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BERNSTEIN–TYPE OPERATORS THAT REPRODUCE EXPONENTIAL FUNCTIONS

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Abstract. In this paper we recover a generalization of the classical Bernstein operators introduced by Morigi and Neamtu in 2000. Specifically, we focus on a sequence of operators that reproduce the exponential functions $\exp(\mu t)$ and $\exp(2\mu t)$, $\mu > 0$. We study its convergence, this including qualitative and quantitative theorems, an asymptotic formula and saturation results. We also show their shape preserving properties by considering generalized convexity. Finally, a comparison is stated, that shows that in a certain sense and for certain family of illustrative functions the new sequence approximates better than the classical Bernstein polynomials.

1. Introduction and preliminaries

The nth classical Bernstein operator assigns to each single function $f \in \mathbb{R}^{[0,1]}$ the polynomial function $B_n f$, defined for $t \in [0,1]$ by

$$B_n f(t) = B_n(f;t) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(t), \quad p_{n,k}(t) := \binom{n}{k} t^k (1-t)^{n-k}.$$

It is well known that if $f \in C[0,1]$, then the sequence of functions $B_n f$ converges to f as n tends to infinity uniformly on $[0,1]$, providing this way a simple and constructive proof to the Weierstrass Approximation Theorem. These operators present a nice estructure, hold fixed the affine functions, interpolate continuous functions at the end points of the interval $[0,1]$ and preserve the classical convexities of all orders.

Due mainly to these basic properties, these operators, and a long list of variations and extensions of them that we do not cite in this note, have been of constant interest in the field of approximation theory. In this paper we consider a special case of a modification introduced in [13] by Morigi and Neamtu, which is associated with classical exponential functions.

We denote respectively by exp and log the natural exponential and logarithmic functions, although we are also writing e^t for the value of $\exp(t)$. As usual, we denote by e_i the polynomial functions defined by $e_i(t) = t^i$.

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Throughout the paper we consider a fixed real parameter $\mu > 0$ and consider the exponential function $\exp_\mu$, defined by $\exp_\mu(t) = e^{\mu t}$. Its inverse function is denoted here by $\log_\mu$, i.e. $\log_\mu$ is the logarithmic function with base e^μ.

The aforesaid modification of our interest in this paper is defined for $f \in \mathbb{R}^{[0,1]}$, $n \in \mathbb{N} = \{1,2,\dots\}$ and $t \in [0,1]$ by

$$\mathscr{G}_n f(t) = \mathscr{G}_n(f;t) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) e^{-\mu k/n} e^{\mu t} p_{n,k}(a_n(t)), \tag{1}$$

where

$$a_n(t) = \frac{e^{\mu t/n} - 1}{e^{\mu/n} - 1}.$$

Its close connection with the Bernstein operators is now displayed:

$$\mathscr{G}_n f(t) = \exp_\mu(t) B_n\left(\frac{f}{\exp_\mu}; a_n(t)\right). \tag{2}$$

Notice that for each $n \in \mathbb{N}$, a_n is an increasing and convex real continuous function satisfying $a_n(0) = 0$, $a_n(1) = 1$ and $a_n(t) > 0$ for $t \in [0,1]$. As a direct consequence, $\mathscr{G}_n$ is a positive operator that also interpolates continuous functions at the endpoints of $[0,1]$. On the other hand, whereas B_n hold fixed the functions e_0 and e_1, as we have just recalled, it can be checked easily that the operators $\mathscr{G}_n$ reproduce $\exp_\mu$ and $\exp_\mu^2$, i.e.

$$\mathscr{G}_n(\exp_\mu; x) = e^{\mu x}, \qquad \mathscr{G}_n(\exp_\mu^2; x) = e^{2\mu x}. \tag{3}$$

Our aim with this paper is to investigate in depth the operators $\mathscr{G}_n$, $n \in \mathbb{N}$, revealing new properties. Firstly, we study the convergence, this including qualitative and quantitative theorems, an asymptotic formula and saturation results. Roughly speaking, we point up in advance that the role played by the so-called Korovkin set $\{e_0, e_1, e_2\}$ for the study of the convergence properties of B_n will now be played by the set $\{e_0, \exp_\mu, \exp_\mu^2\}$, which trivially turns out to be an extended complete Tchebychev system.

We study as well some shape preserving properties, and state a comparison that shows that in a certain sense and for certain family of illustrative functions the new sequence approximates better than the classical Bernstein operators. In this respect, we shall see that it is convenient to consider a notion of generalized convexity related to the functions $\exp_\mu$ and $\exp_\mu^2$.

