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New Integral Type Operators

Emre Deniz[a], Ali Aral[b], Gulsum Ulusoy[c]

[a]*Kırıkkale University Department of Mathematics 71450 Yahşihan, KIRIKKALE - TURKEY*
[b]*Kırıkkale University Department of Mathematics 71450 Yahşihan, KIRIKKALE - TURKEY*
[c]*Department of Mathematics, Faculty of Science, Çankırı Karatekin University, 18100 ÇANKIRI, TURKEY*

Abstract. In this paper we construct new integral type operators including heritable properties of Baskakov Durrmeyer and Baskakov Kantorovich operators. Results concerning convergence of these operators in weighted space and the hypergeometric form of the operators are shown. Voronovskaya type estimate of the pointwise convergence along with its quantitative version based on the weighted modulus of smoothness are given. Moreover, we give a direct approximation theorem for the operators in suitable weighted Lp space on $[0,\infty)$.

1. Introduction

For $f\in C[0,\infty)$, the Baskakov operators $V_n f$ are defined by

$$V_n(f)(x)=\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)p_{n,k}(x),\ \ x\in[0,\infty),\ n\in\mathbb{N}, \tag{1}$$

where

$$p_{n,k}(x)=\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$$

for all functions $f:[0,\infty)\to\mathbb{R}$ for which the series at the right-hand side is absolutely convergent. The space of such functions includes the space $B_{x^2}[0,\infty)$ the set of all functions f defined on $[0,\infty)$ satisfying the condition $\left|f(x)\right|\le M_f\left(1+x^2\right)$ with the constant M_f, which is depending only on f, and is independent of x. These operators are a generalization of Bernstein operators in the space of all continuous functions on $[0,\infty)$ (see [12]).

These operators are meaningful for continuous function whereas for the functions belonging to Lebesgue space, the Durrmeyer and the Kantorovich modification of V_n are more useful. In [9] its Durrmeyer

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Communicated by Dragan S. Djordjević
Email addresses: emredeniz--@hotmail.com (Emre Deniz), aliaral73@yahoo.com (Ali Aral), gulsumulusoy@karatekin.edu.tr (Gulsum Ulusoy)

modification was defined by replacing the values $f(k/n)$ in the definition of (1) with integral over the weight function, in which the same generating kernel $p_{n,k}$ appears, that is

$$B_n(f)(x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} f(t)\, p_{n,k}(t)\,dt, \quad x \in [0,\infty),\ n \in \mathbb{N} \tag{2}$$

where $f : [0,\infty) \to \mathbb{R}$ is an integrable function for which the integrals and the series above are convergent. On the other hand, Kantorovich version of (1) was defined by Ditzian and Totik [15] by replacing the sample values $f(k/n)$ by the mean values of f in the interval $\left[\frac{k}{n}, \frac{k+1}{n}\right]$, that is

$$K_n(f)(x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt, \quad x \in [0,\infty),\ n \in \mathbb{N}$$

where $f : [0,\infty) \to \mathbb{R}$ is an locally integrable function for which the integrals and the series above are convergent. We note that, V_n and K_n are connected by the relation

$$K_n = D \circ V_{n+1} \circ I, \tag{3}$$

where D is the differentiation operator: $D(f) = f^{'}$, $f \in C^1[0,\infty)$ and I is antiderivative operator: $I(f;x) = \int_0^x f(t)\,dt$, $f \in C[0,\infty)$ and $x \in [0,\infty)$. These operators allow to reconstruct a Lebesgue integrable function by means of its mean values on the sets $\left[\frac{k}{n}, \frac{k+1}{n}\right]$.

Both of these construction were applied to other well-known Bernstein type operators such as Bernstein, Szász Mirakyan and others. A number of results concerning B_n, K_n and other Durrmeyer and Kantorovich type operators can be found in references [9, 13–17], [2] . Note that, the main difference between their Durrmeyer and Kantorovich variants respectively is that they have commutativity. On the other hand, a different approach based on the use of Kantorovich sampling and Durrmeyer sampling operators, which significantly use a signal processing and analysis of seismic engineering (see [3], [4] and [5]). Other general approaches of Kantorovich type operators to study in multivariate frame are given in [1] and [6]. Very recently in [10], the author defined a new operator using the structural properties of Durrmeyer and Kantorovich methods for classical Bernstein operators, called Bernstein Durrmeyer Kantorovich operator. Inspired from [10], and applying Kantorovich method given by (3) to Baskakov Durrmeyer operators, we obtain a new positive linear approximation process that we shall call Baskakov Durrmeyer Kantorovich operators. Namely we will deal with a sequence $\left(\widetilde{B}_n\right)_{n\geq 1}$ of the linear positive operators defined by

$$\widetilde{B}_n(f)(x) = (n-1)\sum_{k=0}^{\infty} p_{n+2,k}(x)\int_0^{\infty} f(u)\, p_{n,k+1}(u)\,du, \tag{4}$$

where $f : [0,\infty) \to \mathbb{R}$ is an integrable function for which the integrals and the series above are convergent.

We emphasize that our operators are based on the functions defined on the unbounded interval $[0,\infty)$ while the operators given in [10] are related with functions defined on $[0,1]$. Usually, for the linear positive operators associated with continuous function defined on a compact interval, uniform convergence takes place but for ones defined on unbounded interval only pointwise convergence occurs. For this purpose, we use the weighted spaces of continuous, unbounded functions of polynomial growth given in Section 3 and we give a weighted modulus of continuity for obtaining the rate of convergence on unbounded interval. In the approximation of functions defined on a non compact interval by sequences of positive linear operators, weighted moduli of continuity are useful tools, since usual first modulus of continuity of such type functions does not tend to zero. Hence our approach is different from the approach given in [10]. In this paper, firstly we represent the operators in terms of hypergeometric series and show that our operators are an approximation process on the interval $[0,\infty)$ with respect to the particular weighted norms and give an estimate of convergence of the operators in terms of weighted modulus of continuity. Then, we obtain either Voronovskaya type asymptotic formula for new Kantorovich-Durrmeyer type operators or a quantitative version of the Voronovskaya type formula. In the last section, we prove that our operators are also an approximation process on weighted L_p space and give weighted L_p convergence of these operators using Korovkin type theorem for weighted L_p space given in [18].

2. Construction of Operators

Now we introduce a sequence of positive linear operators applying Kantorovich method to Baskakov Durrmeyer operators.

Our definition of these operators strongly depend on B_n so it is obvious in the following theorem.

