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P-CONTRACTIVE MAPPINGS ON METRIC SPACES

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Abstract. In the present paper, we introduce a new concept of P-contractive mappings on metric spaces. We prove that every ordinary contractive mapping is also P-contractive but the converse may not be true in general. We provide an example to illustrate this fact and also provide some examples to show that nonexpensive mappings and P-contractive mappings are independent on metric spaces. Finally, we present that every continuous P-contractive mapping on compact metric spaces has a unique fixed point. This result includes the famous Edelstein fixed point theorem.

Keywords. Contractive mapping; P-contractive mapping; Fixed point; Compact metric space.

AMS Subject Classification: 54H25, 47H10.

1. Introduction

In a recent paper, Suzuki [1] categorized the metrical fixed point theorems into four type classes as follows according to their provision that whether the fixed point of mappings is unique and whether the Picard sequence converges to a fixed point. Let (X,d) be a metric space and let T be a self mapping on X. Suzuki [1] made the following classification:

(T1) T has a unique fixed point and the Picard sequence $\{T^n x\}$ converges to the fixed point for all $x \in X$. Such a mapping is called a Picard operator in the literature. If a metrical fixed point theorem provides this provision, then it belongs to the Leader type class [2]. Most of fixed point theorems, such as Banach's theorem [3], Ciric's theorem [4], Edelstein's theorem [5], Kannan's theorem [6], Kirk's theorem [7], Matkowski's theorem [8] and Suzuki's theorem [9, 10], are in this class.

(T2) T has a unique fixed point (the Picard sequence $\{T^n x\}$ does not necessarily converge to the fixed point). If a metrical fixed point theorem provides this provision, then it belongs to the Unnamed type class. As Suzuki mentioned that, we can construct a fixed point theorem for Unnamed type class from a

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theorem in Leader type class. However, this becomes a meaningless result. Of course, Theorem 3 of [1] is not such a theorem.

(T3) T has a fixed point (may be more than one) and the Picard sequence $\{T^n x\}$ converges to a fixed point for all $x \in X$. Such a mapping is called a weakly Picard operator in the literature. If a metrical fixed point theorem provides this provision, then it belongs to the Subrahmanyam type class. For example, Subrahmanyam's fixed point theorem [11] is in this class. Fore more results for this class, we refer the readers to [12, 13] and the references therein.

(T4) T has a fixed point (may be more than one and the Picard sequence $\{T^n x\}$ does not necessarily converge to a fixed point). If a metrical fixed point theorem provides this provision, then it belongs to the Caristi type class [14]. Bae's theorems [15], Bae, Cho and Yeom's theorems [16], Caristi and Kirk's theorems [17] and Suzuki's theorems [18] are in this class.

In this paper we introduce a new type contractive mappings (P-contractive mapping) on metric spaces. We claim that every ordinary contractive mapping is also this new type contractive but the converse may not be true in general. We provide some examples to illustrate this fact and provide some examples to show that nonexpensive mappings and P-contractive mappings are independent on metric spaces. We also compare the P-contractive mappings and Suzuki type contractive mappings on metric spaces. Finally, we present a fixed point theorem which belongs to the Unnamed type class. Our theorem includes the famous Edelstein fixed point theorem as properly.

2. Main Results

In [1], Suzuki proved the following theorem, which belongs to the Unnamed type class.

Theorem 2.1. *Let (X,d) be a compact metric space and let T be a self mapping on X. Assume that T is a Suzuki type contractive mapping, that is,*

$$\frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow d(Tx,Ty) < d(x,y), \tag{2.1}$$

for all $x,y \in X$ with $x \neq y$. Then T has a unique fixed point.

It is well known that a self mapping T of a metric space (X,d) is called

- contractive if
$$d(Tx,Ty) < d(x,y)$$
for all $x,y \in X$ with $x \neq y$,
- nonexpensive if
$$d(Tx,Ty) \leq d(x,y)$$
for all $x,y \in X$.

Now we introduce a new type contractive mappings.

Definition 2.2. Let (X,d) be a metric space and let T be a self mapping of X. Then T is said to be P-contractive if

$$d(Tx,Ty) < d(x,y) + |d(x,Tx) - d(y,Ty)| \tag{2.2}$$

for all $x,y \in X$ with $x \neq y$.

To compare the concepts of contractive mappings, nonexpansive mappings, the Suzuki type contractive mappings and the P-contractive mappings, we will denote the class of contractive self mappings of X by $C(X)$, the class of nonexpensive self mappings by $N(X)$, the class of the Suzuki type contractive self mappings by $S(X)$ and the class of P-contractive self mappings by $P(X)$. It is clear that $C(X) \subseteq S(X)$, $C(X) \subseteq P(X)$ and $C(X) \subseteq N(X)$.