We notice that in [6] a particular case of the general King-type operators related to exponential functions was studied, and we point up that very recently in [1] and [10], respective modifications of the Szász-Mirakyan operators and the Phillips Operators that reproduce only one exponential type function have been investigated.

It is also important mentioning that under the setting of approximation theory by linear operators, although the general sequence of operators introduced by Morigi and Neamtu in [13] has received further attention in subsequent papers, see [3], [12] and [2], no results had dealt earlier with the quantitative aspects of the approximation process that they represent, neither had gone so far away concerning the shape preserving properties. On the other hand, although for the sake of completeness we prove the uniform convergence of the particular process we are dealing with, this had been proved earlier. The asymptotic formula had been announced, but the authors have found no proof of it.

We end this introductory section with a lemma that shows some basic identities that we shall use throughout the text. No proof is required as it can be derived by direct calculations, maybe with the eventual use of some mathematical software like Mathematica.

LEMMA 1. *For each $n \in \mathbb{N}$ and $x \in [0,1]$, the following identities hold:*

$$\mathscr{G}_n e_0(x) = e^{\mu(x-1)}\left(e^{\mu/n}+1-e^{\mu x/n}\right)^n,$$

$$\mathscr{G}_n(\exp_\mu^3;x) = e^{\mu x}\left(e^{\mu(x+1)/n}+e^{\mu x/n}-e^{\mu/n}\right)^n,$$

$$\mathscr{G}_n(\exp_\mu^4;x) = e^{\mu x}\left(e^{\mu(x+2)/n}+e^{\mu(x+1)/n}+e^{\mu x/n}-e^{\mu/n}-e^{2\mu/n}\right)^n.$$

2. Convergence properties

Five theorems are stated in this section. The first one shows that $\mathscr{G}_n$ represents an approximation process for the functions of the space $C[0,1]$. Later, the approximation error is estimated by proving a quantitative result. Then, searching for the so-called optimal order of convergence, following this way the classical pattern when studying sequences of linear operators, we prove an asymptotic formula analogous to the classical one stated by Voronovskaja in 1932 for the Bernstein operators, that reads as follows: if $f \in C[0,1]$ and $f''(x)$ exists, then

$$\lim_{n\to\infty} 2n\left(B_n f(x) - f(x)\right) = x(1-x)f''(x). \tag{4}$$

And finally in this section, two results show the so-called trivial class and saturation class of the sequence $\mathscr{G}_n$.

We make use of the first modulus of continuity, defined for $f \in C(I)$ (I being a compact real interval) and $\delta \geqslant 0$ by

$$\omega(f;\delta) = \{\sup|f(t)-f(x)| : t,x \in I, |t-x| \leqslant \delta\}$$

(no confusion will arise about the interval I to be considered when we just write $\omega(f,\delta)$).

Moreover, for each $x \in (0,1)$, we shall consider the functions e_x and $\exp_{\mu,x}$ defined for $t \in [0,1]$ by

$$e_x(t) = t - x, \qquad \exp_{\mu,x}(t) = e^{\mu t} - e^{\mu x}.$$

By elementary calculus, one can prove that for $x \in (0,1)$ and $t \in [0,1]$, whenever $\mu \geqslant 1$,

$$e_x^2(t) \leqslant \exp_{\mu,x}^2(t), \tag{5}$$

and, concerning the images of the first powers of $\exp_{\mu,x}$ under $\mathscr{G}_n$, for $x \in [0,1]$, using (3) and Lemma 1, one finds that

$$\mathscr{G}_n\left(\exp_{\mu,x};x\right) = \mathscr{G}_n\left(\exp_{\mu};x\right) - e^{\mu x}\mathscr{G}_n e_0(x) = e^{\mu x}\left(1-\mathscr{G}_n e_0(x)\right)$$
$$= e^{\mu x}\left(1 - e^{\mu(x-1)}\left(e^{\mu/n}+1-e^{\mu x/n}\right)^n\right), \tag{6}$$

$$\mathscr{G}_n\left(\exp^2_{\mu,x};x\right) = \mathscr{G}_n\left(\exp^2_{\mu};x\right) - 2e^{\mu x}\mathscr{G}_n(\exp_{\mu};x) + e^{2\mu x}\mathscr{G}_n e_0(x)$$
$$= e^{2\mu x}\left(\mathscr{G}_n e_0(x) - 1\right) = e^{2\mu x}\left(e^{\mu(x-1)}\left(e^{\mu/n}+1-e^{\mu x/n}\right)^n - 1\right), \tag{7}$$

