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General Helices with Lightlike Slope Axis

Çetin Camcı[a], Kazım İlarslan[b], Ali Uçum[b]

Dedicated to memory of Academician Professor Dr. Mileva Prvanović

[a]*Onsekiz Mart University, Faculty of Sciences and Arts, Department of Mathematics, Çanakkale, Turkey*
[b]*Kırıkkale University, Faculty of Sciences and Arts, Department of Mathematics, Kırıkkale-Turkey*

Abstract.
In this paper, we investigate general helices with lightlike slope axis. We give necessary and sufficient conditions for a general helix to have a lightlike slope axis. We obtain parametric equation of all general helices with lightlike slope axis. Also we give a nice relation between helix with lightlike slope axis and biharmonic curves in Minkowski 3-space $\mathbb{E}_1^3$.

1. Introduction

Without any doubt, helix is one of the most fascinating curve in science and nature. Scientist have long held a fascination, sometimes bordering on mystical obsession, for helical structures in nature. Helices arise in nanosprings, carbon nanotubes, DNA double and collagen triple helix, lipid bilayers, bacterial flagella in Salmonella and Escherichia coli, aerial hyphae in actinomycetes, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells (helico-spiral structures). Also we can easily see the helix curve or helical structures in fractal geometry, in the fields of computer aided design and computer graphics. Helices can be used for the tool path description, the simulation of kinematic motion or the design of highways, etc.(see [3, 11, 17, 27]).

From the view of differential geometry, a helix is a geometric curve with non- vanishing constant curvature (or first curvature) k_1 and non-vanishing constant torsion (or second curvature) k_2. Indeed a helix is a special case of the general helix. A curve of constant slope or general helix in Euclidean 3-space $\mathbb{E}^3$, is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix). A classical result stated by *M. A. Lancret* in 1802 and first proved by *B. de Saint Venant* in 1845 (for details see [25, 26]) is: A necessary and sufficient condition that for a curve to be a general helix is that the ratio of curvature to torsion is constant. If both k_1 and k_2 are non-zero constants, it is a general helix. We call it a circular helix. It is known that straight line and circle are degenerate-helix examples ($k_1 = 0$, if the curve is straight line and $k_2 = 0$, if the curve is a circle).

The Lancret theorem was revisited and solved by *Barros* ([5]) in 3-dimensional real space forms by using killing vector fields along curves. Also in the same space-forms, a characterization of helices and Cornu spirals is given by *Arroyo, Barros and Garay* in [1]. In [12], the author define a general helix in

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Email addresses: ccamci@comu.edu.tr (Çetin Camcı), kilarslan@yahoo.com (Kazım İlarslan), aliucum05@gmail.com (Ali Uçum)

a Lie group with a bi-invariant metric as a curve whose tangent vector makes a constant angle with a left-invariant vector field. Also, helices or more generally general helices are studied by many authors in different view point in different spaces for example, in Euclidean n-space [8, 18], in Lorentzian space forms [4, 15], in Lorentz-Minkowski spaces [2, 14, 16, 23], in Riemannian manifolds and submanifolds [21], in semi-Riemannian manifolds and submanifolds [13, 22].

On the other hand, with the help of position vector, some properties of helix were studied in [23] and [24].

In [10] (page 714, lemma 3.1), the authors claim that there is no timelike general helix or spacelike general helix of type 1 (spacelike general helix with spacelike principal normal N) in Minkowski 3-space satisfying the condition$|k_1(s)/k_2(s)| = 1$. However, we know that, in [19], Inoguchi proved that every biharmonic Frenet curve in Minkowski 3-space $\mathbb{E}_1^3$ is a helix whose curvature κ and torsion τ satisfy $k_1^2 = k_2^2$. Thus we easily see that there are helices in Minkowski 3-space $\mathbb{E}_1^3$ satisfying the condition $|k_1(s)/k_2(s)| = 1$.

In this paper, we investigate general helices in $\mathbb{E}_1^3$ with lightlike slope axis. We give necessary and sufficient conditions for a general helix to have a lightlike slope axis. We obtain parametric equation of all general helices with lightlike slope axis. Also we give a nice relation between helix with lightlike slope axis and biharmonic curves in $\mathbb{E}_1^3$. Thus we prove that there exist general helices with the curvatures satisfying $|k_1(s)/k_2(s)| = 1$.