Theorem 2.1. *For any $n \in \mathbb{N}$ we have*

$$\widetilde{B}_n(f) = \frac{n-1}{n+1}(D \circ B_{n+1} \circ I)(f).$$

Proof. Changing the order of integration with

$$\left\{ \begin{array}{c} t \in [0,\infty) \\ u \in [0,t] \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} u \in [0,\infty) \\ t \in [u,\infty] \end{array} \right. ,$$

we have

$$\begin{aligned} (B_{n+1} \circ I)(f)(x) &= B_{n+1}I(f;t)(x) = n\sum_{k=0}^{\infty} p_{n+1,k}(x)\int_0^{\infty} p_{n+1,k}(t)\int_0^t f(u)\,dudt \\ &= n\sum_{k=0}^{\infty} p_{n+1,k}(x)\int_0^{\infty} f(u)\int_u^{\infty} p_{n+1,k}(t)\,dudt. \end{aligned} \tag{5}$$

Applying differential operator D to previous equality we get

$$(D \circ B_{n+1} \circ I)(f)(x) = D(B_{n+1} \circ I)(f)(x) = n\sum_{k=0}^{\infty} p'_{n+1,k}(x)\int_0^{\infty} f(u)\int_u^{\infty} p_{n+1,k}(t)\,dudt.$$

Taking into account that

$$p'_{n,k}(x) = n\left(p_{n+1,k-1}(x) - p_{n+1,k}(x)\right), \tag{6}$$

and the convention $p_{n,k}(x) = 0$ for $k < 0$, we obtain

$$\begin{aligned} (D \circ B_{n+1} \circ I)(f)(x) &= n(n+1)\sum_{k=0}^{\infty}\left(p_{n+2,k-1}(x) - p_{n+2,k}(x)\right)\int_0^{\infty} f(u)\int_u^{\infty} p_{n+1,k}(t)\,dtdu \\ &= n(n+1)\sum_{k=1}^{\infty} p_{n+2,k-1}(x)\int_0^{\infty} f(u)\int_u^{\infty} p_{n+1,k}(t)\,dtdu \\ &\quad -n(n+1)\sum_{k=0}^{\infty} p_{n+2,k}(x)\int_0^{\infty} f(u)\int_u^{\infty} p_{n+1,k}(t)\,dtdu \\ &= n(n+1)\sum_{k=0}^{\infty} p_{n+2,k}(x)\int_0^{\infty} f(u)\int_u^{\infty}\left(p_{n+1,k+1}(t) - p_{n+1,k}(t)\right)dtdu. \end{aligned}$$

Using the equality (6), we have

$$\begin{aligned}(D\circ B_{n+1}\circ I)(f)(x) &= (n+1)\sum_{k=0}^{\infty}p_{n+2,k}(x)\int_0^{\infty}f(u)\int_u^{\infty}-p_{n,k+1}^{'}(t)\,dtdu\\ &= (n+1)\sum_{k=0}^{\infty}p_{n+2,k}(x)\int_0^{\infty}f(u)\,p_{n,k+1}(u)\,du.\end{aligned}$$

Therefore, considering the operators (4), we can easily get

$$\widetilde{B}_n(f)(x)=\frac{n-1}{n+1}(D\circ B_{n+1}\circ I)(f)(x),$$

for $n\in\mathbb{N}$. □

Remark 2.2. *When the operators (4) compare with the classical Baskakov Durrmeyer operators (2), it can be seen that different generating kernel $p_{n,k}$ appears in the new operators. Also considering Theorem 2.1, this new operators have a similar equality as in (3) which classical Baskakov Kantorovich operators have. For given a integrable function $f^{'}$ on the interval $[0,\infty)$, using Theorem 2.1, the relation*

$$\widetilde{B}_n\left(f^{'}\right)(x)=\frac{n-1}{n+1}(D\circ B_{n+1})(f)(x)$$

holds for every x at which $\int_0^x f^{'}(u)\,du=f(x)$ (as for the classical Bernstein Kantorovich operators). Under some assumptions of f, using approximation properties of $(D\circ B_{n+1})(f)$, one can obtain similar properties for the operator $\widetilde{B}_n\left(f^{'}\right)$.That is we can say that, from the point of structural properties, this new operators have resemblance to Baskakov Durrmeyer operators and Baskakov Kantorovich operators.

Remark 2.3. *We also represent the operators (4) in hypergeometric form with the notations in [19].*

$$\begin{aligned}\widetilde{B}_n(f)(x) &= n(n-1)\int_0^{\infty}\frac{f(u)\,u(1+x)^{-2}(1+u)^{-1}}{(1+x)^n(1+u)^n}\sum_{k=0}^{\infty}\frac{\Gamma(n+k+2)}{\Gamma(n+2)\,k!}\frac{\Gamma(n+k+1)}{\Gamma(n+1)(k+1)!}\frac{(ux)^k}{[(1+x)(1+u)]^k}du\\ &= n(n-1)\int_0^{\infty}\frac{f(u)\,u(1+x)^{-2}(1+u)^{n}}{(1+x)^n(1+u)^n}\sum_{k=0}^{\infty}\frac{(n+1)_k(n+2)_k}{(k+1)!k!}\frac{(ux)^k}{[(1+x)(1+u)]^k}du\end{aligned}$$

where the Pochhammer symbol is given by $(n)_k=n(n+1)(n+2)\ldots(n+k-1)$.

Using the hypergeometric series ${}_2F_1(a,b;c;x)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k$, we have

$$\widetilde{B}_n(f)(x) = n(n-1)\int_0^{\infty}\frac{f(u)\,u(1+x)^{-2}(1+u)^{-1}}{(1+x)^n(1+u)^n}\times{}_2F_1\left(n+1,n+2;2;\frac{(ux)}{(1+x)(1+u)}\right)du$$

Applying Pfaff-Kummer transformation

$${}_2F_1(a,b;c;x)=(1-x)^{-a}\times{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right),$$

we have

$$\widetilde{B}_n(f)(x) = n(n-1)\int_0^{\infty}\frac{f(u)\,u(1+x)^{-1}}{(1+x+u)^{n+1}}\times{}_2F_1\left(n+1,-n;2;\frac{-ux}{(1+x+u)}\right)du,$$

which is another form of the operator (4) in terms of hypergeometric functions.

3. Auxiliary Results

Now, we give some results which will be necessary for prof of main results.