$$\mathscr{G}_n\left(\exp^4_{\mu,x};x\right) = \mathscr{G}_n\left(\exp^4_{\mu};x\right) - 4e^{\mu x}\mathscr{G}_n(\exp^3_{\mu};x) + 6e^{2\mu x}\mathscr{G}_n(\exp^2_{\mu};x)$$
$$- 4e^{3\mu x}\mathscr{G}_n(\exp_{\mu};x) + e^{4\mu x}\mathscr{G}_n e_0(x)$$
$$= e^{\mu x}\Big[2e^{3\mu x} + e^{\mu(4x-1)}\left(e^{\mu/n}+1-e^{\mu x/n}\right)^n$$
$$- 4e^{\mu x}\left(e^{\mu(x+1)/n}+e^{\mu x/n}-e^{\mu/n}\right)^n$$
$$+ \left(e^{\mu(x+2)/n}+e^{\mu(x+1)/n}+e^{\mu x/n}-e^{\mu/n}-e^{2\mu/n}\right)^n\Big]. \tag{8}$$

THEOREM 1. *If $f \in C[0,1]$, then $\mathscr{G}_n f$ converges to f uniformly on $[0,1]$.*

Proof. As $\mathscr{G}_n$ is a sequence of linear positive operators, and $\{e_0, \exp_{\mu}, \exp^2_{\mu}\}$ is an extended complete Tchebychev system, then the famous Popoviciu–Bohman–Korovkin theorem (see for instante [4]) guaranties the validity of the statement if the thesis is fulfilled for the functions e_0, $\exp_{\mu}$ and $\exp^2_{\mu}$. After (3), this amounts to prove the uniform convergence of $\mathscr{G}_n e_0$ to e_0, and this is what we are checking right now.

The pointwise convergence corresponds to the limit (see Lemma 1)

$$\lim_{n\to\infty} e^{\mu(x-1)}\left(e^{\mu/n}+1-e^{\mu x/n}\right)^n = 1,$$

that can be proved with ease. As for the uniform convergence, we first proceed with a brief analytic study of the function $\mathscr{G}_n e_0$. One has that

$$(\mathscr{G}_n e_0)'(t) = \mu e^{\mu(t-1)}(1+e^{\mu/n}-2e^{\mu t/n})(1+e^{\mu/n}-e^{\mu t/n})^{n-1},$$

$$\mathscr{G}_n e_0(0) = 1 = \mathscr{G}_n e_0(1),$$

$$(\mathscr{G}_n e_0)'(0) = \mu\left(1-e^{-\mu/n}\right) > 0, \qquad (\mathscr{G}_n e_0)'(1) = \mu\left(1-e^{\mu/n}\right) < 0.$$

From that, one obtains that $\mathscr{G}_n e_0(t) > 1$ for $t \in (0,1)$ and that $\mathscr{G}_n e_0$ attains its maximum value within $[0,1]$ at the point $t = \frac{n}{\mu}\log\left(\frac{1+e^{\mu/n}}{2}\right)$. As a consequence,

$$\sup_{t\in[0,1]} \mathscr{G}_n e_0(t) = \mathscr{G}_n e_0\left(\frac{n}{\mu}\log\left(\frac{1+e^{\mu/n}}{2}\right)\right) = 4^{-n}e^{-\mu}\left(1+e^{\mu/n}\right)^{2n}. \tag{9}$$

The easy to check fact that this last sequence tends to 1 as n tends to infinity proves the uniform convergence of $\mathcal{G}_n e_0$ towards e_0 and ends the proof of the theorem. $\square$

THEOREM 2. *Let $f \in C[0,1]$, $x \in (0,1)$ and let $\delta > 0$. Then*

$$|\mathcal{G}_n f(x) - f(x)| \leqslant |f(x)|\,(\mathcal{G}_n e_0(x) - 1) + \left(\mathcal{G}_n e_0(x) + \frac{e^{2\mu x}\,(\mathcal{G}_n e_0(x) - 1)}{\delta^2}\right) \omega\left(f \circ \log_\mu ; \delta\right).$$

If $\mu \geqslant 1$, then $\omega\left(f \circ \log_\mu ; \delta\right)$ can be replaced by $\omega(f;\delta)$.

