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On Approximation Properties of Baskakov-Schurer-Szász Operators Preserving Exponential Functions

Övgü Gürel Yılmaz[a], Murat Bodur[a], Ali Aral[b]

[a] *Ankara University, Faculty of Science, Department of Mathematics, Ankara, Turkey*
[b] *Kırıkkale University, Faculty of Science and Arts, Department of Mathematics, Kırıkkale, Turkey*

Abstract. The goal of this paper is to construct a general class of operators which has known Baskakov-Schurer-Szász that preserving constant and $e^{2ax}, a > 0$ functions. Also, we demonstrate the fact that for these operators, moments can be obtained using the concept of moment generating function. Furthermore, we investigate a uniform convergence result and a quantitative estimate in consideration of given operator, as well. Finally, we discuss the convergence of corresponding sequences in exponential weighted spaces and make a comparison about which one approximates better between classical Baskakov-Schurer-Szász operators and the recent sequence, too.

1. Introduction

The following crucial question is actual in the theory of continuous approximation from the outset: "Is it possible to be roughly represented every arbitrary, continuous function in general by a polynomial with arbitrarily postulated accuracy?" Weierstrass [20] found it possible to give an affirmative response to this in 1885. From his confirmatory answer, there have been done several studies related upon this question.

Positive approximation processes play a vital role in approximation theory and come out in a many way in lots of problems touching on the approximation of continuous functions, especially when one necessitates further qualitative spesifications, such as monotonicity, convexity, shape preservation and etc.

When we take a look into 2003, King [16] described the modified Bernstein operators which preserve e_0 and e_2 test functions on $[0, 1]$ and examined their approximation properties. King accomplished to take an attention in a short time from researchers who perform approximation theory. Since that time, lots of researchers have put forth many relevant studies on this issue. For instance, in 2006, Morales et al. [7] have considered the King-type Bernstein polynomials which reproduce the linear combination of test functions for $\alpha > 0$, $e_2 + \alpha e_1$ and analyzed their shape preserving and approximation properties. They gave information about the order of approximation by comparing Bernstein and its modified operators; hence they obtained more general operators than those of King. In 2007, Duman and Özarslan [10], introduced Szász-Mirakyan type operators preserving the test function e_0 and e_2 and achieved better approximation for the generalization of the classical Szász-Mirakyan operators. A large number of precedents can be

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Email addresses: ogurel@ankara.edu.tr (Övgü Gürel Yılmaz), bodur@ankara.edu.tr (Murat Bodur), aliaral73@yahoo.com (Ali Aral)

given relation with Kings paper, which has been referred at [16]; however, depending on the objective in this study, we will keep it short and focus preserving exponential functions for positive linear operators. Numerous articles can be given interrelated with Kings research ([4], [6], [8], [18], [19], [21]). As far as we know, the development of the preserving exponential functions for positive linear operators is still early ages. Here, we will present a historical backround concerned with preserving exponential functions for positive linear operators.

In 2016 Acar et al. [1] constructed that a modification of Szász-Mirakyan operators which preserving constant e_0 and e^{2ax}, $a > 0$. They considered uniform convergence, order of approximation of their operators and also studied some shape preserving properties of the new operators. After that some researches which connect to this issues raise up day by day. In 2017, Acar et al. [2] inspiring from own paper [1], construct reagain their modified Szász-Mirakyan operators for preserving exponential functions which are e^{ax} and $e^{2ax}, a > 0$. In the same year, Gupta and Tachev [14] applied similar procedure for Phillips operators. Nowadays the number of articles related to this fact has increased ([5], [9], [12], [13], [22]).

The main purpose of this article is to introduce a modern and comprehensive exposition of the main aspects of preserving exponential functions for Baskakov-Schurer-Szász operators.