Lemma 3.1. *For $x \in [0,\infty)$ and $n \geq r+2$, the Baskakov-Durrmeyer-Kantorovich operators (4) have the following property:*

$$\widetilde{B}_n(e_{r+1})(x) = \frac{(r+2+(n+2)x)}{(n-r-2)}\widetilde{B}_n(e_r)(x) + \frac{x(1+x)}{(n-r-2)}\left(\widetilde{B}_n(e_r)(x)\right)', \tag{7}$$

where $e_r(t) = t^r$, $r = 0,1,2,\ldots$ .Consequently,

$$\begin{aligned}
\widetilde{B}_n(e_0)(x) &= 1,\\
\widetilde{B}_n(e_1)(x) &= \frac{2+(n+2)x}{n-2},\\
\widetilde{B}_n(e_2)(x) &= \frac{(n+2)(n+3)x^2+6(n+2)x+6}{(n-3)(n-2)}.
\end{aligned} \tag{8}$$

For $r \geq 1$ and $n \geq r+2$,

$$\widetilde{B}_n(e_r)(x) = \frac{(n+(r+1))(n+r)\ldots(n+2)}{(n-(r+1))(n-r)\ldots(n-2)}x^r + \frac{a_{n,r}}{(n-(r+1))(n-r)\ldots(n-2)}F_{r-1}, \tag{9}$$

where $a_{n,r}$ is a sequence whose degree is at most $r-1$ and F_{r-1} is a positive polynomial of degree $r-1$.

Proof. Using the definition (4), it follows directly:

$$\begin{aligned}
\widetilde{B}_n(e_r)(x) &= (n-1)\sum_{k=0}^{\infty} p_{n+2,k}(x)\int_0^{\infty} u^r p_{n,k+1}(u)\,du\\
&= (n-1)\sum_{k=0}^{\infty} p_{n+2,k}(x)\begin{pmatrix} n+k \\ k+1 \end{pmatrix}\beta(k+r+2,n-r-1),
\end{aligned}$$

where $\beta(\cdot,\cdot)$ is the Euler Beta function. From (6), using the classical properties of Euler Beta and Gamma functions, one can write

$$\begin{aligned}
&\left(\widetilde{B}_n(e_r)(x)\right)'\\
&= (n-1)\sum_{k=0}^{\infty}\frac{k-(n+2)x}{x(1+x)}p_{n+2,k}(x)\int_0^{\infty} u^r p_{n,k+1}(u)\,du\\
&= (n-1)\sum_{k=0}^{\infty}\frac{k-(n+2)x}{x(1+x)}p_{n+2,k}(x)\begin{pmatrix} n+k \\ k+1 \end{pmatrix}\beta(k+r+2,n-r-1)\\
&= \frac{(n-1)}{x(1+x)}\sum_{k=0}^{\infty} p_{n+2,k}(x)\begin{pmatrix} n+k \\ k+1 \end{pmatrix}[(n-r-2)\beta(k+r+3,n-r-2)\\
&\quad -(r+2+(n+2)x)\beta(k+r+2,n-r-1)]\\
&= \frac{1}{x(1+x)}\left\{(n-r-2)\widetilde{B}_n(e_{r+1};x)-(r+2+(n+2)x)\widetilde{B}_n(e_r;x)\right\}.
\end{aligned}$$

Therefore we obtain (7) immediately. The moments $\widetilde{B}_n(e_0)(x)$, $\widetilde{B}_n(e_1)(x)$ and $\widetilde{B}_n(e_2)(x)$ can easily seen from (7).

For the sake of brevity, we omit details of the proof of the equality (9) that can be easily obtained from (7). □

Now, we obtain a recurrence formula which we will use in the proof.

Lemma 3.2. *For $r \in \mathbb{N}_0$ and $n \geq r+2$, if*

$$T_{n,r}(x) = (n-1)\sum_{k=0}^{\infty} p_{n+2,k}(x)\int_0^{\infty}(u-x)^r p_{n,k+1}(u)\,du,\ x \in [0,\infty),$$

then there exists the following recurrence relation:

$$(n-(r+2))\,T_{n,r+1}(x) = \varphi^2(x)\left[T'_{n,r}(x) + 2rT_{n,r-1}(x)\right] + (r+2)(1+2x)\,T_{n,r}(x), \tag{10}$$

where $\varphi^2(x) = x(1+x)$. $T_{n,r}(x)$ is a polynomial in x of degree at most r. Consequently,

$$\begin{aligned} T_{n,0}(x) &= 1, T_{n,1}(x) = \frac{2+4x}{n-2} \\ T_{n,2}(x) &= \frac{2n+24}{(n-3)(n-2)}\left\{\varphi^2(x) + \frac{3}{n+12}\right\}. \end{aligned}$$

Also $T_{n,r}(x) = O\left((n-(r+1))^{-\left[\frac{r+1}{2}\right]}\right)$ for all $x \in [0,\infty)$ where $[\alpha]$ denotes the integer part of α.

Proof. We first prove (10) by using (6). By definition of $T_{n,r}(x)$, we get

$$\begin{aligned} & \varphi^2(x)\left[T'_{n,r}(x) + rT_{n,r-1}(x)\right] \\ = \;& (n-1)\sum_{k=0}^{\infty}\varphi^2(x)\,p'_{n+2,k}(x)\int_0^{\infty}(u-x)^r p_{n,k+1}(u)\,du \\ = \;& (n-1)\sum_{k=0}^{\infty}p_{n+2,k}(x)\int_0^{\infty}(k-(n+2)x)\,p_{n,k+1}(u)(u-x)^r\,du \\ = \;& (n-1)\sum_{k=0}^{\infty}p_{n+2,k}(x)\int_0^{\infty}[[(k+1)-nu] + n(u-x) - (1+2x)]\,p_{n,k+1}(u)(u-x)^r\,du \\ = \;& (n-1)\sum_{k=0}^{\infty}p_{n+2,k}(x)\int_0^{\infty}\varphi^2(u)\,p'_{n,k+1}(u)(u-x)^r\,du + nT_{n,r+1}(x) - (1+2x)\,T_{n,r}(x) \\ = \;& (n-1)\sum_{k=0}^{\infty}p_{n+2,k}(x)\int_0^{\infty}\varphi^2(u)\,p'_{n,k+1}(u)(u-x)^r\,du + nT_{n,r+1}(x) - (1+2x)\,T_{n,r}(x) \\ = \;& (n-1)\sum_{k=0}^{\infty}p_{n+2,k}(x)\int_0^{\infty}\{-(1+2u)(u-x)^r \\ & -r\varphi^2(u)(u-x)^{r-1}\}\,p_{n,k+1}(u)\,du + nT_{n,r+1}(x) - (1+2x)\,T_{n,r}(x) \\ = \;& (n-1)\sum_{k=0}^{\infty}p_{n+2,k}(x)\int_0^{\infty}\left\{-r\varphi^2(x) - (r+1)(1+2x)(u-x)\right. \\ & \left. -(r+2)(u-x)^2\right\}(u-x)^{r-1}\,p_{n,k+1}(u)\,du + nT_{n,r+1}(x) - (1+2x)\,T_{n,r}(x) \\ = \;& -r\varphi^2(x)\,T_{n,r-1}(x) - (r+1)(1+2x)\,T_{n,r}(x) - (r+2)\,T_{n,r+1}(x) + nT_{n,r+1}(x) - (1+2x)\,T_{n,r}(x). \end{aligned}$$

The proofs of the other consequences easily follow by the definition of $T_{n,r}(x)$ and (10).