Proof. It follows directly by applying the already classical Shisha and Mond technique (see [14]). We have that

$$\begin{aligned} |f(t) - f(x)| &= \left|\left(f \circ \log_\mu\right)(e^{\mu t}) - \left(f \circ \log_\mu\right)(e^{\mu x})\right| \\ &\leqslant \omega\left(f \circ \log_\mu ; \left|e^{\mu t} - e^{\mu x}\right|\right) \leqslant \left(1 + \frac{(e^{\mu t} - e^{\mu x})^2}{\delta^2}\right) \omega\left(f \circ \log_\mu ; \delta\right), \end{aligned}$$

and directly from that, using (7) and the fact that $\mathcal{G}_n e_0(t) \geqslant 1$ for $t \in [0,1]$,

$$\begin{aligned} |\mathcal{G}_n f(x) - f(x)| &\leqslant |f(x)|\,(\mathcal{G}_n e_0(x) - 1) + \left(\mathcal{G}_n e_0(x) + \frac{\mathcal{G}_n\left(\exp_{\mu,x}^2 ; x\right)}{\delta^2}\right) \omega\left(f \circ \log_\mu ; \delta\right) \\ &= |f(x)|\,(\mathcal{G}_n e_0(x) - 1) + \left(\mathcal{G}_n e_0(x) + \frac{e^{2\mu x}\,(\mathcal{G}_n e_0(x) - 1)}{\delta^2}\right) \omega\left(f \circ \log_\mu ; \delta\right). \end{aligned}$$

This completes the proof of the main statement. The case when $\mu \geqslant 1$ follows analogously from the following equation, derived from (5):

$$|f(t) - f(x)| \leqslant \left(1 + \frac{e_x^2(t)}{\delta^2}\right) \omega(f;\delta) \leqslant \left(1 + \frac{(e^{\mu t} - e^{\mu x})^2}{\delta^2}\right) \omega(f;\delta). \quad \square$$

REMARK 1. If in the previous theorem we take

$$\delta^2 = \lambda_n(x) := \mathcal{G}_n e_0(x) - 1,$$

then the estimate in the thesis reads as

$$|\mathcal{G}_n f(x) - f(x)| \leqslant |f(x)|\lambda_n(x) + \left(1 + e^{2\mu x} + \lambda_n(x)\right) \omega\left(f \circ \log_\mu ; \sqrt{\lambda_n(x)}\right).$$

From it, taking into account (9), an uniform estimate of the difference between $\mathcal{G}_n f$ and f is easily derived.

Thus, the rapidity of convergence of $\mathcal{G}_n f(x)$ towards $f(x)$ is controlled by the rapidity of convergence of $\mathcal{G}_n e_0(x)$ towards $e_0(x) = 1$, or equivalently, the one of $\lambda_n(x)$

towards 0, and this is given by the following limit, that can be calculated by elementary calculus:

$$\begin{aligned}\lim_{n\to\infty} n\left(\mathscr{G}_n e_0(x)-1\right) &= \lim_{n\to\infty} n\lambda_n(x) \\ &= \lim_{n\to\infty} n\left(e^{\mu(x-1)}\left(e^{\mu/n}+1-e^{\mu x/n}\right)^n-1\right)=\mu^2 x(1-x).\end{aligned} \tag{10}$$

A sort of comparison with the operators B_n goes quickly. It is well-known that for $f\in C[0,1]$

$$|B_nf(x)-f(x)|\leqslant 2\omega\left(f;\sqrt{\tau_n(x)}\right),$$

where $\tau_n(x):=B_ne_2(x)-x^2$. Then the rapidity of convergence of $B_nf(x)$ towards $f(x)$ is controlled by the one of $B_ne_2(x)$ towards $e_2(x)=x^2$. Recall that $\tau_n(x)=x(1-x)/n$, and compare with (10).

THEOREM 3. *If $f\in C[0,1]$ has a second derivative at a point $x\in(0,1)$, then*

$$\lim_{n\to\infty} 2n\left(\mathscr{G}_n\left(f;x\right)-f\left(x\right)\right)=x(1-x)\left(f''(x)-3\mu f'(x)+2\mu^2 f(x)\right). \tag{11}$$