2. Basic Concepts

The Minkowski 3-space $\mathbb{E}_1^3$ is the 3-dimensional linear space provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$. Recall that a vector $v \in \mathbb{E}_1^3 \backslash \{0\}$ is said to be *spacelike* if $g(v, v) > 0$, *timelike* if $g(v, v) < 0$ and *null (lightlike)* if $g(v, v) = 0$ and $v \neq 0$. In particular, the vector $v = 0$ is called a spacelike vector. The norm of a vector v is given by $\|v\| = \sqrt{|g(v, v)|}$, and two vectors v and w are said to be orthogonal, if $g(v, w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^3$, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null ([25]). A spacelike curve in $\mathbb{E}_1^3$ is called *pseudo null curve* if its principal normal vector N is null [6]. A null curve α is said to be parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$. A spacelike or a timelike curve α is said to be parameterized by arc-length s if $g(\alpha'(s), \alpha'(s)) = \pm 1$ ([6]).

Let $\{T, N, B\}$ be the moving Frenet frame along a curve α in $\mathbb{E}_1^3$, consisting of the tangent, the principal normal and the binormal vector fields, respectively. Depending on the causal character of α, the Frenet equations have the following forms.

Case 1. If α is a non-null curve, the Frenet equations are given by ([25]):

$$\begin{aligned} T'(s) &= k_1(s)N(s), \\ N'(s) &= -\epsilon_0\epsilon_1 k_1(s)T(s) + k_2(s)B(s), \\ B'(s) &= -\epsilon_1\epsilon_2 k_2(s)N(s), \end{aligned} \tag{1}$$

where k_1 and k_2 are the first and the second curvature of the curve respectively. Moreover, the following conditions hold:

$$g(T, T) = \epsilon_0 = \pm 1, \quad g(N, N) = \epsilon_1 = \pm 1, \quad g(B, B) = \epsilon_2 = -\epsilon_0\epsilon_1$$

and

$$g(T, N) = g(T, B) = g(N, B) = 0.$$

Case 2. If α is a pseudo null curve, the Frenet formulas have the form ([7])

$$\begin{aligned} T'(s) &= k_1(s)N(s), \\ N'(s) &= k_2(s)N(s), \\ B'(s) &= -k_1(s)T(s) - k_2(s)B(s), \end{aligned} \tag{2}$$

where the first curvature $k_1 = 0$ if α is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$$g(N,N) = g(B,B) = g(T,N) = g(T,B) = 0, \quad g(T,T) = g(N,B) = 1.$$

Case 3. If α is a null curve, the Frenet equations are given by ([6])

$$\begin{aligned} T'(s) &= N(s), \\ B'(s) &= k_2(s)N(s), \\ N'(s) &= -k_2(s)T(s) - B(s), \end{aligned} \tag{3}$$

where the first curvature $k_1 = 0$ if α is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$$g(T,T) = g(B,B) = g(T,N) = g(N,B) = 0, \quad g(N,N) = g(T,B) = 1.$$

3. General Helices with Lightlike Slope Axis in Minkowski 3-Space

In this section, we obtain the necessary and sufficient conditions for a general helix in $\mathbb{E}_1^3$ to have a lightlike slope axis. Also, we give the parametric equations of a general helix with lightlike slope axis.

According to the causal character of the curve $\alpha(s)$, we have the following cases.

Case 1. *Let $\alpha(s)$ be a spacelike curve.* According to the causal character of principal normal vector field N, we have the following subcases.

Case 1.1. Let $\alpha(s)$ be a spacelike curve in $\mathbb{E}_1^3$ with a spacelike principal normal vector field N.

In the following theorem, the necessary and sufficient conditions for a spacelike general helix with spacelike principal normal to have lightlike slope axis are given.

Theorem 3.1. *Let $\alpha(s)$ be a unit speed spacelike general helix in $\mathbb{E}_1^3$ with a spacelike principal normal N and the curvature functions $k_1(s), k_2(s)$. The slope axis of $\alpha(s)$ is a constant lightlike vector if and only if $|k_1(s)| = |k_2(s)|$.*

Proof. Let $\alpha(s)$ be a spacelike general helix in $\mathbb{E}_1^3$ with a spacelike principal normal vector field N and the curvature functions $k_1(s), k_2(s)$. We assume that $\alpha(s)$ has constant lightlike slope axis U given by

$$U = a(s)T(s) + b(s)N(s) + c(s)B(s), \tag{4}$$

where $a(s) = g(U, T(s)), b(s) = g(U, N(s))$ and $c(s) = -g(U, B(s))$. Since $\alpha(s)$ is a general helix, we have

$$g(U, T(s)) = a(s) = a \text{ (constant)}. \tag{5}$$

Here we see that $a \neq 0$. Differentiating (4) with respect to s and using (1) for $\epsilon_0 = 1, \epsilon_1 = 1$ and $\epsilon_2 = -1$, we easily obtain

$$b(s) = g(U, N(s)) = 0, \quad c' = 0, \quad \text{and} \quad ak_1(s) + c(s)k_2(s) = 0. \tag{6}$$

