,0,u_1) + u_1 n(n-m) \\
&\quad \times \left\{\sum_{k=0}^{\infty}\int_0^{\infty}(\upsilon+1)K_{n+m}^{(\upsilon)}(y,0,u_1)(s-y)^r\frac{[-\alpha_n\psi_n(0)]^{\upsilon+1}}{(\upsilon+1)!}\int_0^s K_n^{(\upsilon+1)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^{\upsilon+1}}{(\upsilon+1)!}dxdy\right. \\
&\quad \left. + \sum_{k=0}^{\infty}\int_0^{\infty}u_1 K_{n+m}^{(\upsilon)}(y,0,u_1)(s-y)^r\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!}\int_0^s K_n^{(\upsilon)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!}dxdy\right\} \\
&= s^r K_{n-m}(s,0,u_1) + u_1 n\sum_{k=0}^{\infty}\int_0^{\infty}K_{n+m}^{(\upsilon)}(y,0,u_1)(s-y)^r\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!}\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!} \\
&\quad \times \frac{u_1}{\upsilon+1}\int_0^s (n-m)\left[u_1 K_n^{(\upsilon+1)}(x,0,u_1) + (\upsilon+1)K_n^{(\upsilon)}(x,0,u_1)\right]dxdy.
\end{aligned}$$

Using Conclusion 1 we get

$$\begin{aligned}
H_{n,r}(s) &= s^r K_{n-m}(s,0,u_1) - u_1 n\sum_{k=0}^{\infty}\int_0^{\infty}K_{n+m}^{(\upsilon)}(y,0,u_1)(s-y)^r\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!}\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!} \\
&\quad \times \frac{u_1}{\upsilon+1}\int_0^s \frac{d}{dx}K_{n-m}^{(\upsilon+1)}(x,0,u_1)dxdy \\
&= s^r K_{n-m}(s,0,u_1) - n\frac{u_1^2}{\upsilon+1}\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!}\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!} \\
&\quad \times \sum_{k=0}^{\infty}K_{n-m}^{(\upsilon+1)}(s,0,u_1)\int_0^{\infty}K_{n+m}^{(\upsilon)}(y,0,u_1)(s-y)^r dy.
\end{aligned}$$

We now prove (12). We can easily calculate $H_{n,1}(s)$ and $H_{n,2}(s)$. Then we get from (11) that

$$\begin{aligned}
H'_{n,r}(s) &= s^r K'_{n-m}(s,0,u_1) + rH_{n,r-1}(s) - n\frac{u_1^2}{\upsilon+1}\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!}\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!} \\
&\quad \times \sum_{k=0}^{\infty}\frac{d}{ds}K_{n-m}^{(\upsilon+1)}(s,0,u_1)\int_0^{\infty}K_{n+m}^{(\upsilon)}(y,0,u_1)(s-y)^r dy.
\end{aligned}$$

Thus by using (5) we get

$$\begin{aligned}
\varphi^2(s)[H'_{n,r}(s) - rH_{n,r-1}(s)] &= \varphi^2(s)s^r K'_{n-m}(s,0,u_1) - n\frac{u_1^2}{\upsilon+1}\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!}\frac{[-\alpha_n\psi_n(0)]^{\upsilon}}{(\upsilon)!} \\
&\quad \times \sum_{k=0}^{\infty}K_{n-m}^{(\upsilon+1)}(s,0,u_1)\int_0^{\infty}(\upsilon+1-su_1(n-m))K_{n+m}^{(\upsilon)}(y,0,u_1)(s-y)^r dy.
\end{aligned}$$

Since

$$(\upsilon+1-(n-m)su_1) = (\upsilon-(n+m)yu_1) - (n+m)u_1(s-y) + (1+2msu_1)$$

using (5) we can also write

$$\begin{aligned}
&\varphi^2(s)[H'_{n,r}(s) - rH_{n,r-1}(s)] + (n+m)u_1H_{n,r+1}(s) - (1+2msu_1)H_{n,r}(s)\\
=\ &\varphi^2(s)s^rK'_{n-m}(s,0,u_1) - n\frac{u_1^2}{\upsilon+1}\frac{[-\alpha_n\psi_n(0)]^\upsilon}{(\upsilon)!}\frac{[-\alpha_n\psi_n(0)]^\upsilon}{(\upsilon)!}\\
&\times\sum_{k=0}^{\infty}K_{n-m}^{(\upsilon+1)}(s,0,u_1)\int_0^\infty[(\upsilon-(n+m)yu_1)]K_{n+m}^{(\upsilon)}(y,0,u_1)(s-y)^r\,dy\\
&+[u_1(n+m)s^{r+1} - (1+2msu_1)s^r]K_{n-m}(s,0,u_1).
\end{aligned}$$