From the recurrence relation, it easily verify that for $x \in [0,\infty)$ we have $T_{n,r}(x) = O\left((n-(r+1))^{-\left[\frac{r+1}{2}\right]}\right)$. □

4. Weighted Approximation Properties

In this section we give some weighted approximation properties of the $\widetilde{B}_n(f)$ operators.
$B_{x^2}[0,\infty)$ is called weighted space and it is a Banach space endowed with the norm

$$\|f\|_{x^2} = \sup_{x\in[0,\infty)} \frac{f(x)}{1+x^2}.$$

Let $C_{x^2}[0,\infty) = C[0,\infty) \cap B_{x^2}[0,\infty)$ be the subspace of $B_{x^2}[0,\infty)$ containing continuous functions. By $C^k_{x^2}[0,\infty)$, we denote subspace of all continuous functions $f \in B_{x^2}[0,\infty)$ for which $\lim_{x\to\infty} \frac{f(x)}{1+x^2} = k$, k is finite.
We know that the usual first modulus of continuity $\omega(\delta)$ does not tend to zero, as $\delta \to 0$, on infinite interval. Thus we use weighted modulus of continuity $\Omega(f;\delta)$ defined on infinite interval $[0,\infty)$ (see [11]). Let

$$\mathbf{\Omega}(f;\delta) = \sup_{|h|<\delta,\ x\in[0,\infty)} \frac{\left|f(x+h)-f(x)\right|}{(1+h^2)(1+x^2)} \text{ for each } f\in C_{x^2}[0,\infty).$$

We call $\Omega(f;\delta)$ the weighted modulus of continuity of the function $f \in C_{x^2}[0,\infty)$.
Now some elementary properties of $\Omega(f;\delta)$ are collected in the following Lemma.

Lemma 4.1. *Let $f \in C_{x^2}[0,\infty)$. Then,*

i) $\Omega(f;\delta)$ is a monotonically increasing function of δ, $\delta \geq 0$.
ii) for every $f \in C^k_{x^2}[0,\infty)$, $\lim\limits_{\delta\to 0}\Omega(f;\delta) = 0$.
iii) for each $\lambda > 0$,

$$\Omega(f;\lambda\delta) \leq 2(1+\lambda)\left(1+\delta^2\right)\Omega(f;\delta). \tag{11}$$

From the inequality (11) and definition of $\Omega(f;\delta)$ we get

$$\left|f(t)-f(x)\right| \leq 2\left(1+x^2\right)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\Omega(f;\delta) \tag{12}$$

for every $f \in C_{x^2}[0,\infty)$ and $x,t \in [0,\infty)$. (See [11])

Lemma 4.2. *For $n > 3$, $(\widetilde{B}_n)_{n\in N}$ is a uniformly bounded sequence of positive linear operators from $C^k_{x^2}[0,\infty)$ into $C^k_{x^2}[0,\infty)$, satisfying the inequality*

$$\left\|\widetilde{B}_n f\right\|_{C^k_{x^2}} \leq 2 + \frac{10}{n-2} + \frac{6}{n-3} + \frac{60}{(n-3)(n-2)}.$$

Proof. It is enough to show the proof for the space $C^0_{x^2}[0,\infty)$. Because of (8),

$$\frac{\widetilde{B}_n\left(1+t^2\right)(x)}{1+x^2} \leq 2 + \frac{10}{n-2} + \frac{6}{n-3} + \frac{60}{(n-3)(n-2)} \tag{13}$$

for every $x \geq 0$. Let $n > 3$ and $f \in C^0_{x^2}[0,\infty)$, for $\varepsilon > 0$ there exist $t_0 \geq 0$ such that

$$\left|f(t)\right| \leq \varepsilon\left(1+t^2\right) \quad (t \geq t_0).$$

Then, set $M := \sup_{0\leq t\leq t_0}\left|f(t)\right|$, we have

$$\left|f(t)\right| \leq M + \varepsilon\left(1+t^2\right) \quad t \geq 0.$$

Accordingly, for every $x \geq 0$

$$\begin{aligned}\left|\widetilde{B}_n(f)(x)\right| &\leq \widetilde{B}_n\left(\left|f\right|\right)(x) \\ &\leq M + \varepsilon \widetilde{B}_n\left(1+t^2\right)(x).\end{aligned}$$

Choosing $x_0 \geq 0$ such that $\frac{M}{1+x^2} \leq \varepsilon$, for $x \geq x_0$, we get

$$\frac{\left|\widetilde{B}_n(f)(x)\right|}{1+x^2} \leq 79\varepsilon.$$

Since ε is arbitrary, we have $\widetilde{B}_n(f) \in C^0_{x^2}[0,\infty)$.

Finally, if $f \in C^0_{x^2}[0,\infty)$, then

$$\left|f(x)\right| \leq \left\|f\right\|_{x^2}\left(1+x^2\right).$$

Taking into consideration of (13), we get

$$\begin{aligned}\left\|\widetilde{B}_n(f)\right\|_{x^2} &\leq \left\|f\right\|_{x^2}\left\|\widetilde{B}_n\left(1+t^2\right)\right\|_{x^2} \\ &\leq \left\|f\right\|_{x^2}\left(2+\frac{10}{n-2}+\frac{6}{n-3}+\frac{60}{(n-3)(n-2)}\right).\end{aligned}$$

Thus, we have the desired result. □

Theorem 4.3. *If $f \in C^k_{x^2}[0,\infty)$, then for a sufficiently large n we get*

$$\sup_{x\geq 0}\frac{\left|\widetilde{B}_n(f)(x)-f(x)\right|}{(1+x^2)^{\frac{5}{2}}} \leq K\Omega\left(f;\frac{1}{\sqrt{n-2}}\right),$$

where $K = \sup_{x\geq 0}\frac{4\left\{1+\left(x^2+x+1\right)+\sqrt{(x^2+x+1)}\left(1+\sqrt{\left(x^4+x^3+x^2+x+1\right)}\right)\right\}}{(1+x^2)^{\frac{5}{2}}}$.