By using (6), from (4), we get

$$U = aT(s) - \frac{ak_1(s)}{k_2(s)}B(s). \tag{7}$$

Since U is a lightlike vector, we have $g(U,U) = 0$ which implies that

$$|k_1(s)| = |k_2(s)|. \tag{8}$$

Conversely, let $\alpha(s)$ be a spacelike general helix in $\mathbb{E}_1^3$ with a spacelike principal normal vector field N and the curvature functions $k_1(s)$, $k_2(s)$. By using $|k_1(s)| = |k_2(s)|$ in (7), we find that the slope axis is a lightlike vector. This completes the proof of the theorem. □

In the following theorem, we give the parametric equation of a spacelike general helix with a spacelike principal normal vector field N and a constant lightlike slope axis.

Theorem 3.2. *Let $\alpha(s)$ be a unit speed spacelike general helix in $\mathbb{E}_1^3$ with a spacelike principal normal N and a constant lightlike slope axis U. The parametric equation of $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_2(s))$ is given by*

$$\begin{aligned}
\alpha_1(s) &= \int \sigma(s)ds + c_1, \\
\alpha_2(s) &= \cos\varphi \int \sigma(s)ds + s\lambda\cos\varphi - \sin\varphi \int \sqrt{1-\lambda^2-2\lambda\sigma(s)}ds + c_2, \\
\alpha_3(s) &= \sin\varphi \int \sigma(s)ds + s\lambda\sin\varphi + \cos\varphi \int \sqrt{1-\lambda^2-2\lambda\sigma(s)}ds + c_3,
\end{aligned}$$

and the lightlike slope axis is given by $U = (u_1, u_1\cos\varphi, u_1\sin\varphi)$ where $c_0, c_1, c_2, c_3, \varphi \in \mathbb{R}$, $\lambda \in \mathbb{R}/\{0\}$, $\varepsilon_0 = \pm 1$ and

$$\sigma(s) = \frac{-1}{2\lambda}\left[\left(\varepsilon_0 \int k_1(s)ds + \lambda c_0\right)^2 + \lambda^2 - 1\right].$$

Proof. Let $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ be a unit speed spacelike general helix in $\mathbb{E}_1^3$ with a spacelike principal normal vector field N and the curvature functions $k_1(s) = 1, k_2(s)$. We assume that $\alpha(s)$ has constant lightlike slope axis $U = (u_1, u_2, u_3)$. Since $\alpha(s)$ is a unit speed curve, then its tangent vector field T is given by

$$T(s) = \alpha'(s) = \left(\alpha_1'(s), \alpha_2'(s), \alpha_3'(s)\right), \tag{9}$$

Since $g(T(s), T(s)) = 1$, we have

$$-\left(\alpha_1'(s)\right)^2 + \left(\alpha_2'(s)\right)^2 + \left(\alpha_3'(s)\right)^2 = 1. \tag{10}$$

Putting $\sigma(s) = \alpha_1'(s)$ in (10), we get

$$\begin{aligned}
\alpha_2'(s) &= \sqrt{1+\sigma^2(s)}\cos\theta, \\
\alpha_3'(s) &= \sqrt{1+\sigma^2(s)}\sin\theta
\end{aligned} \tag{11}$$

where $\theta = \theta(s)$. On the other hand, since U is a constant lightlike vector, we have $-u_1^2 + u_2^2 + u_3^3 = 0$, which implies that

$$\begin{aligned}
u_2 &= u_1\cos\varphi, \\
u_3 &= u_1\sin\varphi
\end{aligned} \tag{12}$$

where $\varphi \in \mathbb{R}$. Since $\alpha(s)$ is a general helix, we have

$$g(U,T) = -u_1\alpha_1'(s) + u_2\alpha_2'(s) + u_3\alpha_3'(s) = c \tag{13}$$

where $c \in \mathbb{R}/\{0\}$. By using (11) and (12) in (13), we find

$$\cos(\theta - \varphi) = \frac{u_1 \sigma(s) + c}{u_1 \sqrt{1 + \sigma^2(s)}},$$