The following examples show that $C(X)$ is the proper subclass of $P(X)$.

Example 2.3. Let $X = \{(0,0),(3,4),(2,1),(0,1),(3,0)\} \subset \mathbb{R}^2$ with the metric

$$d(x,y) = d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

for $x = (x_1,x_2), y = (y_1,y_2) \in X$. Define a mapping $T : X \to X$ by

$$T = \begin{pmatrix} (0,0) & (3,4) & (2,1) & (0,1) & (3,0) \\ \\ (0,0) & (3,0) & (0,1) & (0,0) & (0,0) \end{pmatrix}.$$

Since

$$d(T(3,4),T(2,1)) = 4 = d((3,4),(2,1)),$$

we see that T is not a contractive mapping. On the other hand, we can see that (for $x \neq y$)

$$d(Tx,Ty) < d(x,y)$$

except for $(x,y) = ((3,4),(2,1))$ and $(x,y) = ((2,1),(3,4))$. Also,

$$d(T(3,4),T(2,1)) = 4 < 6 = d(x,y) + |d(x,Tx) - d(y,Ty)|.$$

Therefore T is a P-contractive mapping. Hence, $T \in P(X)\backslash C(X)$.

Example 2.4. Let $X = [0,2]$ with the usual metric. Define a mapping $T : X \to X$ by

$$Tx = \begin{cases} 1, & x \leq 1, \\ \\ 0, & x > 1. \end{cases}$$

Then $T \in P(X)\backslash C(X)$.

Remark 2.5. The classes of $P(X)$ and $N(X)$ of a metric space are different. For example, let T be the identity mapping on a metric space X. It is clear that T is a nonexpensive mapping but not a P-contractive mapping. On the other side, let $X = [0,1]$ with the usual metric d and let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{1}{2}, & x = 0, \\ \\ \frac{x}{2}, & x \neq 0. \end{cases}$$

Since T is not continuous, we find that T is not nonexpensive. Without loss of generality, we may assume $y < x$. It follows that

$$d(Tx,Ty) = \frac{1}{2} d(x,y)$$

for $y > 0$. Also, we have

$$d(T0,Tx) = \left| \frac{1-x}{2} \right| < x + \left| \frac{1-x}{2} \right| = d(0,x) + |d(0,T0) - d(x,Tx)|.$$

Therefore T is P-contractive.

Remark 2.6. The classes of $P(X)$ and $S(X)$ of a metric space are different. The mapping T in Example 2.3 (or in Example 2.4) is P-contractive, but not Suzuki contractive. Now let

$$X = \{(0,0),(4,0),(0,4),(4,5),(5,4)\} \subset \mathbb{R}^2$$

with the metric

$$d(x,y) = d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

for $x = (x_1,x_2), y = (y_1,y_2) \in X$. Define a mapping $T : X \to X$ by

$$T = \begin{pmatrix} (0,0) & (4,0) & (0,4) & (4,5) & (5,4) \\ \\ (0,0) & (0,0) & (0,0) & (4,0) & (0,4) \end{pmatrix}.$$

Since

$$d(Tx,Ty) = 8 > 2 = d(x,y) + |d(x,Tx) - d(y,Ty)|$$

for $x = (4,5)$ and $y = (5,4)$, we find that T is not a P-contractive mapping. On the other hand, we can see that (for $x \neq y$)

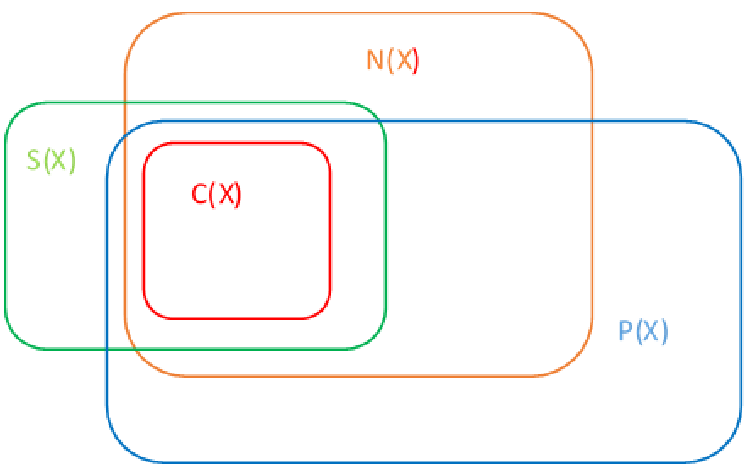
$$d(Tx,Ty) < d(x,y)$$

except for $(x,y) = ((4,5),(5,4))$ and $(x,y) = ((5,4),(4,5))$. Also, since

$$\frac{1}{2}d(x,Tx) = \frac{5}{2} > 2 = d(x,y)$$

for $(x,y) = ((4,5),(5,4))$ or $(x,y) = ((5,4),(4,5))$, we obtain that (2.1) is satisfied.