First using integration by parts and then using the identity, we get

$$r\varphi^2(y) - (1+2u_1my)(s-y) = r\varphi^2(s) - (r+1)(1+2u_1ms)(s-y) + u_1m(r+2)(s-y)^2.$$

$$\begin{aligned}
&\varphi^2(s)[H'_{n,r}(s) - rH_{n,r-1}(s)] + (n+m)u_1H_{n,r+1}(s) - (1+2msu_1)H_{n,r}(s)\\
=\ &\varphi^2(s)s^rK'_{n-m}(s,0,u_1) - n\frac{u_1^2}{\upsilon+1}\frac{[-\alpha_n\psi_n(0)]^\upsilon}{(\upsilon)!}\frac{[-\alpha_n\psi_n(0)]^\upsilon}{(\upsilon)!}\\
&\times\sum_{k=0}^{\infty}K_{n-m}^{(\upsilon+1)}(s,0,u_1)\int_0^\infty[r\varphi^2(y) - (1+2msu_1)(s-y)]K_{n+m}^{(\upsilon)}(y,0,u_1)(s-y)^{r-1}\,dy\\
&+[u_1(n+m)s^{r+1} - (1+2msu_1)s^r]K_{n-m}(s,0,u_1)
\end{aligned}$$

$$\begin{aligned}
=\ &u_1m(r+2)H_{n,r+1}(s) - (r+1)(1+2u_1ms)H_{n,r}(s) + r\varphi^2(s)H_{n,r-1}(s)\\
&+\varphi^2(s)s^rK'_{n-m}(s,0,u_1) + u_1(n-m)s^{r+1}K_{n-m}(s,0,u_1).
\end{aligned}$$

According to (5) we have

$$\begin{aligned}
&\varphi^2(s)s^rK'_{n-m}(s,0,u_1) + u_1(n-m)s^{r+1}K_{n-m}(s,0,u_1)\\
=\ &s^r(0 - u_1(n-m)s)K_{n-m}(s,0,u_1) + u_1(n-m)s^{r+1}K_{n-m}(s,0,u_1) = 0.
\end{aligned}$$

The representations (13) can be easily derived from the recursion formula by induction. □

From the above mentioned lemma we get the following corollary which can be proved in the same way as we did in Corollary 1.

Corollary 2. *For all $n, r \in \mathbb{N}$, $n > 2(r+1)m$, $s \in \mathbb{R}^+$, we have*

$$\left|H_{n,2r}(s)\right| \le C\left[(n-(2r+1)m)u_1\right]^{-r}\left(\varphi^2(s) + ((n-(2r+1)m)u_1)^{-1}\right)^r,$$

where C denotes a constant independent of n and s.

In addition to the conditions (1)-(6), throughout the rest of paper we also assume the following condition for $K_n(x,t,u)$.

(7)

$$(1+u_1mx)^{-r}K_n^{(v)}(x,0,u_1) = K_{n+rm}^{(v)}(x,0,u_1)\alpha_{n,r},$$

where $(\alpha_{n,r})$ is a sequence of n which convergence a positive real number.

Lemma 7. *Let the functions $K_n(x,t,u)$ satisfy the condition (1)-(7) and $f(t) = (1+u_1mt)^{-r}$, $t \in \mathbb{R}^+$, $n, r \in \mathbb{N}$.*

$$M_n\left((1+u_1mt)^{-r}; x\right) \le C(1+u_1mx)^{-r}, \quad x \in \mathbb{R}^+$$

holds for $n > mr$, where C denotes a constant independent of n.

Proof. For $m = 0$ it is trivial. For $m > 0$ we obtain

$$M_n(f;x) = (n-m)u_1 \sum_{v=0}^{\infty} K_n^{(v)}(x,0,u_1)\frac{[-u_1]^v}{v!}\alpha_{n,r}\int_0^{\infty} K_{n+rm}^{(v)}(y,0,u_1)\frac{[-u_1]^v}{v!}dy$$

$$= (n-m)\sum_{v=0}^{\infty} K_n^{(v)}(x,0,u_1)\frac{[-u_1]^v}{v!}\alpha_{n,r}\frac{1}{(n-m+rm)}.$$

Thus we get

$$\begin{aligned} M_n(f;x) &= \frac{(n-m)}{(n-m+rm)}\sum_{v=0}^{\infty}(1+u_1mx)^{-r}K_{n-rm}^{(v)}(x,0,u_1)\frac{[-u_1]^v}{v!}\frac{\alpha_{n,r}}{\alpha_{n-rm,r}} \\ &\leq C(1+u_1mx)^{-r}\sum_{v=0}^{\infty}K_{n-rm}^{(v)}(x,0,u_1)\frac{[-u_1]^v}{v!} \\ &= C(1+u_1mx)^{-r}. \end{aligned}$$

□

3 Main Results

For the consideration of the connections between the smoothness of a function and the rate of approximation we use the Ditzian Totik modulus of smoothness(see [13]) which in our case is given by

$$w_{\varphi}^r(f,t)_p = \sup_{0<h\leq r}\left\|\Delta_h^r\varphi f\right\|_p, \;\; f\in\mathbb{R}^+, \;\; 1\leq p\leq\infty, \;\; \varphi(x) := \sqrt{x\left(1+xm\alpha_n\psi_n(0)\right)},$$

where $\Delta_H^r f(x) = \sum_{k=0}^{r}\binom{r}{k}(-1)^k f\left(x+\left(\frac{r}{2}-k\right)H\right)$whenever $\left[x-\frac{r}{2}H, x+\frac{r}{2}H\right]\subset\mathbb{R}^+$ and $\Delta_H^r f(x) = 0$ otherwise. Ditzian and Totik proved (see [[13], Chapter 1, 2, 3]) the equivalence of this modulus to the following K-functional

$$\overline{K}_{\varphi}^r(f,t^r)_p = \inf_{g\in\overline{W}_p^r(\varphi,[0,\infty))}\left\{\|f-g\|_p + t^r\left\|\varphi^r g^{(r)}\right\|_p + t^{2r}\left\|g^{(r)}\right\|_p\right\},$$

where

$$\overline{W}_p^r(\varphi,[0,\infty)) = \left\{g\in L_p\left(\mathbb{R}^+\right) : g^{(r-1)}\in AC_{loc}(0,\infty); \varphi^r g^{(r)}\in L_p\left(\mathbb{R}^+\right)\right\}$$

denote the corresponding weighted Sobolev spaces.