Proof. By Taylor's theorem, we have

$$\begin{aligned}f\left(t\right)=\left(f\circ\log_\mu\right)\left(e^{\mu t}\right)&=\left(f\circ\log_\mu\right)\left(e^{\mu x}\right)+\left(f\circ\log_\mu\right)'\left(e^{\mu x}\right)\exp_{\mu,x}(t)\\ &+\frac{\left(f\circ\log_\mu\right)''\left(e^{\mu x}\right)}{2}\exp^2_{\mu,x}(t)+h_x(t)\exp^2_{\mu,x}(t),\end{aligned}$$

where $h_x(t):=h(t-x)$ and h is a continuous function which vanishes at 0. Applying the operator $\mathscr{G}_n$ and then evaluating at the point x we obtain

$$\begin{aligned}\mathscr{G}_nf(x)&=f\left(x\right)\mathscr{G}_ne_0(x)+\left(f\circ\log_\mu\right)'\left(e^{\mu x}\right)\mathscr{G}_n\left(\exp_{\mu,x};x\right)\\ &+\frac{\left(f\circ\log_\mu\right)''\left(e^{\mu x}\right)}{2}\mathscr{G}_n\left(\exp^2_{\mu,x};x\right)+\mathscr{G}_n\left(h_x\exp^2_{\mu,x};x\right).\end{aligned}$$

Since

$$\begin{aligned}\left(f\circ\log_\mu\right)'\left(e^{\mu x}\right)&=e^{-\mu x}\mu^{-1}f'\left(x\right),\\ \left(f\circ\log_\mu\right)''\left(e^{\mu x}\right)&=e^{-2\mu x}\left(\mu^{-2}f''\left(x\right)-\mu^{-1}f'\left(x\right)\right),\end{aligned}$$

directly from (6) and (7) we have

$$\mathscr{G}_nf(x)-f\left(x\right)=\left(\mathscr{G}_ne_0(x)-1\right)\left(f(x)-\frac{3}{2\mu}f'(x)+\frac{1}{2\mu^2}f''(x)\right)+\mathscr{G}_n\left(h_x\exp^2_{\mu x};x\right).$$

If we multiply by n the previous expression and make use of the limit (10), the proof of the theorem will be over if we prove that

$$\lim_{n\to\infty} n\mathscr{G}_n\left(h_x \exp^2_{\mu,x};x\right) = 0.$$

From Cauchy-Schwarz inequality we can write

$$n\left|\mathscr{G}_n\left(h_x \exp^2_{\mu,x};x\right)\right| \leqslant \sqrt{\mathscr{G}_n\left(h_x^2;x\right)}\sqrt{n^2\mathscr{G}_n\left(\exp^4_{\mu,x};x\right)}.$$

Now, Theorem 1 allows to write

$$\lim_{n\to\infty} \mathscr{G}_n(h_x^2;x) = h_x^2(x) = 0,$$

and from (8), some calculations, maybe with the aid of some mathematical software, give us

$$\lim_{n\to\infty} n^2\mathscr{G}_n\left(\exp^4_{\mu x};x\right) = 3e^{4\mu x}\mu^4 x^2(1-x)^2. \tag{12}$$

This completes the proof of the theorem. □

A natural step after stating an asymptotic formula is to solve the saturation problem. With that purpose, we first take a look at the differential operator that appears in the right-hand side of (11). It is an easy exercise to get the following expression related to it:

$$x(1-x)\left(f''(x) - 3\mu f'(x) + 2\mu^2 f(x)\right) = \frac{1}{w_2(x)}\left(\frac{1}{w_1(x)}\left(\frac{f(x)}{w_0(x)}\right)'\right)',$$

where

$$w_0(x) = w_1(x) = e^{\mu x}, \qquad w_2(x) = \frac{e^{-2\mu x}}{x(1-x)}.$$

With the obvious modifications the results in [9, Section 5] apply to the operators $\mathscr{G}_n$, and immediately the following two results appear. Notice that a fundamental system of solutions of the second order differential equation $f'' - 3\mu f' + 2\mu^2 f = 0$ is given by the functions $\exp_\mu$ and $\exp^2_\mu$.

THEOREM 4. *Let $f \in C[0,1]$ and let $0 < a < b < 1$. Then for each $x \in (a,b)$*

$$2n\left(\mathscr{G}_n f(x) - f(x)\right) = o(1)$$

if and only if f is a solution of the differential equation $f'' - 3\mu f' + 2\mu^2 f = 0$ in (a,b).

THEOREM 5. *Let $f \in C[0,1]$, let $0 < a < b < 1$ and let $M \geqslant 0$. Then for each $x \in (a,b)$*

$$2n\left|\mathscr{G}_n f(x) - f(x)\right| \leqslant M + o(1)$$

if and only if, for almost every $t \in (a,b)$

$$|f''(t) - 3\mu f'(t) + 2\mu^2 f(t)| \leqslant M.$$

3. Shape preserving properties

It was already pointed out in the introduction that, for each $n \in \mathbb{N}$, the operator $\mathscr{G}_n$ is positive, hold fixed the functions $\exp_\mu$ and $\exp_\mu^2$, and interpolates the continuous functions at the end points of the interval $[0,1]$.