2. Construction of the Operators

Let $p, k, n \in \mathbb{N}$ and f real valued continuous functions on $[0, \infty)$. They considered the Baskakov-Schurer-Szász operators [17] as

$$L_{n,p}(f;x) = (n+p)\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}x^k(1+x)^{-n-p-k}\int_0^{\infty} f(t)e^{-(n+p)t}\frac{((n+p)t)^k}{k!}dt. \tag{1}$$

The authors, firstly, calculate the moments and central moments for $e_i(t) = t^i$, $i = 0, 1, 2$. Then they investigated the approximation behavior of the operators for real valued functions belonging to weighted space. After that they presented a Stancu type generalization and gave a result on approximation properties of the operators, as well. Finally, the article has been completed by examining the characteristics of the linear positive operator defined using quantum calculus.

We consider the following modified form of Baskakov-Schurer-Szász operators as

$$L^*_{n,p}(f;x) = (n+p)\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\frac{(\theta_n(x))^k}{(1+\theta_n(x))^{n+p+k}}\int_0^{\infty} f(t)e^{-(n+p)t}\frac{((n+p)t)^k}{k!}dt. \tag{2}$$

We concerned about that our operators preserve e_0 and e^{2ax}. Suppose these operators (2) preserve e^{2ax}, then

$$\begin{aligned}
L^*_{n,p}(e^{2at};x) &= (n+p)\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\frac{(\theta_n(x))^k}{(1+\theta_n(x))^{n+p+k}}\int_0^{\infty} e^{-[(n+p)-2a]t}\frac{((n+p)t)^k}{k!}dt \\
&= \frac{n+p}{n+p-2a}\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\left(\frac{(n+p)\theta_n(x)}{n+p-2a}\right)^k\left(\frac{1}{1+\theta_n(x)}\right)^{n+p+k} \\
&= \frac{n+p}{n+p-2a}\left(-\frac{(n+p)\theta_n(x)}{n+p-2a}+\theta_n(x)+1\right)^{-n-p} \\
&= \frac{n+p}{n+p-2a}\left(-2a\frac{\theta_n(x)}{n+p-2a}+1\right)^{-n-p}.
\end{aligned}$$

Taking into account $L^*_{n,p}(e^{2at};x) = e^{2ax}$, then we can surely identify

$$\theta_n(x) = \frac{n+p-2a}{2a}\Big(1 - \Big(\frac{n+p-2a}{n+p}e^{2ax}\Big)^{-1/n+p}\Big). \tag{3}$$

Apparently the function which is $\theta_n(x)$ satisfies the situtation

$$\theta_n(x) = \Big(L^*_{n,p}(e^{2at};x)\Big)^{-1} \circ e^{2ax}.$$

3. Auxiliary Results

In this part, we shall present the moments and the central moments of the operators (2) which will be necessary to prove our main results.

Lemma 3.1. *Let $e_i(t) := t^i,\ i = 0,1,2$. Then the Baskakov-Schurer-Szász operators $L^*_{n,p}$ satisfies*

$$\begin{aligned}
L^*_{n,p}(e_0;x) &= 1,\\
L^*_{n,p}(e_1;x) &= \theta_n(x) + \frac{1}{n+p},\\
L^*_{n,p}(e_2;x) &= \Big(1+\frac{1}{n+p}\Big)(\theta_n(x))^2 + \frac{4}{n+p}\theta_n(x) + \frac{2}{(n+p)^2}.
\end{aligned}$$

Lemma 3.2. *Let $\mu_{n,r}(x) = L^*_{n,p}((t-x)^r, x),\ r = 0,1,2.$ Then by considering Lemma 3.1, we possess*

$$\begin{aligned}
\mu_{n,0}(x) &= 1,\\
\mu_{n,1}(x) &= \theta_n(x) + \frac{1}{n+p} - x,\\
\mu_{n,2}(x) &= (\theta_n(x)-x)^2 + \frac{1}{n+p}\Big((\theta_n(x)+2)^2 - 2(x+2)\Big) + \frac{2}{(n+p)^2}.
\end{aligned}$$