Remark 2.7. One can see from the above examples that the class of $S(X)$ and $N(X)$ are different. Hence, we have the following diagram:



Remark 2.8. Let (X,d) be a metric space and let T be a self mapping of X. It is well known that if T is continuous, then $x \to d(x,Tx)$ is also continuous (and so it is lower semicontinuous). However, if $f(x) = d(x,Tx)$ is lower semicontinuous, then T may not be continuous. For example, let $(\mathbb{R},d)$ be usual metric space and let $T : \mathbb{R} \to \mathbb{R}$ be a mapping defined by

$$Tx = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

In this case, we have

$$f(x)=\begin{cases}|x-1|, & x>0,\\ 0, & x=0,\\ |x+1|, & x<0.\end{cases}$$

Therefore, f is lower semicontinuous, but T is not continuous.

Now we are in a position to present our main theorems.

Theorem 2.9. *Let (X,d) be a metric space. Let T be a P-contractive self mapping on X and let f be a function defined by $f(x)=d(x,Tx)$. If there exists $x_0\in X$ such that $f(x_0)\le f(Tx_0)$, then T has a unique fixed point in X.*

Proof. Consider the function $f:X\to\mathbb{R}$ as $f(x)=d(x,Tx)$ and suppose that $x_0\in X$ is a point as $f(x_0)\le f(Tx_0)$. Then, x_0 is a fixed point of f. Indeed, if $x_0\ne Tx_0$, then we find from the P-contractivity of T that

$$\begin{aligned} f(Tx_0) &= d(Tx_0,TTx_0)\\ &< d(x_0,Tx_0)+|d(x_0,Tx_0)-d(Tx_0,TTx_0)|\\ &= f(x_0)+|f(x_0)-f(Tx_0)|\\ &= f(x_0)+f(Tx_0)-f(x_0)\\ &= f(Tx_0),\end{aligned}$$

which is a contradiction. Hence, $x_0=Tx_0$. Using (2.2), we find the uniqueness of fixed points immediately. This completes the proof. □

Remark 2.10. If $f(x)=d(x,Tx)$ attains the minimum, then there exists $x_0\in X$ such that $f(x_0)=\inf f(X)$. In this case, we have $f(x_0)\le f(Tx_0)$. From Theorem 2.9, we find that x_0 is the unique fixed point of T.

Theorem 2.11. *Let (X,d) be a compact metric space and T be a P-contractive self mapping on X. If the function $f(x)=d(x,Tx)$ is lower semicontinuous, then T has a unique fixed point in X.*

Proof. By Theorem 2.5.4 of [19], f attains the minimum. From Theorem 2.9 and Remark 2.10, we find that T has a unique fixed point. □

In Example 2.4, T is P-contractive and $f(x)=d(x,Tx)$ is lower semicontinuous. From Theorem 2.11, we see that T has a unique fixed point. Furthermore, the Picard sequence $\{T^nx\}$ converges to the fixed point for all $x\in X$. However, the following example shows that Theorem 2.11 (and also Theorem 2.9) belongs to the Unnamed type class.

Example 2.12. Let $X=[-2,-1]\cup\{0\}\cup[1,2]$ with the usual metric. Define a mapping $T:X\to X$ by

$$Tx=\begin{cases}\frac{1-x}{2}, & x\in[-2,-1),\\ -x, & x\in(1,2],\\ 0, & x\in\{-1,0,1\}.\end{cases}$$

Then T is a P-contractive mapping (in fact $T \in P(X) \setminus S(X)$) and the function

$$f(x) = d(x, Tx) = \begin{cases} \frac{1-3x}{2}, & x \in [-2,-1), \\ 2x, & x \in (1,2], \\ 1, & x \in \{-1,1\}, \\ 0, & x = 0 \end{cases}$$

is lower semicontinuous. By Theorem 2.11 (and also by Theorem 2.9), we find that T has a fixed point. However, the Picard iteration sequence with the initial point $x_0 = 2$ does not converge to the fixed point of T.

Remark 2.13. From Remark 2.8, we can present the following theorem, which is more general than the Edelstein fixed point theorem.

Theorem 2.14. *Let (X,d) be a compact metric space and let T be a continuous P-contractive self mapping on X. Then T has a unique fixed point in X.*

Proof. Since T is continuous, $f(x) = d(x, Tx)$ is also lower semicontinuous. Therefore, by Theorem 2.11, T has a unique fixed point. This completes the proof. □

Problem 2.15. It is an open question that Theorem 2.14 belongs to which class (the Leader type or the Unnamed type).

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