Now we are searching for further shape preserving properties. We begin by computing the first two derivatives of $\mathscr{G}_n f/\exp_\mu$:

$$\left(\frac{\mathscr{G}_n f}{\exp_\mu}\right)'(t) = n a_n'(t) \sum_{k=0}^{n-1} \left[\frac{f}{\exp_\mu}\left(\frac{k+1}{n}\right) - \frac{f}{\exp_\mu}\left(\frac{k}{n}\right)\right] p_{n-1,k}(a_n(t)), \tag{13}$$

$$\begin{aligned} \left(\frac{\mathscr{G}_n f}{\exp_\mu}\right)''(t) &= n a_n''(t) \sum_{k=0}^{n-1} \left[\frac{f}{\exp_\mu}\left(\frac{k+1}{n}\right) - \frac{f}{\exp_\mu}\left(\frac{k}{n}\right)\right] p_{n-1,k}(a_n(t)) \\ &\quad + n(n-1) a_n'(x)^2 \sum_{k=0}^{n-2} \left[\frac{f}{\exp_\mu}\left(\frac{k+2}{n}\right) - 2\frac{f}{\exp_\mu}\left(\frac{k+1}{n}\right) \right. \\ &\quad \left. + \frac{f}{\exp_\mu}\left(\frac{k}{n}\right)\right] p_{n-2,k}(a_n(t)). \end{aligned} \tag{14}$$

As $a_n(t)$ is increasing and convex, we deduce from the previous expressions that if for $f \in C[0,1]$, $f/\exp_\mu$ is increasing, then so is $\mathscr{G}_n f/\exp_\mu$; and if in addition $f/\exp_\mu$ is convex, then $\mathscr{G}_n f/\exp_\mu$ is also convex. However, the sole convexity of $f/\exp_\mu$ does not imply the convexity of $\mathscr{G}_n f/\exp_\mu$. Counterexamples appear easily after taking $f = e_0$.

It is convenient to recover these shape preserving properties, and add some others, in terms of generalized convexities with respect to the functions $\exp_\mu$ and $\exp_\mu^2$. After [11], we consider the following definition and results in this direction. Notice that $\{\exp_\mu, \exp_\mu^2\}$ is an extended complete Tchebychev system.

DEFINITION 1. A function $f \in \mathbb{R}^{(0,1)}$ is said to be convex with respect to $\{\exp_\mu\}$, denoted by $f \in \mathscr{C}(\exp_\mu)$, if

$$\begin{vmatrix} e^{\mu t_0} & e^{\mu t_1} \\ f(t_0) & f(t_1) \end{vmatrix} \geqslant 0, \quad 0 < t_0 < t_1 < 1.$$

f is said to be convex with respect to $\{\exp_\mu, \exp_\mu^2\}$, denoted by $f \in \mathscr{C}(\exp_\mu, \exp_\mu^2)$, if

$$\begin{vmatrix} e^{\mu t_0} & e^{\mu t_1} & e^{\mu t_2} \\ e^{2\mu t_0} & e^{2\mu t_1} & e^{2\mu t_2} \\ f(t_0) & f(t_1) & f(t_2) \end{vmatrix} \geqslant 0, \quad 0 < t_0 < t_1 < t_2 < 1.$$

Note that if $f \in C[0,1]$, then both previous non-strict inequalities with the determinants will hold, by continuity, respectively for $0 \leqslant t_0 < t_1 \leqslant 1$ and $0 \leqslant t_0 < t_1 < t_2 \leqslant 1$.

PROPOSITION 1. *Let $f \in C^1[0,1]$. Then the following items are equivalent:*

1. $f \in \mathscr{C}(\exp_\mu)$,

2. $f/\exp_\mu$ *is increasing,*

3. $f'(t) \geqslant \mu f(t)$ *for* $t \in [0,1]$.

Proof. It suffices to use the definition of convexity with respect to $\{\exp_\mu\}$ and work out the derivative of $f/\exp_\mu$. $\square$

PROPOSITION 2. *Let* $f \in C^2[0,1]$. *Then the following items are equivalent:*

1. $f \in \mathscr{C}(\exp_\mu, \exp_\mu^2)$,

2. $(f/\exp_\mu)''(t) \geqslant \mu (f/\exp_\mu)'(t)$ *for* $t \in [0,1]$,

3. $f''(t) - 3\mu f'(t) + 2\mu^2 f(t) \geqslant 0$ *for* $t \in [0,1]$.