Proof. From (12), we can write

$$\begin{aligned}&\left|\widetilde{B}_n(f)(x)-f(x)\right| \\ &\leq 2\left(1+x^2\right)\left(1+\delta_n^2\right)\Omega(f;\delta_n)(n-1)\sum_{k=0}^{\infty}p_{n+2,k}(x)\int_0^{\infty}\left\{1+(u-x)^2+\frac{|u-x|}{\delta}+(u-x)^2\frac{|u-x|}{\delta}\right\}p_{n,k+1}(u)\,du \\ &\leq 2\left(1+x^2\right)\left(1+\delta_n^2\right)\Omega(f;\delta_n)\left\{1+\widetilde{B}_n\left((t-x)^2;x\right)+\frac{1}{\delta_n}\widetilde{B}_n\left(|t-x|;x\right)+\frac{1}{\delta_n}\widetilde{B}_n\left((t-x)^2|t-x|;x\right)\right\}.\end{aligned}$$

Applying Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}&\left|\widetilde{B}_n(f)(x)-f(x)\right| \\ &\leq 2\left(1+x^2\right)\left(1+\delta_n^2\right)\Omega(f;\delta_n)\left\{1+\widetilde{B}_n\left((t-x)^2\right)(x)+\frac{1}{\delta_n}\widetilde{B}_n\left((t-x)^2\right)^{\frac{1}{2}}(x)+\frac{1}{\delta_n}\widetilde{B}_n\left((t-x)^2\right)^{\frac{1}{2}}(x)\,\widetilde{B}_n\left((t-x)^4\right)(x)^{\frac{1}{2}}\right\} \\ &= 2\left(1+x^2\right)\left(1+\delta_n^2\right)\Omega(f;\delta)\left\{1+I_1+\frac{1}{\delta_n}\sqrt{I_1}+\frac{1}{\delta_n}\sqrt{I_1}\sqrt{I_2}\right\},\end{aligned}$$

where

$$I_1 = \widetilde{B}_n\left((t-x)^2\right)(x)$$

and

$$I_2 = \widetilde{B}_n\left((t-x)^4\right)(x).$$

From Lemma 3.2, we obtain

$$I_1 = x^2\left\{\frac{2n+24}{(n-3)(n-2)}\right\} + x\left\{\frac{2n+24}{(n-3)(n-2)}\right\} + \frac{6}{(n-3)(n-2)} \tag{14}$$

and

$$\begin{aligned} I_2 &= x^4\frac{12\left(n^2+44n+144\right)}{(n-5)(n-4)(n-3)(n-2)} + x^3\frac{12\left(2n^2+91n+304\right)}{(n-5)(n-4)(n-3)(n-2)} \\ &\quad + x^2\frac{\left(12n^3+852n^2+10888n+28752\right)}{(n-5)(n-4)(n-3)(n-2)(n+12)} + x\frac{144n^2+2164n+5712}{(n-5)(n-4)(n-3)(n-2)(n+12)} \\ &\quad + \frac{120}{(n-5)(n-4)(n-3)(n-2)(n+12)}. \end{aligned} \tag{15}$$

Using (14) and (15) for a sufficiently large n, we get

$$I_1 = \left(x^2+x+1\right)O\left(\frac{1}{n-2}\right)$$

and

$$I_2 = \left(x^4+x^3+x^2+x+1\right)O\left(\frac{1}{n-2}\right),$$

where by O we mean that usual Landau symbol. Thus we can write

$$\begin{aligned} \left|\widetilde{B}_n(f)(x) - f(x)\right| &\leq 2\left(1+x^2\right)\left(1+\delta_n^2\right)\Omega(f;\delta_n)\left\{1+\left(x^2+x+1\right)O\left(\frac{1}{n-2}\right)\right. \\ &\quad + \frac{1}{\delta_n}\sqrt{(x^2+x+1)\,O\left(\frac{1}{n-2}\right)} \\ &\quad \left. + \frac{1}{\delta_n}\sqrt{(x^2+x+1)\,O\left(\frac{1}{n-2}\right)}\sqrt{(x^4+x^3+x^2+x+1)\,O\left(\frac{1}{n-2}\right)}\right\}. \end{aligned}$$

Choosing $\delta_n = \frac{1}{\sqrt{n-2}}$, we have

$$\begin{aligned} \left|\widetilde{B}_n(f)(x) - f(x)\right| &\leq 2\left(1+x^2\right)\left(1+\delta_n^2\right)\Omega(f;\delta_n)\left\{1+\delta_n^2\left(x^2+x+1\right)\right. \\ &\quad \left. + \sqrt{(x^2+x+1)} + \delta_n\sqrt{(x^2+x+1)}\sqrt{(x^4+x^3+x^2+x+1)}\right\} \end{aligned}$$

and since $\delta_n < 1$ for a sufficiently large n, we can write

$$\begin{aligned} \left|\widetilde{B}_n(f)(x) - f(x)\right| &\leq 4\left(1+x^2\right)\Omega(f;\delta_n)\left\{1+\left(x^2+x+1\right)+\sqrt{(x^2+x+1)}\right. \\ &\quad \left. + \sqrt{(x^2+x+1)}\sqrt{(x^4+x^3+x^2+x+1)}\right\}. \end{aligned}$$

Thus for a sufficiently large n, we get

$$\sup_{x\geq 0}\frac{\left|\widetilde{B}_n(f)x - f(x)\right|}{(1+x^2)^{\frac{5}{2}}} \leq K\Omega\left(f;\frac{1}{\sqrt{n-2}}\right).$$

□

Theorem 4.4. *For each $f \in C_{x^2}^{k}[0,\infty)$, we have*

$$\lim_{n\to\infty}\left\|\widetilde{B}_n(f)-f\right\|_{x^2}=0$$

Proof. Using the theorem in [7] we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\left\|\widetilde{B}_n(e_r)-e_r\right\|_{x^2}=0,\ r=0,1,2. \tag{16}$$

Since $\widetilde{B}_n(e_0)(x)=1$, the first condition of (16) is fulfilled for $r=0$.

By Lemma 3.1, the second condition of (16) holds for $r=1$ as $n\to\infty$.