In order to present a quantitative type theorem giving the rate of convergence we consider the following Ditzian-Totik modulus of continuity and the corresponding K-functional. (More details see [13]).

Lemma 8. *Let* $n\in\mathbb{N}, n>m, f\in L_p\left(\mathbb{R}^+\right), 1\leq p<\infty$. *The inequality*

$$\|M_n f\|_p \leq \|f\|_p \tag{17}$$

holds.

Proof. By Riesz Thorin Theorem (see [[14], Theorem 1.1.1]), it is enough to give the proof for $p = 1$ and $p = \infty$. In the case of $p = 1$ and by using Lemma 2 and (2) we have

$$\begin{aligned}\|M_n f\|_1 &\le (n-m)\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}\int_0^{\infty}\frac{\partial^\nu}{\partial u^\nu}K_n(x,0,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\frac{\left[-\alpha_n\psi_n(0)\right]^\nu}{\nu!}dx\\ &\quad\times\int_0^{\infty}|f(y)|\,\frac{\partial^\nu}{\partial u^\nu}K_n(y,0,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\frac{\left[-\alpha_n\psi_n(0)\right]^\nu}{\nu!}dy\\ &\le\int_0^{\infty}|f(y)|\sum_{\nu=0}^{\infty}\frac{\partial^\nu}{\partial u^\nu}K_n(y,0,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\frac{\left[-\alpha_n\psi_n(0)\right]^\nu}{\nu!}dy = \|f\|_1 .\end{aligned}$$

For $p = \infty$, we have

$$\begin{aligned}\|M_n f\|_\infty &\le \|f\|_\infty\left\|(n-m)\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}\frac{\partial^\nu}{\partial u^\nu}K_n(x,0,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\frac{\left[-\alpha_n\psi_n(0)\right]^\nu}{\nu!}\right.\\ &\quad\left.\times\int_0^{\infty}\frac{\partial^\nu}{\partial u^\nu}K_n(y,0,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\frac{\left[-\alpha_n\psi_n(0)\right]^\nu}{\nu!}dy\right\|_\infty = \|f\|_\infty\end{aligned}$$

□

We are ready to give our main result.

Theorem 1. *Let* $n \in \mathbb{N}$, $n > 5m$. *For every* $f \in L_p\left(\mathbb{R}^+\right)$, $1 \le p < \infty$, *we have*

$$\|M_n f - f\|_p \le C\left\{w_\varphi^2(f,((n-3m)u_1)^{-1/2})_p + ((n-3m)u_1))^{-1}\|f\|_p\right\}.$$

Proof. If we use the equivalence of $w_\varphi^2(f,((n-3m)u_1)^{-1/2})_p$ and the modified K-functional $\overline{K}_\varphi^2(f,((n-3m)u_1)^{-1})_p$, it is sufficient to give the proof of inequality.

$$\|M_n f - f\|_p \le C\overline{K}_\varphi^2(f,((n-3m)u_1)^{-1})_p + ((n-3m)u_1))^{-1}\|f\|_p .$$

For all $g \in \overline{W}_p^2(\varphi,\mathbb{R}^+)$, by Lemma 8, we can write

$$\begin{aligned}\left\|M_n(f-g+g)-(f-g+g)\right\|_p &\le \left\|M_n(f-g)+(f-g)+M_n g-g\right\|_p\\ &\le \left\|M_n(f-g)\right\|_p+\left\|(f-g)\right\|_p+\|M_n g-g\|_p\\ &\le \|f-g\|_p+\|f-g\|_p+\|M_n g-g\|_p\\ &\le 2\|f-g\|_p+\|M_n g-g\|_p .\end{aligned}\tag{18}$$

Let us estimate the second term in above inequality. By the Taylor expansion of g,

$$g(t) = g(x) + g'(x)(t-x) + R_2(g,t,x),$$

where

$$R_2(g,t,x) = \int_x^t (t-u)g''(u)du$$

we can write

$$M_n(g;x) - g(x) = M_n\left[(t-x)g'(x)\right] + [M_n(R_2(g,t,x))](x).\tag{19}$$

We shall show the validity of following inequality

$$\left\|M_n(R_2(g,.,x))\right\|_p \le C((n-3m)(u_1))^{-1}\left\|(\varphi^2+((n-3m)u_1))^{-1})g''\right\|_p .\tag{20}$$

To do this, we benefit from Riesz Thorin Theorem, which directs us to give the proof just for $p = 1$ and $p = \infty$. In the case of $p = \infty$ we have

$$\begin{aligned} \left|R_2(g,t,x)\right| &\leq \left\|(\varphi^2 + ((n-3m)(u_1))^{-1})g''\right\|_\infty \\ &\times \left|\int_x^t |t-u| \left[\varphi(u)^2 + ((n-3m)(u_1))^{-1}\right]^{-1} du\right|. \end{aligned} \tag{21}$$