Putting $\lambda = c/u_1$, we get

$$\cos(\theta - \varphi) = \frac{\sigma(s) + \lambda}{\sqrt{1 + \sigma^2(s)}} \tag{14}$$

and

$$\sin(\theta - \varphi) = \frac{\sqrt{1 - \lambda^2 - 2\lambda\sigma(s)}}{\sqrt{1 + \sigma^2(s)}}. \tag{15}$$

Considering (14) and (15) together, we obtain

$$\begin{aligned} \cos\theta &= \frac{\sigma(s) + \lambda}{\sqrt{1 + \sigma^2(s)}} \cos\varphi - \frac{\sqrt{1 - \lambda^2 - 2\lambda\sigma(s)}}{\sqrt{1 + \sigma^2(s)}} \sin\varphi, \\ \sin\theta &= \frac{\sigma(s) + \lambda}{\sqrt{1 + \sigma^2(s)}} \sin\varphi + \frac{\sqrt{1 - \lambda^2 - 2\lambda\sigma(s)}}{\sqrt{1 + \sigma^2(s)}} \cos\varphi. \end{aligned} \tag{16}$$

Thus we get

$$\begin{aligned} \alpha_1'(s) &= \sigma(s), \\ \alpha_2'(s) &= (\sigma(s) + \lambda)\cos\varphi - \left(\sqrt{1 - \lambda^2 - 2\lambda\sigma(s)}\right)\sin\varphi, \\ \alpha_3'(s) &= (\sigma(s) + \lambda)\sin\varphi + \left(\sqrt{1 - \lambda^2 - 2\lambda\sigma(s)}\right)\cos\varphi. \end{aligned} \tag{17}$$

We know that $k_1(s) = \|\alpha''(s)\|$. If we use the equalities given in (17), we obtain

$$k_1^2(s) = \frac{(\sigma'(s))^2 \lambda^2}{1 - \lambda^2 - 2\lambda\sigma(s)}$$

and

$$\varepsilon_0 \frac{k_1(s)}{\lambda} = \frac{\sigma'(s)}{\sqrt{1 - \lambda^2 - 2\lambda\sigma(s)}} \tag{18}$$

where $\varepsilon_0 = \pm 1$. By integrating (18), we find

$$\sigma(s) = \frac{-1}{2\lambda}\left[\left(\varepsilon_0 \int k_1(s)ds + \lambda c_0\right)^2 + \lambda^2 - 1\right]$$

where $c_0 \in \mathbb{R}$. By integrating (17), we find

$$\begin{aligned} \alpha_1(s) &= \int \sigma(s)ds + c_1, \\ \alpha_2(s) &= \cos\varphi \int \sigma(s)ds + s\lambda\cos\varphi - \sin\varphi \int \sqrt{1 - \lambda^2 - 2\lambda\sigma(s)}ds + c_2, \\ \alpha_3(s) &= \sin\varphi \int \sigma(s)ds + s\lambda\sin\varphi + \cos\varphi \int \sqrt{1 - \lambda^2 - 2\lambda\sigma(s)}ds + c_3, \end{aligned}$$

where $c_0, c_1, c_2, c_3, \varphi \in \mathbb{R}$, $\lambda \in \mathbb{R}/\{0\}$. This completes the proof of the theorem. □

Case 1.2. Let $\alpha(s)$ be a spacelike curve in $\mathbb{E}_1^3$ with a timelike principal normal vector field N. Then we give the following theorem:

Theorem 3.3. *There exists no unit speed spacelike general helix in $\mathbb{E}_1^3$ with a timelike principal normal N and a constant lightlike slope axis.*

Proof. Assume that $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ is a unit speed spacelike general helix in $\mathbb{E}_1^3$ with a timelike principal normal vector field N and a constant lightlike slope axis $U = (u_1, u_2, u_3)$. Then U is given by

$$U = a(s)T(s) + b(s)N(s) + c(s)B(s) \tag{19}$$

where $a(s) = g(U, T(s)), b(s) = -g(U, N(s))$ and $c(s) = g(U, B(s))$. Since $\alpha(s)$ is a general helix, we have

$$g(U, T(s)) = a(s) = a \text{ (constant} \neq 0). \tag{20}$$

Differentiating (19) with respect to s and using (1) for $\epsilon_0 = 1, \epsilon_1 = -1$ and $\epsilon_2 = 1$, we easily obtain

$$b(s) = -g(U, N(s)) = 0, \quad c' = 0, \quad \text{and} \quad a(s)k_1(s) + c(s)k_2(s) = 0. \tag{21}$$