Besides, the consecutive limits hold

$$\lim_{n\to\infty} n\Big(\theta_n(x) - x + \frac{1}{n+p}\Big) = -ax(x+2)$$

and

$$\lim_{n\to\infty} n\Big((\theta_n(x)-x)^2 + \frac{1}{n+p}\Big((\theta_n(x)+2)^2 - 2(x+2)\Big) + \frac{2}{(n+p)^2}\Big) = x(x+2).$$

4. Main Results

In this main section, we would like to show that the constructed operators are meticulously discussed linked with a uniform convergence result and a quantitative estimate. And also, we debate the convergence of corresponding sequences in exponential weighted spaces. We consider by $C^*[0,\infty)$ the class of real-valued continuous functions f, possessing finite limit for x sufficiently large and equipped with the uniform norm.
Theorem A. [15] *Take into account a sequence of positive linear operators $L_n : C^*[0,\infty) \to C^*[0,\infty)$ and set*

$$\|L_n(e_0) - 1\|_{[0,\infty)} = \alpha_n,$$

$$\left\|L_n(e^{-t}) - e^{-x}\right\|_{[0,\infty)} = \beta_n,$$

$$\left\|L_n(e^{-2t}) - e^{-2x}\right\|_{[0,\infty)} = \gamma_n.$$

then for each $f \in C^*[0,\infty)$

$$\left\|L_n(f) - f\right\|_{[0,\infty)} \leq \alpha_n \left\|f\right\|_{[0,\infty)} + (2+\alpha_n)\omega^*(f;\ \sqrt{\alpha_n + 2\beta_n + \gamma_n}),$$

where the modulus of continuity is defined as

$$\omega^*(f;\delta) := \sup_{\substack{x,t\geq 0 \\ |e^{-x}-e^{-t}|\leq\delta}} \left|f(t) - f(x)\right|.$$

Now, we put to a test below Theorem A for Baskakov-Schurer-Szász operators and also, give out quantitative results.

Theorem 4.1. *For each function* $f \in C^*[0,\infty)$, *we own*

$$\left\|L^*_{n,p}f - f\right\|_{[0,\infty)} \leq 2\omega^*(f;\ \sqrt{2\beta_n + \gamma_n}),$$

where

$$\left\|L^*_{n,p}(e^{-t}) - e^{-x}\right\|_{[0,\infty)} = \beta_n,$$

$$\left\|L^*_{n,p}(e^{-2t}) - e^{-2x}\right\|_{[0,\infty)} = \gamma_n.$$

Proof. According the definition of the operators, since they preserve constants, we reach

$$\left\|L^*_{n,p}(e_0) - 1\right\|_{[0,\infty)} = \alpha_n = 0$$

and for $f(t) = e^{-t}$, we get

$$\begin{aligned} L^*_{n,p}(e^{-t};x) &= (n+p)\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\frac{(\theta_n(x))^k}{(1+\theta_n(x))^{n+p+k}}\int_0^{\infty} e^{-(n+p+1)t}\frac{((n+p)t)^k}{k!}dt \\ &= \frac{n+p}{n+p+1}\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\left(\frac{(n+p)\theta_n(x)}{n+p+1}\right)^k\left(\frac{1}{1+\theta_n(x)}\right)^{n+p+k} \\ &= \frac{n+p}{n+p+1}\left(\frac{\theta_n(x)}{n+p+1}+1\right)^{-n-p}. \end{aligned} \tag{4}$$

When using the well-known software program that name is Maple to make a calculation of the right hand side which is found (4), we obtain

$$L^*_{n,p}\left(e^{-t};x\right) = e^{-x} + \frac{e^{-x}\left((2a+1)x^2 + 2(2a+1)x\right)}{2n} + O(\frac{1}{n^2}),$$

$$\left\|L^*_{n,p}(e^{-t}) - e^{-x}\right\|_{[0,\infty)} = \frac{2(2a+1)}{ne^2} + \frac{(2a+1)}{ne} + O(\frac{1}{n^2}) = \beta_n.$$