Similarly we can write for $n>3$

$$\begin{aligned}
\left\|\widetilde{B}_n(e_2)-e_2\right\|_{x^2} &= \sup_{x\in[0,\infty)}\frac{\left|\widetilde{B}_n(e_2)(x)-x^2\right|}{1+x^2}\\
&\le \left(\frac{(n+2)(n+3)}{(n-3)(n-2)}-1\right)\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}+\frac{6(n+2)}{(n-3)(n-2)}\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{6}{(n-3)(n-2)}\\
&\le \left(\frac{(n+2)(n+3)}{(n-3)(n-2)}-1\right)+\frac{6(n+2)}{(n-3)(n-2)}+\frac{6}{(n-3)(n-2)}
\end{aligned}$$

and the third condition of (16) holds for $r=2$ as $n\to\infty$. This proves the theorem. □

5. Voronovskaya Type Theorems

In this section, we obtain some asymptotic estimates of the pointwise convergence of operators (4). Then , we give an estimate of this pointwise convergence in terms of weighted modulus of continuity $\mathbf{\Omega}(f;\cdot)$.

Theorem 5.1. *Let $f\in C_{x^2}[0,\infty)$ be a function such that f'' exist at a point $x\in[0,\infty)$. Then, we have*

$$\lim_{n\to\infty}n\left[\widetilde{B}_n(f)(x)-f(x)\right]=2(1+2x)f'(x)+2x(1+x)f''(x). \tag{17}$$

Proof. Using the Taylor formula of the second order, we have

$$n\left[\widetilde{B}_n(f)(x)-f(x)\right]=nf'(x)\widetilde{B}_n(t-x)(x)+n\frac{f''(x)}{2!}\widetilde{B}_n\left((t-x)^2\right)(x)+n\widetilde{B}_n\left(r(t,x)(t-x)^2\right)(x),$$

where $r(t,x)$ is the Peano form of the remainder and $r(\cdot,x)$ is a bounded function such that $\lim\limits_{t\to x}r(t,x)=0$. By Lemma 3.2 and passing to limit with $n\to\infty$, we have

$$\begin{aligned}
\lim_{n\to\infty}n\left[\widetilde{B}_n(f)(x)-f(x)\right] &= f'(x)\lim_{n\to\infty}nT_{n,1}(x)+\frac{f''(x)}{2!}\lim_{n\to\infty}nT_{n,2}(x)+\lim_{n\to\infty}n\widetilde{B}_n\left(r(t,x)(t-x)^2\right)(x)\\
&= 2(1+2x)f'(x)+2x(1+x)f''(x)+F,
\end{aligned} \tag{18}$$

where $F=\lim\limits_{n\to\infty}n\widetilde{B}_n\left(r(t,x)(t-x)^2\right)(x)$. Now we have to show that $F=0$ as $n\to\infty$. Let $\varepsilon>0$ be given and θ_x be open set containing x such that $t\in\theta_x$, $|r(t,x)|<\varepsilon$.

Then if we define $\vartheta(t):=\left\{0,|r(t,x)-\varepsilon|(t-x)^2\right\}$ we have for $t\ge 0$

$$|r(t,x)|(t-x)^2\le\varepsilon(t-x)^2+\vartheta(t).$$

Since $r(t,x)\to 0$ as $t\to x$, then ϑ vanishes on θ_x, so there is a constant D such that

$$|\vartheta(t)|\le D\frac{(t-x)^4}{\delta^2}.$$

Thus, we have

$$|r(t,x)|(t-x)^2 \leq \varepsilon (t-x)^2 + D\frac{(t-x)^4}{\delta^2} \text{ for all } x,t \in [0,\infty).$$

So we can deduce

$$\begin{aligned} F &\leq \lim_{n\to\infty} n\widetilde{B}_n\left((t-x)^2\left(\varepsilon + D\frac{(t-x)^2}{\delta^2}\right)\right)(x) \\ &\leq \varepsilon \lim_{n\to\infty} nT_{n,2}(x) + \frac{D}{\delta^2}\lim_{n\to\infty} nT_{n,4}(x) \\ &= 0. \end{aligned}$$

Hence, we have $F = 0$. Finally it is easily seen from (18) that

$$\lim_{n\to\infty} n\left[\widetilde{B}_n(f)(x) - f(x)\right] = 2(1+2x) f'(x) + 2x(1+x) f''(x).$$

□

The quantitative version of Theorem 5.1 can be expressed as following.

Theorem 5.2. *Let $f'' \in C^k_{x^2}[0,\infty)$, $x > 0$ be fixed and $n > 7$. Then we have*

$$\begin{aligned} &\left| n\left[\widetilde{B}_n(f)(x) - f(x)\right] - 2(1+2x) f'(x) - 2x(1+x) f''(x)\right| \\ &\leq f'(x)\left|nT_{n,1}(x) - 2(1+2x)\right| + \frac{f''(x)}{2}\left|nT_{n,2}(x) - 2x(1+x)\right| \\ &\quad + 8\left(1+x^2\right) T_{n,2}(x)\,\Omega\left(f''; \frac{1}{\sqrt{n-7}}\right)\left(x^4+x^3+1\right). \end{aligned}$$

Proof. By the local Taylor formula, there exists η lying between x and y such that

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2}(y-x)^2 + h(y,x)(y-x)^2,$$

where

$$h(y,x) := \frac{f''(\eta) - f''(x)}{2}$$

and h is a continuous function which vanishes at 0. Applying the operator $\widetilde{B}_n$ to above equality, we obtain

$$\widetilde{B}_n(f)(x) - f(x) = f'(x) T_{n,1}(x) + \frac{f''(x)}{2} T_{n,2}(x) + \widetilde{B}_n\left(h(y,x)(y-x)^2\right)(x).$$

Also, we can write that

$$\begin{aligned} &\left|\widetilde{B}_n(f)(x) - f(x) - \frac{f'(x)}{n}2(1+2x) - \frac{f''(x)}{n}2x(1+x)\right| \\ &\leq f'(x)\left|\widetilde{B}_n(y-x)(x) - \frac{2(1+2x)}{n}\right| + \frac{f''(x)}{2}\left|\widetilde{B}_n\left((y-x)^2\right)(x) - \frac{2x(1+x)}{n}\right| + \widetilde{B}_n\left(\left|h(y,x)\right|(y-x)^2\right)(x). \end{aligned}$$