By using (21) , from 19, we get

$$U = a(s)T(s) - \frac{ak_1(s)}{k_2(s)}B(s).$$

Since U is a null vector, we have $g(U, U) = 0$. This gives

$$\left(\frac{k_1(s)}{k_2(s)}\right)^2 = -1,$$

which is contradiction. This completes the proof of the theorem. □

Case 1.3. Let $\alpha(s)$ be a spacelike curve in $\mathbb{E}_1^3$ with a lightlike principal normal vector field N (these curves are also known as pseudo null curves). Then we give the following theorem:

Theorem 3.4. *Let $\alpha(s)$ be a unit speed pseudo null general helix in $\mathbb{E}_1^3$ with the curvature functions $k_1(s) = 1$, $k_2(s)$. The slope axis of $\alpha(s)$ is a lightlike vector if and only if the binormal component of the slope axis is given by $g(U, B) = c_0 e^{-\int k_2(s)ds}$, $c_0 \in \mathbb{R}/\{0\}$.*

Proof. Let $\alpha(s)$ be a unit speed pseudo null general helix in $\mathbb{E}_1^3$ with the curvature functions $k_1(s) = 1, k_2(s)$. We assume that $\alpha(s)$ has constant lightlike slope axis U given by

$$U = a(s)T(s) + b(s)N(s) + c(s)B(s), \tag{22}$$

where $a(s) = g(U, T(s))$, $b(s) = g(U, B(s))$ and $c(s) = g(U, N(s))$. Since α is a helix, we have

$$g(U, T(s)) = a(s) = a \text{ (constant)}. \tag{23}$$

Differentiating (23) with respect to s and using (2), we easily obtain $c(s) = g(U, N(s)) = 0$. Also U is a lightlike vector, we get $\|U\| = a = 0$. Thus the constant null slope axis U has the following form:

$$U = b(s)N(s). \tag{24}$$

Differentiating (24) with respect to s and using (2), we obtain that $\dfrac{db(s)}{ds} + k_2(s)b(s) = 0$. Thus we get

$$b(s) = g(U, B) = c_0 e^{-\int k_2(s)ds}, \quad c_0 \in \mathbb{R}/\{0\}.$$

The converse of the proof is clear. □

In the following theorem, we obtain the parametric equation of a pseudo null general helix with a constant lightlike slope axis.

Theorem 3.5. *Let $\alpha(s)$ be a unit speed pseudo null general helix in $\mathbb{E}_1^3$ with the curvature functions $k_1(s) = 1$, $k_2(s)$ and a constant lightlike slope axis. The parametric equation of $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_2(s))$ is given by*

$$\begin{aligned}
\alpha_1(s) &= \int \sigma(s)\,ds + c_1, \\
\alpha_2(s) &= \cos\varphi \int \sigma(s)\,ds - (\sin\varphi)\,s + c_2, \\
\alpha_3(s) &= \sin\varphi \int \sigma(s)\,ds + (\cos\varphi)\,s + c_3
\end{aligned} \tag{25}$$

where $\sigma(s)$ is a differentiable function, φ, c_1, c_2, $c_3 \in \mathbb{R}$ and the lightlike slope axis $U = (1, \cos\varphi, \sin\varphi)$.

Proof. Let $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ be a unit speed pseudo null general helix in $\mathbb{E}_1^3$ with the curvature functions $k_1(s) = 1$, $k_2(s)$. We assume that $\alpha(s)$ has constant lightlike slope axis $U = (u_1, u_2, u_3)$. Then we find

$$-\left(\alpha_1'(s)\right)^2 + \left(\alpha_2'(s)\right)^2 + \left(\alpha_3'(s)\right)^2 = 1. \tag{26}$$

and

$$-u_1^2 + u_2^2 + u_3^3 = 0. \tag{27}$$

Putting $\sigma(s) = \alpha_1'(s)$ in (26), we get

$$\begin{aligned}
\alpha_2'(s) &= \sqrt{1 + \sigma^2(s)}\cos\theta(s), \\
\alpha_3'(s) &= \sqrt{1 + \sigma^2(s)}\sin\theta(s),
\end{aligned}$$

where $\theta(s)$ is a differentiable function. From (27), we have

$$\begin{aligned}
u_2 &= u_1 \cos\varphi, \\
u_3 &= u_1 \sin\varphi
\end{aligned}$$

where $\varphi \in \mathbb{R}$. Since $g(U, T) = 0$, then

$$-u_1\alpha_1'(s) + u_2\alpha_2'(s) + u_3\alpha_3'(s) = 0$$

which implies that

$$\begin{aligned}
\cos(\theta(s) - \varphi) &= \frac{\sigma(s)}{\sqrt{1 + \sigma^2(s)}}, \\
\sin(\theta(s) - \varphi) &= \frac{1}{\sqrt{1 + \sigma^2(s)}}.
\end{aligned}$$