Also, for $f(t) = e^{-2t}$

$$\begin{aligned} L^*_{n,p}(e^{-2t};x) &= (n+p)\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\frac{(\theta_n(x))^k}{(1+\theta_n(x))^{n+p+k}}\int_0^{\infty} e^{-(n+p+2)t}\frac{((n+p)t)^k}{k!}dt \\ &= \frac{n+p}{n+p+2}\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\left(\frac{(n+p)\theta_n(x)}{n+p+2}\right)^k\left(\frac{1}{1+\theta_n(x)}\right)^{n+p+k} \\ &= \frac{n+p}{n+p+2}\left(\frac{2\theta_n(x)}{n+p+2}+1\right)^{-n-p}. \end{aligned} \tag{5}$$

If the procedure applied for previous equality is performed again, we receive

$$L^*_{n,p}\left(e^{-2t};x\right) = e^{-2x} + \frac{e^{-2x}\Big(2(a+1)x^2 + 4(a+1)x\Big)}{n} + O(\frac{1}{n^2}),$$

$$\left\|L^*_{n,p}(e^{-2t}) - e^{-2x}\right\|_{[0,\infty)} = \frac{2(a+1)}{ne^2} + \frac{2(a+1)}{ne} + O(\frac{1}{n^2}) = \gamma_n.$$

Here, β_n and γ_n tend to zero as $n \to \infty$ so this completes the proof. □

We are now in a position to prove the Voronovskaya type theorem for the operators $L^*_{n,p}$.

Theorem 4.2. *Let $f, f'' \in C^*[0,\infty)$ then for any $x \in [0,\infty)$ we have*

$$\begin{aligned} &\left|n[L^*_{n,p}(f;x) - f(x)] + ax(x+2)f'(x) - \frac{x(x+2)}{2}f''(x)\right| \\ \leq\ & |p_n||f'(x)| + |q_n||f''(x)| + 2(2q_n + x(x+2) + r_n)\omega^*\Big(f'', \frac{1}{\sqrt{n}}\Big), \end{aligned}$$

where

$$\begin{aligned} p_n &= n\mu_{n,1}(x) + ax(x+2), \\ q_n &= 2^{-1}(n\mu_{n,2}(x) - x(x+2)), \\ r_n &= n^2\sqrt{L^*_{n,p}((e^{-x} - e^{-t})^4;x)}\sqrt{\mu_{n,4}(x)}. \end{aligned}$$

Proof. By Taylor's expansion,

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + h(t,x)(t-x)^2, \tag{6}$$

where

$$h(t,x) = \frac{f''(\eta) - f''(x)}{2}$$

and η is a number lying between x and t. If applied the operator $L^*_{n,p}$ to both sides of the above Taylor's expansion and made a simple calculations, we obtain

$$\left|n[L^*_{n,p}(f;x) - f(x)] + ax(x+2)f'(x) - \frac{x(x+2)}{2}f''(x)\right| \leq |p_n||f'(x)| + |q_n||f''(x)| + |nL^*_{n,p}(h(t,x)(t-x)^2;x)|.$$

So as to complete the proof taking consideration of $|nL^*_{n,p}(h(t,x)(t-x)^2;x)|$. When using the property as shown below,

$$|f(t) - f(x)| \leq \Big(1 + \frac{(e^{-x} - e^{-t})^2}{\delta^2}\Big)\omega^*(f,\delta), \quad \delta > 0$$

then we are capable of reaching

$$|h(t;x)| \leq \Big(1 + \frac{(e^{-x} - e^{-t})^2}{\delta^2}\Big)\omega^*(f'',\delta).$$

Moreover,

$$|h(t;x)| \leq \begin{cases} 2\omega^*(f'',\delta) & , \quad |e^{-x} - e^{-t}| < \delta \\ 2\Big(\frac{(e^{-x}-e^{-t})^2}{\delta^2}\Big)\omega^*(f'',\delta) & , \quad |e^{-x} - e^{-t}| \geq \delta \end{cases}$$

and thence

$$|h(t;x)| \le 2\Big(1 + \frac{(e^{-x} - e^{-t})^2}{\delta^2}\Big)\omega^*(f'',\delta).$$