Let $x < t$ we can write

$$\frac{|t-u|}{[\varphi(u)^2 + ((n-3m)(u_1))^{-1}]} \leq \frac{|t-x|}{[\varphi(x)^2 + ((n-3m)u_1))^{-1}]}, \tag{22}$$

If we replace (22) in (21), we get

$$\left|R_2(g,t,x)\right| \leq \left\|(\varphi^2 + ((n-3m)u_1))^{-1})g''\right\|_\infty (t-x)^2((n-3m)u_1). \tag{23}$$

For $x > t$, since $|t-u|\, x \leq |t-x|\, u$ and $\frac{u|t-u|}{\varphi(u)^2+((n-3m)u_1))^{-1}} \leq \frac{x|t-x|}{\varphi(x)^2+((n-3m)u_1))^{-1}}$, we have

$$\frac{|t-u|}{\left[\varphi(u)^2 + ((n-3m)u_1))^{-1}\right]} \leq \frac{|t-x|}{x}\left\{\frac{x}{\varphi(x)^2 + ((n-3m)u_1))^{-1}} + \frac{t}{\varphi(t)^2 + ((n-3m)u_1))^{-1}}\right\}. \tag{24}$$

Using (24) in (21) we get

$$\begin{aligned} \left|R_2(g,t,x)\right| &\leq \left\|(\varphi^2 + ((n-3m)(u_1))^{-1})g''\right\|_\infty \\ &\times \frac{(t-x)^2}{x}\left\{\frac{x}{\varphi(x)^2 + ((n-3m)u_1))^{-1}} + \frac{t}{\varphi(t)^2 + ((n-3m)u_1))^{-1}}\right\}. \end{aligned} \tag{25}$$

On the other hand, the inequality $\frac{|t-u|}{[\varphi(u)^2+((n-3m)u_1))^{-1}]} \leq \frac{|t-u|}{((n-3m)u_1)^{-1}}$ holds.

For $x \in [0, \frac{1}{u_1(n-3m)}]$, we obtain $((n-3m)u_1)[\varphi(x)^2 + ((n-3m)u_1))^{-1}] \leq C_1$ where C_1 independent of n. Thus from (23) we see, by using Corollary 1, that

$$\begin{aligned} \left|M_n(R_2(g,t,x);x)\right| &\leq \left\|(\varphi^2 + ((n-3m)(u_1))^{-1})g''\right\|_\infty (M_n(t-x)^2)(x)((n-3m)u_1) \\ &\leq \left\|(\varphi^2 + ((n-3m)(u_1))^{-1})g''\right\|_\infty C_1[\varphi(x)^2 + ((n-3m)(u_1))^{-1}]^{-1} T_{n,2}(x) \\ &\leq C((n-3m)u_1)^{-1}\left\|(\varphi^2 + ((n-3m)(u_1))^{-1})g''\right\|_\infty. \end{aligned} \tag{26}$$

For $x \in \left[\frac{1}{u_1(n-3m)}, \infty\right)$ from (25) using Corollary 1 and $\frac{t}{\varphi(t)^2+((n-3m)u_1))^{-1}} \leq \frac{1}{1+u_1mt}$, we have

$$\begin{aligned} \left|M_n(R_2(g,t,x);x)\right| &\leq \left\|(\varphi^2 + ((n-3m)u_1))^{-1})g''\right\|_\infty \left\{\frac{1}{[\varphi(x)^2 + ((n-3m)u_1))^{-1}]} T_{n,2}(x)\right. \\ &\quad \left. + \frac{1}{x}\left[M_n\left[(t-x)^2\frac{t}{\varphi(t)^2 + ((n-3m)u_1))^{-1}}\right]\right](x)\right\} \\ &\leq \left\|(\varphi^2 + ((n-3m)u_1))^{-1})g''\right\|_\infty \left\{C((n-3m)u_1)^{-1}\right. \\ &\quad \left. + \frac{1}{x}\left[M_n\left[(t-x)^2\frac{1}{1+u_1mt}\right]\right](x)\right\}. \end{aligned}$$

Using Cauchy Schwarz inequality and Lemma 7 we can write

$$\begin{aligned} \left|M_n(R_2(g,t,x);x)\right| &\leq \left\|(\varphi^2 + ((n-3m)u_1))^{-1})g''\right\|_\infty \left\{C((n-3m)u_1)^{-1}\right. \\ &\quad \left. + \frac{1}{x}\left[\left[M_n(t-x)^4\right](x)\right]^{1/2}\left[\left[M_n(1+u_1mt)^{-2}\right](x)\right]^{1/2}\right\} \end{aligned}$$

$$\begin{aligned}
&\left|M_n(R_2(g,t,x);x)\right| \\
\leq\ & \left\|(\varphi^2+((n-3m)u_1))^{-1})g''\right\|_\infty \left\{C((n-3m)u_1)^{-1}+\frac{1}{x}\left[T_{n,4}(x)\right]^{1/2}C_2(1+u_1mx)^{-1}\right\}.
\end{aligned} \tag{27}$$