From above, we find

$$\begin{aligned}
\cos\theta(s) &= \frac{\sigma(s)}{\sqrt{1 + \sigma^2(s)}}\cos\varphi - \frac{1}{\sqrt{1 + \sigma^2(s)}}\sin\varphi, \\
\sin\theta(s) &= \frac{\sigma(s)}{\sqrt{1 + \sigma^2(s)}}\sin\varphi + \frac{1}{\sqrt{1 + \sigma^2(s)}}\cos\varphi.
\end{aligned}$$

Then

$$\begin{aligned}\alpha_1'(s) &= \sigma(s),\\ \alpha_2'(s) &= \sigma(s)\cos\varphi - \sin\varphi,\\ \alpha_3'(s) &= \sigma(s)\sin\varphi + \cos\varphi.\end{aligned}$$

Differentiating above equations, we obtain

$$\begin{aligned}\alpha_1''(s) &= \sigma'(s),\\ \alpha_2''(s) &= \sigma'(s)\cos\varphi,\\ \alpha_3''(s) &= \sigma'(s)\sin\varphi,\end{aligned}$$

which implies that $-\left(\alpha_1''(s)\right)^2+\left(\alpha_2''(s)\right)^2+\left(\alpha_3''(s)\right)^2=0$. Then the parametric equation of a pseudo null general helix with a lightlike slope axis can be given as

$$\begin{aligned}\alpha_1(s) &= \int \sigma(s)\,ds + c_1,\\ \alpha_2(s) &= \cos\varphi\int \sigma(s)\,ds - (\sin\varphi)\,s + c_2,\\ \alpha_3(s) &= \sin\varphi\int \sigma(s)\,ds + (\cos\varphi)\,s + c_3\end{aligned}$$

where $c_1, c_2, c_3 \in \mathbb{R}$. Also the lightlike slope axis $U = (1, \cos\varphi, \sin\varphi)$. □

Corollary 3.6. *Let $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_2(s))$ be a pseudo null general helix in $\mathbb{E}_1^3$ parametrized by* (25) *. Then*

$$k_2(s) = \frac{\sigma''(s)}{\sigma'(s)}.$$

Case 2. Let $\alpha(s)$ be a timelike curve in $\mathbb{E}_1^3$. Then we give the following theorems without their proofs since they are similar to above proofs.

Theorem 3.7. *Let $\alpha(s)$ be a unit speed timelike general helix in $\mathbb{E}_1^3$ with the curvature functions $k_1(s)$, $k_2(s)$. The slope axis of α is a lightlike vector if and only if $|k_1| = |k_2|$.*

Theorem 3.8. *Let $\alpha(s)$ be a unit speed timelike general helix in $\mathbb{E}_1^3$ with the curvature functions $k_1(s)$, $k_2(s)$ and a constant lightlike slope axis. The parametric equation of $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_2(s))$ is given by*

$$\begin{aligned}\alpha_1(s) &= \int \sigma(s)ds + c_1,\\ \alpha_2(s) &= \cos\varphi\int \sigma(s)ds + s\lambda\cos\varphi - \sin\varphi\int \sqrt{-1-\lambda^2-2\lambda\sigma(s)}ds + c_2,\\ \alpha_3(s) &= \sin\varphi\int \sigma(s)ds + s\lambda\sin\varphi + \cos\varphi\int \sqrt{-1-\lambda^2-2\lambda\sigma(s)}ds + c_3,\end{aligned}$$

and the constant lightlike slope axis U is given by $U = (1, \cos\varphi, \sin\varphi)$ where

$$\sigma(s) = \frac{-1}{2\lambda}\left[\left(\varepsilon_0\int k_1(s)ds + \lambda c_0\right)^2 + \lambda^2 + 1\right],$$

and $c_0, c_1, c_2, c_3, \varphi \in \mathbb{R}$, $\lambda \in \mathbb{R}/\{0\}$.