To estimate the last inequality, using the inequality (12) and the inequality $\left|\eta - x\right| \leq \left|y - x\right|$, we can write that

$$\begin{aligned} \left|h(y,x)\right| &= \frac{\left|f''(\eta) - f''(x)\right|}{2} \\ &\leq \frac{1}{2}\left(1 + (\eta - x)^2\right)\left(1 + x^2\right)\left(1 + \delta_n^2\right)\Omega\left(f''; \left|\eta - x\right|\right) \\ &\leq \frac{1}{2}\left(1 + (y - x)^2\right)\left(1 + x^2\right)\left(1 + \delta_n^2\right)\Omega\left(f''; \left|y - x\right|\right). \end{aligned}$$

Considering (11), we can write

$$\left|h(y,x)\right| \leq \left(1 + (y - x)^2\right)\left(1 + x^2\right)\left(1 + \frac{\left|y - x\right|}{\delta_n}\right)\left(1 + \delta_n^2\right)\Omega\left(f''; \delta_n\right).$$

Since

$$\left|h(y,x)\right| \leq \begin{cases} 2\left(1 + x^2\right)\left(1 + \delta_n^2\right)^2\Omega\left(f''; \delta_n\right), & \left|y - x\right| < \delta_n \\ 2\left(1 + x^2\right)\left(1 + \delta_n^2\right)^2 \frac{(y-x)^4}{\delta^4}\Omega\left(f''; \delta_n\right), & \left|y - x\right| \geq \delta_n \end{cases}$$

choosing $\delta_n < 1$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \left|h(y,x)\right| &\leq 2\left(1 + x^2\right)\left(1 + \frac{(y - x)^4}{\delta_n^4}\right)\left(1 + \delta_n^2\right)^2\Omega\left(f''; \delta_n\right) \\ &\leq 8\left(1 + x^2\right)\left(1 + \frac{(y - x)^4}{\delta_n^4}\right)\Omega\left(f''; \delta_n\right). \end{aligned}$$

Therefore, we deduce that

$$\begin{aligned} \widetilde{B}_n\left(\left|h(y,x)\right|(y - x)^2\right)(x) &\leq 8\left(1 + x^2\right)\Omega\left(f''; \delta_n\right)\left\{T_{n,2}(x) + \frac{1}{\delta_n^4}T_{n,6}(x)\right\} \\ &\leq 8\left(1 + x^2\right)\Omega\left(f''; \delta_n\right)T_{n,2}(x)\left\{1 + \frac{1}{\delta_n^4}\frac{T_{n,6}(x)}{T_{n,2}(x)}\right\}. \end{aligned}$$

From Lemma 3.2, we know that

$$T_{n,2}(x) = O\left(\frac{1}{n - 3}\right)\left(x^2 + x + 1\right)$$

and

$$\begin{aligned} T_{n,6}(x) &= O\left(\frac{1}{(n - 7)^3}\right)\left(x^6 + ... + x^2 + x + 1\right) \\ &= O\left(\frac{1}{(n - 7)^3}\right)\left(x^4 + x^3 + 1\right)\left(x^2 + x + 1\right) \end{aligned}$$

Choosing $\delta_n = \frac{1}{\sqrt{n-7}}$ for $n > 7$, we have

$$\widetilde{B}_n\left(\left|h(y,x)\right|(y - x)^2\right)(x) \leq 8\left(1 + x^2\right)T_{n,2}(x)\,\Omega\left(f''; \frac{1}{\sqrt{n - 7}}\right)\left(x^4 + x^3 + 1\right),$$

which completes the prove. □

6. Weighted L_p Approximation

Finally, we deal with a direct approximation result in Theorem 6.3 by means of the weighted Korovkin type theorem given in [18]. Now we recall this theorem.

Let w be positive continuous function on real axis $[0,\infty)$, satisfying the condition

$$\int_0^\infty x^{2p} w(x)dx < \infty.$$

We denote by $L_{p,w}[0,\infty)$ the linear space of p-absolutely integrable on $[0,\infty)$ with respect to the weight function w, i.e. for $1 \le p < \infty$,

$$L_{p,w}[0,\infty) = \left\{ f:[0,\infty)\to\mathbb{R}; \left\|f\right\|_{p,w} = \left(\int_0^\infty \left|f(x)\right|^p w(x)dx \right)^{\frac{1}{p}} < \infty \right\}.$$

Theorem 6.1. *Let $(L_n)_{n\in N}$ be a uniformly bounded sequence of positive linear operators from $L_{p,w}[0,\infty)$ into $L_{p,w}[0,\infty)$, satisfying the conditions*

$$\lim_{n\to\infty} \|L_n(e_i)(x) - e_i\|_{p,w} = 0, \quad i = 0,1,2. \tag{19}$$

Then for every $f \in L_{p,w}[0,\infty)$

$$\lim_{n\to\infty} \left\|L_n f - f\right\|_{p,w} = 0.$$

Now we choose $w(x) = \frac{1}{(1+x^{2r})^p}$, $1 \le p < \infty$ and consider analogue weighted Lp-space

$$L_{p,2r}[0,\infty) := \left\{ f:[0,\infty)\to\mathbb{R}; \left\|f\right\|_{p,2r} = \left(\int_0^\infty \left|\frac{f(x)}{1+x^{2r}}\right|^p dx \right)^{\frac{1}{p}} < \infty \right\}.$$

Before stating the main result, some preliminary lemma is needed. Also this lemma show that our operators are an approximation process on $L_{p,2r}[0,\infty)$

Theorem 6.2. *Let $\left(\widetilde{B}_n\right)_{n\ge1}$ be a sequence of operators defined by (4) and $f \in L_{p,2r}[0,\infty)$, $r > 1, n \ge 2r+1$, $1 \le p \le \infty$.Then $\left(\widetilde{B}_n\right)_{n\ge1}$ are positive linear operators from $L_{p,2r}[0,\infty)$ into $L_{p,2r}[0,\infty)$ such that*

$$\left\|\widetilde{B}_n f\right\|_{L_{p,2r}} \le C_n \left\|f\right\|_{L_{p,2r}}, \tag{20}$$

$$C_n = 1 + \frac{(n+(2r+1))(n+2r)\dots(n+2)}{(n-(2r+1))(n-2r)\dots(n-2)} + \frac{a_{n,2r}}{(n-(2r+1))(n-2r)\dots(n-2)}$$

Proof. Consider $n \ge 2r+1$ and $f \in L_{p,2r}[0,\infty)$. We prove (20) for $p = 1$ and $p = \infty$. For $1 < p < \infty$, it follows by the Riesz-Thorin Theorem.