For $x\in\left[\frac{1}{u_1(n-3m)},\infty\right)$, we have $(1+u_1mx)^{-1}\leq 2\frac{x}{\varphi(x)^2+((n-3m)(u_1))^{-1}}$, using (27) and Corollary 1 we obtain

$$\begin{aligned}
\left|M_n(R_2(g,t,x));x\right| &\leq \left\|(\varphi^2+((n-3m)u_1))^{-1})g''\right\|_\infty \Big\{C((n-3m)u_1)^{-1} \\
&\quad+\frac{2C_2}{[\varphi(x)^2+((n-3m)u_1))^{-1}]}\left[T_{n,4}(x)\right]^{1/2}\Big\} \\
&\leq C((n-3m)u_1)^{-1}\left\|(\varphi^2+((n-3m)u_1))^{-1})g''\right\|_\infty,
\end{aligned} \tag{28}$$

where C, C_2 are constants and may be different at each occurrence. Hence, (26) and (28) implies (20) in the case of $p=\infty$.

For $p=1$ we derive (20) by applying Fubini's theorem, using Corollary 2

$$\begin{aligned}
\left\|M_n(R_2(g,t,x);x)\right\|_1 &\leq (n-m)\,\alpha_n\psi_n(0)\int_0^\infty\sum_{k=0}^\infty K_n^{(v)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^v}{v!} \\
&\quad\times\int_0^\infty K_n^{(v)}(y,0,u_1)\frac{[-\alpha_n\psi_n(0)]^v}{v!}\left|\int_x^t(t-u)g''(u)du\right|dydx \\
&\leq (n-m)\alpha_n\psi_n(0)\int_0^\infty\sum_{k=0}^\infty K_n^{(v)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^v}{v!} \\
&\quad\times\int_0^x K_n^{(v)}(y,0,u_1)\frac{[-\alpha_n\psi_n(0)]^v}{v!}\int_t^x(u-t)\left|g''(u)\right|dudydx \\
&\quad+(n-m)\alpha_n\psi_n(0)\int_0^\infty\sum_{k=0}^\infty K_n^{(v)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^v}{v!} \\
&\quad\times\int_x^\infty K_n^{(v)}(y,0,u_1)\frac{[-\alpha_n\psi_n(0)]^v}{v!}\left|\int_x^t(t-u)g''(u)\right|dudydx
\end{aligned}$$

$$\begin{aligned}
\left\|M_n(R_2(g,t,x);x)\right\|_1 &= (n-m)\alpha_n\psi_n(0)\int_0^\infty\left|g''(u)\right|\left\{\int_u^\infty\int_0^u-\int_0^u\int_u^\infty\right\}(u-t) \\
&\quad\times\sum_{k=0}^\infty K_n^{(v)}(x,0,u_1)\frac{[-\alpha_n\psi_n(0)]^v}{v!}K_n^{(v)}(y,0,u_1)\frac{[-\alpha_n\psi_n(0)]^v}{v!}dydxdu \\
&= \frac{1}{2}\int_0^\infty\left|g''(u)\right|\left|H_{n,2}(u)\right|du \\
&\leq C\int_0^\infty\left|g''(u)\right|((n-3m)u_1)^{-1}[\varphi^2+((n-3m)u_1))^{-1}]du \\
&= C((n-3m)u_1)^{-1}\left\|(\varphi^2+((n-3m)u_1))^{-1})g''\right\|_1.
\end{aligned}$$

For $1<p<\infty$ we get (20) by means of the Riesz Thorin theorem. And thus we get (18).

On the other hand, for all $g \in \overline{W}_p^2(\varphi, \mathbb{R}^+)$ we get

$$\begin{aligned}\|M_n g - g\|_p &\le C((n-3m)u_1)^{-1}\left\{\|g'\|_p^{[0,1]} + \|(1+2u_1 m\cdot)g'\|_p^{[1,\infty)} + \left\|(\varphi^2 + ((n-3m)u_1)^{-1})g''\right\|_p\right\} \\ &\le C((n-3m)u_1)^{-1}\left\{\|g\|_p + \left\|(\varphi^2 g''\right\|_p + \left\|(\varphi^2 + ((n-3m)u_1)^{-1})g''\right\|_p\right\},\end{aligned}$$

where the last term of above inequality is obtained from (a) and (c) in the proof of Theorem 9.5.3 in [13]. Together with (18) this leads to

$$\begin{aligned}\|M_n f - f\|_p &\le 2\|f-g\|_p + C((n-3m)u_1)^{-1} \\ &\quad \times\left\{\|f-g\|_p + \|f\|_p + \left\|\varphi^2 g''\right\|_p + \left\|(\varphi^2 + ((n-3m)u_1)^{-1}g''\right\|_p\right\} \\ &\le C\left\{\|f-g\|_p + ((n-3m)u_1)^{-1}\|f\|_p\right. \\ &\quad \left. + ((n-3m)u_1)^{-1}\left\|\varphi^2 g''\right\|_p + ((n-3m)u_1)^{-2}\|g''\|_p\right\}.\end{aligned}$$