Case 3. Let $\alpha(s)$ be a Cartan null curve in $\mathbb{E}_1^3$. Then we give the following theorems:

Theorem 3.9. *Let $\alpha(s)$ be a null Cartan general helix in $\mathbb{E}_1^3$ parameterized by pseudo arc s with the curvature functions $k_1 = 1$ and k_2. The slope axis of $\alpha(s)$ is a lightlike vector if and only if $k_2 = 0$.*

Therefore, the slope axis of a null Cartan general helix is a lightlike vector if and only if it is the generalized null cubic.

Theorem 3.10. *Let $\alpha(s)$ be a null Cartan general helix in $\mathbb{E}_1^3$ parameterized by pseudo arc s with a constant lightlike slope axis. The parametric equation of $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_2(s))$ is given by*

$$\begin{aligned}
\alpha_1(s) &= -\frac{c}{2u_1}s - \frac{u_1}{6c}s^3 + c_1,\\
\alpha_2(s) &= \left(\frac{c}{2u_1}s - \frac{u_1}{6c}s^3\right)\cos\varphi + \frac{m}{2}s^2\sin\varphi + c_2,\\
\alpha_3(s) &= \left(\frac{c}{2u_1}s - \frac{u_1}{6c}s^3\right)\sin\varphi - \frac{m}{2}s^2\cos\varphi + c_3
\end{aligned}$$

and the lightlike slope axis U is given by $U = (u_1, u_1\cos\varphi, u_1\sin\varphi)$ where c_1, c_2, c_3, c, $u_1 \in \mathbb{R}$ and $m = |cu_1| / cu_1$.

Proof. Let $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ be a null Cartan general helix in $\mathbb{E}_1^3$ parameterized by pseudo arc s with the curvature functions $k_1(s) = 1$, $k_2(s)$. We assume that $\alpha(s)$ has constant lightlike slope axis $U = (u_1, u_2, u_3)$. Then we find

$$-\left(\alpha_1'(s)\right)^2 + \left(\alpha_2'(s)\right)^2 + \left(\alpha_3'(s)\right)^2 = 0 \tag{28}$$

and

$$-u_1^2 + u_2^2 + u_3^3 = 0. \tag{29}$$

Putting $\sigma(s) = \alpha_1'(s)$ in (28), we get

$$\begin{aligned}
\alpha_2'(s) &= \sigma(s)\cos\theta(s),\\
\alpha_3'(s) &= \sigma(s)\sin\theta(s),
\end{aligned}$$

where $\theta(s)$ is a differentiable function. From (29), we have

$$\begin{aligned}
u_2 &= u_1\cos\varphi,\\
u_3 &= u_1\sin\varphi
\end{aligned}$$

where $\varphi \in \mathbb{R}$. Since $g(U, T) = c$ (constant), then

$$-u_1\alpha_1'(s) + u_2\alpha_2'(s) + u_3\alpha_3'(s) = c$$

which implies that

$$\begin{aligned}
\cos(\theta(s) - \varphi) &= \frac{c + \sigma(s)u_1}{\sigma(s)u_1},\\
\sin(\theta(s) - \varphi) &= \frac{m_1\sqrt{-2cu_1\sigma(s) - c^2}}{\sigma(s)u_1}
\end{aligned}$$

where $m_1 = |\sigma(s)u_1| / \sigma(s)u_1$. From above, we find

$$\begin{aligned}
\cos\theta(s) &= \frac{c + \sigma(s)u_1}{\sigma(s)u_1}\cos\varphi - \frac{m_1\sqrt{-2cu_1\sigma(s) - c^2}}{\sigma(s)u_1}\sin\varphi,\\
\sin\theta(s) &= \frac{c + \sigma(s)u_1}{\sigma(s)u_1}\sin\varphi + \frac{m_1\sqrt{-2cu_1\sigma(s) - c^2}}{\sigma(s)u_1}\cos\varphi.
\end{aligned}$$

Then

$$\begin{aligned}
\alpha_1'(s) &= \sigma(s), \\
\alpha_2'(s) &= \frac{c+\sigma(s)\,u_1}{u_1}\cos\varphi - \frac{m_1\sqrt{-2cu_1\sigma(s)-c^2}}{u_1}\sin\varphi, \\
\alpha_3'(s) &= \frac{c+\sigma(s)\,u_1}{u_1}\sin\varphi + \frac{m_1\sqrt{-2cu_1\sigma(s)-c^2}}{u_1}\cos\varphi.
\end{aligned}$$