It is now no longer difficult to prove the given theorem. From Cauchy-Schwarz inequality we can write

$$|n(L^*_{n,p}h(t,x)(t-x)^2;x)| \le 2n\omega^*(f'',\delta)\mu_{n,2}(x) + \frac{2n}{\delta^2}\omega^*(f'',\delta)\sqrt{L^*_{n,p}((e^{-x} - e^{-t})^4;x)}\sqrt{\mu_{n,4}(x)}.$$

Lastly, choosing $\delta = \frac{1}{\sqrt{n}}$ and bearing in mind some simple calculations

$$\Big|n[L^*_{n,p}(f;x) - f(x)] + ax(x+2)f'(x) - \frac{x(x+2)}{2}f''(x)\Big| \le |p_n||f'(x)| + |q_n||f''(x)| + 2\omega^*\Big(f'', \frac{1}{\sqrt{n}}\Big)(2q_n + x(x+2) + r_n).$$

Hence, the proof is completed. □

Corollary 4.3. *Let $f, f'' \in C^*[0,\infty)$. Then the inequality*

$$\lim_{n\to\infty} n[L^*_{n,p}(f;x) - f(x)] = -ax(x+2)f'(x) + \frac{x(x+2)}{2}f''(x) \tag{7}$$

holds for any $x \in [0,\infty)$.

Corollary 4.4. *Let $f \in C^2[0,\infty)$ be an decreasing and convex function. Then there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$, we have $f(x) < L^*_{n,p}(f;x)$ for all $x \in [0,\infty)$.*

Presently, we also examine the behavior of the operators on some weighted spaces. Set $\varphi(x) = 1 + e^{2ax}$, $x \in \mathbb{R}^+$ and turn the following weighted spaces over in our mind:

$$\begin{aligned} B_\varphi(\mathbb{R}^+) &= \{f : \mathbb{R}^+ \to \mathbb{R} : \big|f(x)\big| \le M_f\varphi(x), \quad x \ge 0\}, \\ C_\varphi(\mathbb{R}^+) &= \{C(\mathbb{R}^+) \cap B_\varphi(\mathbb{R}^+)\}, \\ C^k_\varphi(\mathbb{R}^+) &= \{f \in C_\varphi(\mathbb{R}^+) : \lim_{x\to\infty}\frac{f(x)}{\varphi(x)} = k_f\}. \end{aligned}$$

where M_f and k_f are constants depending on f. All three spaces are normed spaces with the norm

$$\big\|f\big\|_\varphi = \sup_{x\in\mathbb{R}^+}\frac{|f(x)|}{\varphi(x)}.$$

It appears that for any $f \in C_\varphi(\mathbb{R}^+)$ the inequality

$$\big\|L^*_{n,p}(f)\big\|_\varphi \le \big\|f\big\|_\varphi$$

holds and we complete that $L^*_{n,p}(f)$ maps $C_\varphi(\mathbb{R}^+)$ to $C_\varphi(\mathbb{R}^+)$ [11].

Theorem 4.5. *For each function $f \in C^k_\varphi(\mathbb{R}^+)$*

$$\lim_{n\to\infty}\big\|L^*_{n,p}(f) - f\big\|_\varphi = 0.$$

Proof. Using the general result shown in [11], the following three conditional approximations are sufficient.

$$\lim_{n\to\infty}\big\|L^*_{n,p}(e^{\nu a\cdot}) - e^{\nu a\cdot}\big\|_\varphi = 0, \quad \nu = 0,1,2. \tag{8}$$

We know that for the given operator which is represented with $L^*_{n,p}$, $L^*_{n,p}(e_0) = 1$ and $L^*_{n,p}(e^{2at}) = e^{2ax}$ occurs. Presently, if we take into consideration the situation for $\nu = 1$

$$L^*_{n,p}(f;x) = (n+p)\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\frac{(\theta_n(x))^k}{(1+\theta_n(x))^{n+p+k}}\int_0^{\infty} e^{-((n+p)-a)t}\frac{((n+p)t)^k}{k!}dt$$

and also on the assumption that the simple calculations are made, we reach

$$L^*_{n,p}\left(e^{at}\right) = \frac{n+p}{n+p-a}\left(-\frac{\theta_n(x)(n+p)}{n+p-a} + \theta_n(x) + 1\right)^{-n-p}. \tag{9}$$