In the case that $p = 1$ and in view of the definition of the operators (4), we have

$$\begin{aligned}
\int_0^\infty \frac{\left|\widetilde{B}_n(f)(x)\right|}{1+x^{2r}}dx &\le (n-1)\sum_{k=0}^\infty \int_0^\infty p_{n+2,k}(x)\,dx \int_0^\infty \left|f(u)\right| p_{n,k+1}(u)\,du \\
&\le (n-1)\sum_{k=0}^\infty \int_0^\infty p_{n+2,k}(x)\,dx \int_0^\infty \frac{\left|f(u)\right|}{1+u^{2r}} p_{n,k+1}(u)\,du \\
&\quad +(n-1)\sum_{k=0}^\infty \int_0^\infty p_{n+2,k}(x)\,dx \int_0^\infty \frac{\left|f(u)\right|}{1+u^{2r}} u^{2r} p_{n,k+1}(u)\,du
\end{aligned}$$

Since $\int_0^\infty p_{n,k}(x)\,dx = \frac{1}{n-1}$ and $\lim_{x\to\infty} x^k P_{n,k+1} = 0$, we get

$$\begin{aligned}
\int\limits_0^\infty \frac{\left|\widetilde{B}_n(f)(x)\right|}{1+x^{2r}}dx &\leq \frac{(n-1)}{n+1}\int_0^\infty \frac{\left|f(u)\right|}{1+u^{2r}}\sum_{k=0}^\infty p_{n,k+1}(u)\,du + \frac{(n-1)}{n+1}\int_0^\infty \frac{\left|f(u)\right|}{1+u^{2r}}u^{2r}\sum_{k=0}^\infty p_{n,k+1}(u)\,du \\
&= \frac{(n-1)}{n+1}\left\{\int_0^\infty \frac{\left|f(u)\right|}{1+u^{2r}} + \sum_{k=0}^\infty \sup_{u\geq 0} u^{2r}p_{n,k+1}(u)\int_0^\infty \frac{\left|f(u)\right|}{1+u^{2r}}du\right\} \\
&\leq \left\|f\right\|_{1,2r} \leq C_n\left\|f\right\|_{1,2r}.
\end{aligned}$$

For $p=\infty$, we have

$$\begin{aligned}
\left|\widetilde{B}_n(f)(x)\right| &= (n-1)\sum_{k=0}^\infty p_{n+2,k}(x)\int_0^\infty \frac{f(u)}{1+u^{2r}}(1+u^{2r})p_{n,k+1}(u)\,du \\
&\leq \sup_{u\geq 0}\frac{\left|f(u)\right|}{1+u^{2r}}\Bigg\{(n-1)\sum_{k=0}^\infty p_{n+2,k}(x)\int_0^\infty p_{n,k+1}(u)\,du \\
&\quad + (n-1)\sum_{k=0}^\infty p_{n+2,k}(x)\int_0^\infty u^{2r}p_{n,k+1}(u)\,du\Bigg\}.
\end{aligned}$$

Using (4), we can write

$$\left|\widetilde{B}_n(f)(x)\right| \leq \left\|f\right\|_{\infty,2r}\left\{\widetilde{B}_n(e_1)(x) + \widetilde{B}_n(e_{2r})(x)\right\}.$$

Using Lemma 3.1

$$\begin{aligned}
\left\|\widetilde{B}_n(f)\right\|_{\infty,2r} &= \sup_{x\geq 0}\frac{\left|\widetilde{B}_n(f)(x)\right|}{1+x^{2r}} \\
&\leq \left(1+\frac{(n+(2r+1))(n+2r)\ldots(n+2)}{(n-(2r+1))(n-2r)\ldots(n-2)}x^{2r} + \frac{a_{n,2r}}{(n-(2r+1))(n-2r)\ldots(n-2)}F_{2r-1}\right)\left\|f\right\|_{\infty,2r} \\
&= (1+C)\left\|f\right\|_{\infty,2r},
\end{aligned}$$

where $C_n \geq \sup_{x\geq 0}\left(1+\frac{(n+(2r+1))(n+2r)\ldots(n+2)}{(n-(2r+1))(n-2r)\ldots(n-2)}x^{2r} + \frac{a_{n,2r}}{(n-(2r+1))(n-2r)\ldots(n-2)}F_{2r-1}\right)\left(1+x^{2r}\right)^{-1}$. Finally by the Riesz-Thorin theorem we get (20). □

We are now in a position to get the desired weighted Lp approximation.

Theorem 6.3. *Let $\left(\widetilde{B}_n\right)_{n\geq 1}$ be a sequence of operators defined by (4). For every $f\in L_{p,2r}[0,\infty)$, $r>1$, we have*

$$\lim_{n\to\infty}\left\|\widetilde{B}_n(f)-f\right\|_{p,2r} = 0.$$

Proof. Using the Theorem 6.1, we see that it is sufficient to verify the three conditions (19). Since $\widetilde{B}_n(1)=1$, the first condition is obvious for $i=0$. By Lemma 3.1 , for $i=1$, we have

$$\begin{aligned}
\left(\int\limits_0^\infty \frac{\left|\widetilde{B}_n(t)(x)-x\right|^p}{1+x^{2r}}dx\right)^{\frac{1}{p}} &= \left(\int\limits_0^\infty \frac{1}{1+x^{2rp}}\left|\left(\frac{4}{n-2}\right)x+\frac{2}{n-2}\right|^p dx\right)^{\frac{1}{p}} \\
&\leq \left(\frac{4}{n-2}\right)\left(\int\limits_0^\infty \frac{|x|^p}{1+x^{2rp}}dx\right)^{\frac{1}{p}} + \frac{2}{n-2}\left(\int\limits_0^\infty \frac{dx}{1+x^{2rp}}\right)^{\frac{1}{p}} \longrightarrow 0, \quad n\to\infty.
\end{aligned}$$

For $i = 2$, we can write

$$\left(\int_0^\infty \frac{\left|\widetilde{B}_n\left(t^2\right)(x) - x^2\right|^p}{1+x^{2r}}dx\right)^{\frac{1}{p}} = \left(\frac{(n+2)(n+3)}{(n-2)(n-3)} - 1\right)\left(\int_0^\infty \frac{x^{2p}}{1+x^{2rp}}dx\right)^{\frac{1}{p}} + \left(\frac{6(n+2)}{(n-2)(n-3)}\right)\left(\int_0^\infty \frac{|x|^p}{1+x^{2rp}}dx\right)^{\frac{1}{p}}$$
$$+\left(\frac{6}{(n-2)(n-3)}\right)\left(\int_0^\infty \frac{dx}{1+x^{2rp}}\right)^{\frac{1}{p}} \longrightarrow 0, \quad n \to \infty$$

which shows the theorem. □

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