Taking the infimum over all $g \in \overline{W}_p^2(\varphi, \mathbb{R}^+)$ we can write

$$\|M_n f - f\|_p \le C\left\{\overline{K}_\varphi^2(f, ((n-3m)u_1)_p^{-1} + ((n-3m)u_1)^{-1}\|f\|_p\right\},$$

which completes the proof. □

Theorem 2. *Let* $n \in \mathbb{N}$, $n > 5m$, $f \in L_p\left(\mathbb{R}^+\right)$, $1 \le p < \infty$ *we have*

$$\lim_{n\to\infty}\|M_n f - f\|_p = 0.$$

Proof. According to equivalence $\overline{K}_\varphi^2(f, ((n-3m)u_1)^{-1})_p$ and $w_\varphi^2(f, ((n-3m)u_1)^{-1/2})_p$, the proof is immediately obtained. □

4 Weighted approximation

This section is devoted to obtain weighted approximation properties of the operators $M_n(f;x)$ in weighted $L_{p,2r}(\mathbb{R}^+)$. Let us recall the mentioned space and the corresponding norm. We denote the $L_{p,2r}(\mathbb{R}^+)$, $(1 \le p < \infty)$ by

$$L_{p,2r}(\mathbb{R}^+) = \left\{f : \mathbb{R}^+ \to \mathbb{R}; \|f\|_{p,2r} := \left(\int_0^\infty \frac{|f(t)|^p}{1+t^{2r}}dt\right)^{\frac{1}{p}} < \infty\right\}.$$

No doubtly, we must show that the operators $M_n(f;x)$ are acting from $L_{p,2r}(\mathbb{R}^+)$ to $L_{p,2r}(\mathbb{R}^+)$.

Lemma 9. *Let* $f \in L_{p,2r}(\mathbb{R}^+)$, $r \in \mathbb{N}$, $1 \le p \le \infty$ *we have*

$$\|M_n f\|_{p,2r} \le \|f\|_{p,2r}. \tag{29}$$

Proof. In view of the definition of the operators and using (2) we have

$$\begin{aligned}\int_0^\infty \frac{|M_n(f;x)|}{1+x^{2r}}dx &\le (n-m)\alpha_n\psi_n(0)\sum_{v=0}^\infty \int_0^\infty \frac{\partial^v}{\partial u^v}K_n(x,0,u)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \frac{[-\alpha_n\psi_n(0)]^v}{v!}dx \\ &\times \int_0^\infty \frac{|f(y)|}{1+y^{2r}}\frac{\partial^v}{\partial u^v}K_n(y,0,u)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \frac{[-\alpha_n\psi_n(0)]^v}{v!}dy \\ &+(n-m)\alpha_n\psi_n(0)\sum_{v=0}^\infty \int_0^\infty \frac{\partial^v}{\partial u^v}K_n(x,0,u)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \frac{[-\alpha_n\psi_n(0)]^v}{v!}dx \\ &\times \int_0^\infty \frac{|f(y)|}{1+y^{2r}}y^{2r}\frac{\partial^v}{\partial u^v}K_n(y,0,u)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \frac{[-\alpha_n\psi_n(0)]^v}{v!}dy.\end{aligned}$$

From Lemma 2, we can write

$$\begin{aligned}\int_0^\infty \frac{|M_n(f;x)|}{1+x^{2r}}dx &\le \int_0^\infty \frac{|f(y)|}{1+y^{2r}}\sum_{v=0}^\infty \frac{\partial^v}{\partial u^v}K_n(y,0,u)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \frac{[-\alpha_n\psi_n(0)]^v}{v!}dy \\ &+\int_0^\infty \frac{|f(y)|}{1+y^{2r}}y^{2r}\sum_{v=0}^\infty \frac{\partial^v}{\partial u^v}K_n(y,0,u)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \frac{[-\alpha_n\psi_n(0)]^v}{v!}dy\end{aligned}$$

and using the fact (2) with the condition (4) we get

$$\begin{aligned}\int_0^\infty \frac{|M_n(f;x)|}{1+x^{2r}}dx &\le \int_0^\infty \frac{|f(y)|}{1+y^{2r}}dy \\ &+\sum_{v=0}^\infty \sup_{y\ge 0} y^{2r}\frac{\partial^v}{\partial u^v}K_n(y,0,u)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \frac{[-\alpha_n\psi_n(0)]^v}{v!}\int_0^\infty \frac{|f(y)|}{1+y^{2r}}dy \\ &\le \|f\|_{1,2r}(1+C),\end{aligned}$$

which means that (29) is true for $p=1$, where C is a constant independent of n.

Let us now give the proof for $p=\infty$. We can easily get

$$\begin{aligned}|M_n(f;x)| &= (n-m)\alpha_n\psi_n(0)\sum_{v=0}^\infty \frac{\partial^v}{\partial u^v}K_n(x,0,u)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \frac{[-\alpha_n\psi_n(0)]^v}{v!} \\ &\times \int_0^\infty \frac{|f(y)|}{1+y^{2r}}(1+y^{2r})\frac{\partial^v}{\partial u^v}K_n(y,0,u)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \frac{[-\alpha_n\psi_n(0)]^v}{v!}dy \\ &\le \|f\|_{\infty,2r}\Bigg\{(n-m)\alpha_n\psi_n(0)\sum_{v=0}^\infty \frac{\partial^v}{\partial u^v}K_n(x,0,u)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \frac{[-\alpha_n\psi_n(0)]^v}{v!} \\ &\times \int_0^\infty \frac{\partial^v}{\partial u^v}K_n(y,0,u)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \frac{[-\alpha_n\psi_n(0)]^v}{v!}dy \\ &+(n-m)\alpha_n\psi_n(0)\sum_{v=0}^\infty \frac{\partial^v}{\partial u^v}K_n(x,0,u)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \frac{[-\alpha_n\psi_n(0)]^v}{v!} \\ &\times \int_0^\infty y^{2r}\frac{\partial^v}{\partial u^v}K_n(y,0,u)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \frac{[-\alpha_n\psi_n(0)]^v}{v!}dy\Bigg\}.\end{aligned}$$