Keeping an account of $\theta_n(x)$ and computing (9) with Maple,

$$\begin{aligned} L^*_{n,p}\left(e^{at}\right) &= \sqrt{e^{2ax}} - \frac{\sqrt{e^{2ax}}(\ln(e^{2ax}))^2}{8n} - \frac{\sqrt{e^{2ax}}\ln(e^{2ax})a}{2n} + O(\frac{1}{n^2}) \\ &= e^{ax} - \frac{e^{ax}4a^2x^2}{8n} - \frac{e^{ax}2a^2x}{2n} + O(\frac{1}{n^2}) \\ &= e^{ax} - \frac{e^{ax}a^2x^2}{2n} - \frac{e^{ax}a^2x}{n} + O(\frac{1}{n^2}). \end{aligned}$$

Conclusively,

$$L^*_{n,p}\left(e^{at}\right) - e^{ax} = -\frac{e^{ax}a^2x^2}{2n} - \frac{e^{ax}a^2x}{n} + O(\frac{1}{n^2}) = \frac{-a^2x(x+2)e^{ax}}{2n} + O(\frac{1}{n^2})$$

and

$$\frac{L^*_{n,p}\left(e^{at}\right) - e^{ax}}{1+e^{2ax}} = \frac{-a^2x(x+2)e^{ax}}{2n(1+e^{2ax})} + O(\frac{1}{n^2}).$$

And this circumstance guarantees uniform continuity. Since $L^*_{n,p}(e_0) = 1$ and $L^*_{n,p}(e^{2at}) = e^{2ax}$, the conditions (8) are implemented for $\nu = 0$ and $\nu = 2$. Hence, the proof is completed. □

Immediately, we desire to demonstrate that our modified operators approximate better than classical Baskakov-Schurer-Szász operators. This part, we take into consideration of article which is Aral et al. [3]. Ultimate theorem which would like to be given as below:

Theorem 4.6. *Let $f \in C^2[0,\infty)$. Assume that there exists $n_0 \in \mathbb{N}$ such that*

$$f(x) \le L^*_{n,p}(f;x) \le L_{n,p}(f;x), \text{ for all } n \ge n_0,\ x \in (0,\infty). \tag{10}$$

Then

$$\frac{x(x+2)}{2}f''(x) \ge (ax^2 + 2ax + 1)f'(x) \ge 0,\ x \in (0,\infty). \tag{11}$$

Particularly $f'(x) \ge 0$ and $f''(x) \ge 0$.

Contrarily, if (11) holds with strict inequalities at a given point $x \in (0,\infty)$, there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$

$$f(x) < L^*_{n,p}(f;x) < L_{n,p}(f;x).$$

Proof. From (10) we have that

$$0 \leq n(L^*_{n,p}(f;x) - f(x)) \leq n(L_{n,p}(f;x) - f(x)), \text{ for all } n \geq n_0, \ x \in (0,\infty).$$

From classical Baskakov-Schurer-Szász operators [17] satisfies an asymptotic formula which is shown as below,

$$\lim_{n\to\infty} n(L_{n,p}(f;x) - f(x)) = f'(x) + \frac{x(x+2)}{2} f''(x). \tag{12}$$

Using (7) and (12)

$$0 \leq (ax^2 + 2ax + 1) f'(x) \leq \frac{x(x+2)}{2} f''(x)$$

from which follows (11) directly.

Contrarily, if (11) holds with strict inequalities for a given $x \in (0,\infty)$ then

$$0 < (ax^2 + 2ax + 1) f'(x) < \frac{x(x+2)}{2} f''(x)$$

and using again (7) and (12) the proof is completed. □

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