Proof. It suffices to work out the first two derivatives of $f/\exp_\mu$, and use the definition of convexity with respect to $\{\exp_\mu, \exp_\mu^2\}$ together with a characterization given in [5] that ensures its equivalence with the classical convexity of the function $(f/\exp_\mu)\circ \log_\mu$ on the interval $[1, e^\mu]$. $\square$

REMARK 2. The functions $\exp_\mu$ and $\exp_\mu^2$ are frequently said to form a Haar system or the Haar pair $(\exp_\mu, \exp_\mu^2)$. Under this denomination, convexity with respect to $\{\exp_\mu, \exp_\mu^2\}$ is usually called $(\exp_\mu, \exp_\mu^2)$-convexity. Moreover, the monotony of $f/\exp_\mu$ is usually named as the $\exp_\mu$-monotony of f.

On the other hand, following [7], as $\exp_\mu$ and $\exp_\mu^2$ form a fundamental system of solutions of the differential equation $f'' - 3\mu f' + 2\mu^2 f = 0$, then the $(\exp_\mu, \exp_\mu^2)$-convexity is equivalent to the fact of being sub-L, L being the differential operator defined as $Lf := f'' - 3\mu f' + 2\mu^2 f$.

We are coming back now to the shape preserving properties of $\mathscr{G}_n$. First of all, following [15],

$$f \in \mathscr{C}(\exp_\mu, \exp_\mu^2) \quad \Rightarrow \quad \mathscr{G}_n f(t) \geqslant f(t), \quad 0 \leqslant t \leqslant 1. \tag{15}$$

Secondly, we wrote after (13) and (14) with other terminology that

$$f \in \mathscr{C}(\exp_\mu) \quad \Rightarrow \quad \mathscr{G}_n f \in \mathscr{C}(\exp_\mu).$$

Moreover, using (13), (14), the easy to check inequality $a_n''(t) \leqslant a_n'(t)$ for $t \in [0,1]$, and Proposition 2, we can write in short that

$$\left.\begin{array}{c} -f \in \mathscr{C}(\exp_\mu) \\ (f/\exp_\mu) \text{ convex} \end{array}\right\} \quad \Rightarrow \quad \left(\frac{\mathscr{G}_n f}{\exp_\mu}\right)'' \geqslant \mu \left(\frac{\mathscr{G}_n f}{\exp_\mu}\right)' \quad \Leftrightarrow \quad \mathscr{G}_n f \in \mathscr{C}(\exp_\mu, \exp_\mu^2).$$

Finally, making items (3) in both Proposition 1 and Proposition 2 enter the scene and using the fact that the classical convexity of $f/\exp_\mu$ amounts to $f''(t) - 2\mu f'(t) + \mu^2 f(t) \geqslant 0$ for $t \in [0,1]$, we have that

$$\left.\begin{array}{r} -f \in \mathscr{C}(\exp_\mu) \Leftrightarrow \mu f \geqslant f' \\ \frac{f}{\exp_\mu} \text{ convex} \Leftrightarrow f'' - 2\mu f' + \mu^2 f \geqslant 0 \end{array}\right\} \Rightarrow f'' - 3\mu f' + 2\mu^2 f \geqslant 0 \Leftrightarrow f \in \mathscr{C}(\exp_\mu, \exp_\mu^2).$$

4. Comparison with Bernstein polynomials

Next theorem uses the asymptotic formulae fulfilled by B_n and $\mathscr{G}_n$ to state a sort of weak result that shows that for certain family of illustrative functions the new sequence approximates better than the classical Bernstein operators.

THEOREM 6. *Let $f \in C^2[0,1]$. Suppose that there exists $n_0 \in \mathbb{N}$ such that*

$$f(t) \leqslant \mathscr{G}_n f(t) \leqslant B_n f(t), \quad \text{for all } n \geqslant n_0,\ t \in (0,1). \tag{16}$$

Then

$$f''(t) \geqslant 3\mu f'(t) - 2\mu^2 f(t) \geqslant 0, \quad t \in (0,1). \tag{17}$$

In particular, $f''(x) \geqslant 0$.