Using (1) we can write

$$|M_n(f;x)| \leq \|f\|_{\infty,2r}\left\{M_n(1);x) + M_n\left(t^{2r}\right);x\right\}.$$

Also using Lemma 4

$$\|M_n f\|_{\infty,2r} = \sup_{x\geq 0}\frac{|M_n(f;x)|}{1+x^{2r}} \leq C\|f\|_{\infty,2r},$$

where C is a constant independent of n. Finally by the Riesz-Thorin theorem we get (29). □

The weighted Korovkin type theorem in weighted L_p spaces was presented in [5]. Considering that the theorem we investigate the uniform convergence of the operators in weighted L_p spaces.

Theorem 3. *(see [5]) For a fixed $p \in [1,\infty)$, let w be positive continuous function on the whole real axis satisfying the condition*

$$\int_{\mathbb{R}} t^{2p} w(t)dt < \infty.$$

Let $(L_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of positive linear operators from $L_{p,w}(\mathbb{R})$ into $L_{p,w}(\mathbb{R})$, satisfying the conditions

$$\lim_{n\to\infty}\left\|L_n(t^i;x) - x^i\right\|_{p,w} = 0, \qquad i = 0,1,2.$$

Then for every $f \in L_{p,w}(\mathbb{R})$, we have

$$\lim_{n\to\infty}\|L_n f - f\|_{p,w} = 0.$$

If we choose $w(x) = (1+x^{2r)^{-1}}$, we can give following theorem.

Theorem 4. *Let $f \in L_{p,2r}(\mathbb{R}^+)$, $1 \leq p \leq \infty$ and $r \in \mathbb{N}$, $r - p > 1/2$. Then we have*

$$\lim_{n\to\infty}\|M_n f - f\|_{p,2r} = 0.$$

Proof. According to Theorem 3 it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\|M_n(t^\nu;x) - x^\nu\|_{p,w} = 0, \quad \nu = 0,1,2. \tag{30}$$

Since $M_n(1;x) = 1$, the first condition of (30) is fulfilled for $\nu = 0$. By Lemma 4 we have for $n > 2m$ that

$$\begin{aligned}
\left(\int_0^\infty \frac{|M_n(t;x) - x|^p}{1+x^{2r}}dx\right)^{\frac{1}{p}} &= \left(\int_0^\infty \frac{1}{1+x^{2r}}\left|\frac{n^2}{(n-2m)\alpha_n}\left(\frac{\alpha_n}{n}x + \frac{1}{n^2\varphi_n(0)}\right) - x\right|^p dx\right)^{\frac{1}{p}}\\
&= \left(\int_0^\infty \frac{1}{1+x^{2r}}\left|\left(\frac{n}{(n-2m)} - 1\right)x + \frac{1}{(n-2m)\alpha_n\varphi_n(0)}\right|^p dx\right)^{\frac{1}{p}}\\
&\leq \left(\frac{n}{(n-2m)} - 1\right)\left(\int_0^\infty \frac{x^p}{1+x^{2r}}dx\right)^{\frac{1}{p}} + \frac{1}{(n-2m)\alpha_n\varphi_n(0)}\left(\int_0^\infty \frac{dx}{1+x^{2r}}\right)^{\frac{1}{p}}
\end{aligned}$$

and the second condition of (30) holds for $\nu = 1$ as $n \to \infty$. Similarly we can write for $n > 3m$ that

$$\begin{aligned}
\left(\int_0^\infty \frac{\left|M_n(t^2;x) - x^2\right|^p}{1+x^{2r}}dx\right)^{\frac{1}{p}} &= \left(\frac{n(m+n)}{(n-2m)(n-3m)} - 1\right)\left(\int_0^\infty \frac{x^{2p}}{1+x^{2r}}dx\right)^{\frac{1}{p}}\\
&+ \left(\frac{4n}{(n-2m)(n-3m)\alpha_n\psi_n(0)}\right)\left(\int_0^\infty \frac{x^p}{1+x^{2r}}dx\right)^{\frac{1}{p}}\\
&+ \frac{2}{(n-2m)(n-3m)(\alpha_n\psi_n(0))^2}\left(\int_0^\infty \frac{1}{1+x^{2r}}dx\right)^{\frac{1}{p}}
\end{aligned}$$

and the third condition of (30) holds for $v = 2$ as $n \to \infty$. □

5 Examples

The operators $M_n(f)$ reduce to following well-known operators in special case as shown in the following table:

$M_n(f;x)$	$K_n(x,t,u)$	α_n	$\psi_n(0)$
Baskakov-Durrmeyer	$[1+t+ux]^{-n}$	n	$1/n$
Szasz-Durrmeyer	$e^{-n(t+ux)}$	n	$1/n$
Generalized Baskakov-Durrmeyer	$K_n(t+ux)$	n	$1/n$