Conversely, if (17) holds with strict inequalities at a given point $x \in (0,1)$, then there exists $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$

$$f(x) < \mathscr{G}_n f(x) < B_n f(x).$$

Proof. From (16) we have that

$$0 \leqslant 2n\left(\mathscr{G}_n f(t) - f(t)\right) \leqslant 2n\left(B_n f(t) - f(t)\right), \quad n \geqslant n_0,\ t \in (0,1).$$

Then, using (4) and (11),

$$0 \leqslant f''(t) - 3\mu f'(t) + 2\mu^2 f(t) \leqslant f''(t)$$

from which (17) follows directly.

Conversely, if (17) holds with strict inequalities for a given $x \in (0,1)$, then directly

$$0 < f''(x) - 3\mu f'(x) + 2\mu^2 f(x) < f''(x),$$

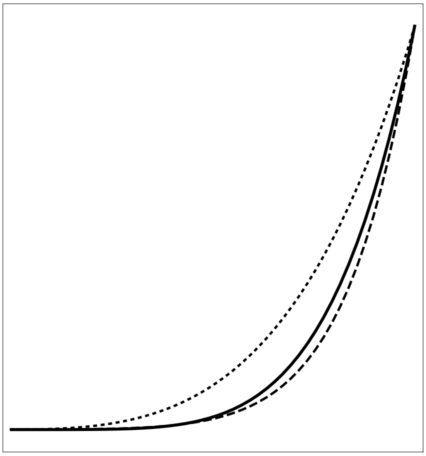
and using again (4) and (11), the proof follows. □

REMARK 3. It is of interest to point up that, according to Proposition 1 and Proposition 2, condition (17) is satisfied by the functions f such that $f \in \mathscr{C}(\exp_\mu)$ and $f \in \mathscr{C}(\exp_\mu, \exp_\mu^2)$. Among them one finds, after an elementary analysis, the illustrative families of functions given by

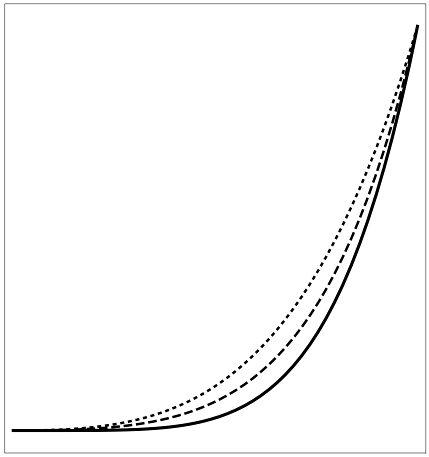
$$f(t) = t^p, \quad p \geqslant \frac{3\mu + 1 + \sqrt{\mu^2 + 6\mu + 1}}{2},$$

and

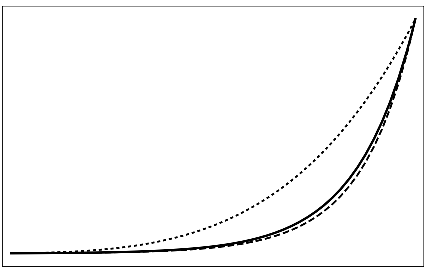
$$f(t) = \exp(qt), \quad q \geqslant 2\mu.$$



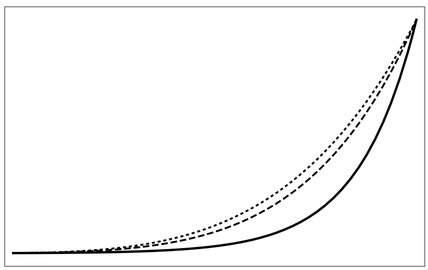
$f(x)=x^5,\ n=5,\ \mu=4$



$f(x)=x^5,\ n=5,\ \mu=1$



$f(x)=\exp(7x),\ n=4,\ \mu=5$



$f(x)=\exp(7x),\ n=4,\ \mu=0.5$

Figure 1.

The graphics given in Figure 1 intend to show that a stronger version of Theorem 6 could be stated. We finish the paper with it in the form of a conjecture, insisting on the subsequent fact that the sequence $\mathscr{G}_n$ seems to represent a nice approximation tool to bear in mind. The thick line, the dotted line and the dashed one represent respectively the function f to be approximated, B_nf and G_nf according to the information captioned at the bottom of each single graph.

CONJECTURE 1. *If $f \in C[0,1]$ is such that $f \in \mathscr{C}(\exp_\mu)$ and $f \in \mathscr{C}(\exp_\mu, \exp_\mu^2)$, then for all $n \in \mathbb{N}$ and all $t \in [0,1]$, one has that $f(t) \leqslant \mathscr{G}_n f(t) \leqslant B_n(t)$.*

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