Other classical sequences of linear positive operators can be obtained by making an adequate selection of K_n.

a) If $K_n(z)$ is an entire analytic function and $K_n(x,t,u) = K_n(t+ux)$, $\alpha_n = n$, $\psi_n(0) = 1/n$, $m = 1$, the operators (1) reduce to generalized Baskakov-Durrmeyer operators given in [2]

$$G_n(f;x) = (n-c)\sum_{k=0}^{\infty} w_{n,k}(x)\int_0^{\infty} w_{n,k}(t) f(t)\,dt,$$

where $w_{n,k}(x) = (-1)^k \frac{x^k}{k!}\phi_n^k(x)$.
$\phi_n(x) = (1-x)^n$ for the interval $[0,1]$ with $c = 1$
$\phi_n(x) = e^{-nx}$ for the interval $[0,\infty)$ with $c = 0$
$\phi_n(x) = (1+cx)^{-n/c}$ for the interval $[0,\infty)$ with $c > 0$.

b) If we choose $K_n(x,t,u) = [1+t+ux]^{-n}$, $\alpha_n = n$, $\psi_n(0) = 1/n$, $m = 1$, the operators (1) reduce to Baskakov-Durrmeyer operators given in [16]

$$\mathcal{B}_n(f;x) = (n-1)\sum_{k=0}^{\infty} v_{n,k}(x)\int_0^{\infty} v_{n,k}(t) f(t)\,dt,$$

where $v_{n,k}(x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \dfrac{x^k}{(1+x)^{n+k}}$.

c) If we choose $K_n(x,t,u) = e^{-n(t+ux)}$, $\alpha_n = n$, $\psi_n(0) = 1/n$, $m = 0$ the operators (1) reduce to Szasz-Durrmeyer operators given in [15]

$$S_n(f;x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t) f(t)dt,$$

where $p_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$.

Under the assumptations of Theorem 1 we can give following results for Szasz-Durrmeyer and Baskakov-Durrmeyer operators.

Theorem 5. *If $f \in L_p(\mathbb{R}^+)$, $1 \le p < \infty$, we have*

$$\|S_n f - f\|_p \le C\left\{w_\varphi^2(f,(n)^{-1/2})_p + (n)^{-1}\|f\|_p\right\}.$$

If $f \in L_p(\mathbb{R}^+)$, $1 \le p < \infty$, $n > 3$ we have

$$\|B_n f - f\|_p \le C\left\{w_\varphi^2(f,((n-3))^{-1/2})_p + ((n-3)))^{-1}\|f\|_p\right\}.$$

References

[1] Ditzian Z., Ivanov K., Bernstein-type operators and their derivatives, J. Approx. Theory, 1989, 56, 72-90
[2] Heilmann M., Direct and converse results for operators of Baskakov-Durrmeyer type, Approx. Theory Appl., 1989, 5 (1), 105–127
[3] Agrawal P. N., Mohammad A. J., On L_p-approximation by a linear combination of a new sequence of linear positive operators, Turk. J. Math, 2003, 27, 389-405
[4] Dzjadyk V. K., Approximation of functions by positive linear operators and singular integrals, (Russ.) Mat. Sb. (N,S), 1966, 112 (70), 508–517
[5] Gadjiev A. D., Aral A., Weighted L_p -approximation with positive linear operators on unbounded sets, Appl. Math. Lett., 2007, 20 (10), 1046–1051
[6] Gadjiev A. D., Ibragimov I.I., Ibragimov I.I., On a sequence of linear positive operators, 1970, 11, 1092-1095
[7] Aral A., Approximation by Ibragimov-Gadjiyev operators in polynomial weighted space, Proc. of IMM of NAS of Azerbaijan, 2003, XIV, 35–44
[8] Coskun T., On a Construction of positive linear operators for approximation of continuous functions in the weighted spaces, J. Comp. Anal. and Appl., 2011, 13 (4), 756-770
[9] Doğru O., On a certain family of linear positive operators, Turkish J. Math., 1997, 21 (4), 387–399
[10] Dogru, O., On the order of approximation of unbounded functions by the family of generalized linear positive operators, Commun Fac. Sci. Univ. Ankara, Series A1, 1997, 46, 173-181
[11] Gadjiev A. D., Ispir N., On a sequence of linear positive operators in weighted spaces, Proc. of IMM of Azerbaijan AS, 1999, Vol. XI(XIX), 45-56
[12] Aral A., Acar T., Modern mathematical methods and high performance computing in science and technology, 1–15, Springer Proc. Math. Stat., 171, (Springer, Singapore, 2016)
[13] Ditzian, Z., Totik, V., Moduli of Smoothness, Springer Series in Computational Mathematics 9, (Springer-Verlag, Berlin, Heidelberg, Newyork, 1987)
[14] Bergh J., Löfström J., Interpolation Spaces, An Introduction, Springer-Verlag, Berlin, heidelberg, New York, 1976
[15] Mazhar S. M., Totik V., Approximation by modified Szasz operators, Acta Sci. Math., 1985, 49 , 257-269
[16] Sahai A., Prasad G., On simultaneous approximation by modified Lupas operators, J. Approx. Theory, 1985, 45 (12), 122–128