Differentiating above equations, we obtain

$$\begin{aligned}
\alpha_1''(s) &= \sigma'(s), \\
\alpha_2''(s) &= \sigma'(s)\cos\varphi + \frac{cm_1\sigma'(s)}{\sqrt{-2cu_1\sigma(s)-c^2}}\sin\varphi, \\
\alpha_3''(s) &= \sigma'(s)\sin\varphi - \frac{cm_1\sigma'(s)}{\sqrt{-2cu_1\sigma(s)-c^2}}\cos\varphi.
\end{aligned}$$

Since $-\left(\alpha_1''(s)\right)^2 + \left(\alpha_2''(s)\right)^2 + \left(\alpha_3''(s)\right)^2 = 1$, we find

$$\sigma(s) = -\frac{c^2+u_1^2s^2}{2cu_1}.$$

Then the parametric equation of a null helix with a constant lightlike slope axis can be given as

$$\begin{aligned}
\alpha_1(s) &= -\frac{c}{2u_1}s - \frac{u_1}{6c}s^3 + c_1, \\
\alpha_2(s) &= \left(\frac{c}{2u_1}s - \frac{u_1}{6c}s^3\right)\cos\varphi + \frac{m}{2}s^2\sin\varphi + c_2, \\
\alpha_3(s) &= \left(\frac{c}{2u_1}s - \frac{u_1}{6c}s^3\right)\sin\varphi - \frac{m}{2}s^2\cos\varphi + c_3
\end{aligned}$$

where $m = |cu_1|/cu_1$, $c_1, c_2, c_3 \in \mathbb{R}$. Also the lightlike slope axis $U = (1, \cos\varphi, \sin\varphi)$. □

4. Relation Between Biharmonic Curves and Helices with Lightlike Slope Axis

It is well-known that a unit speed curve $\gamma : I \to M$ in a Lorentz 3-manifold M is said to be biharmonic if $\Delta H = 0$, where H is the mean curvature vector field. If M is the semi-Euclidean 3-space, then γ is biharmonic if and only if $\Delta\Delta\gamma = 0$ [19]. Chen and Ishikawa [9] classified biharmonic curves in semi-Euclidean space $\mathbb{E}_v^n$ and they showed that every biharmonic curve lies in a three dimensional totally geodesic subspace. All kinds of biharmonic curves in Minkowski 3-space were classified by Inoguchi [19, 20]. He has shown that every biharmonic curve in $\mathbb{E}_1^3$ is a helix whose curvature k_1 and torsion k_2 satisfy $k_1^2 = k_2^2$.

Corollary 4.1. *[19, 20] Let γ be a Frenet curve in a Lorentz 3-manifold. Then, γ is a nongeodesic biharmonic curve if and only if one of the following holds:*
(i) γ is a spacelike helix with a spacelike principal normal such that $k_1 = \pm k_2$(=constant).
(ii) γ is a spacelike helix with a lightlike principal normal such that $k_2' + k_2^2 = 0$.
(iii) γ is a timelike helix such that $k_1 = \pm k_2$(=constant).

In this section, we show that there is an interesting relation between biharmonic curves and helix with lightlike axis. We give the following corollaries to show the relations.

Corollary 4.2. *Let α be a spacelike helix in $\mathbb{E}_1^3$ with a spacelike principal normal. Then the slope axis of α is a lightlike vector if and only if α is a biharmonic curve in $\mathbb{E}_1^3$.*

Proof. By assuming that k_1 and k_2 are constants, from theorem 3.1 and corollary 4.1, the proof is clear. □

Corollary 4.3. *Let α be a timelike helix in $\mathbb{E}_1^3$. Then the slope axis of α is a lightlike vector if and only if α is a biharmonic curve in $\mathbb{E}_1^3$.*

Proof. By assuming that k_1 and k_2 are constants, from theorem 3.7 and corollary 4.1, the proof is clear. □

In the following corollary, we obtain the parametric equation of a bihormanic helix with lightlike slope axis

Corollary 4.4. *Let $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_2(s))$ be a pseudo null helix in $\mathbb{E}_1^3$ parametrized by* (25) *. Then α is biharmonic if and only if the function $\sigma(s)$ in theorem 3.5 satisfies $\sigma(s) = As^2 + Bs + C$ where $A, B, C \in \mathbb{R}$ and $(A, B, C) \neq (0, 0, 0)$.*

Proof. α is biharmonic if and only if $k_2'(s) + (k_2(s))^2 = 0$ if and only if $\sigma(s) = As^2 + Bs + C$ where $A, B, C \in \mathbb{R}$ and $(A, B, C) \neq (0, 0, 0)$. □

5. Examples

In this section, we give some helix examples for the above theorems.

Example 5.1. *Taking $c_0 = c_1 = c_2 = c_3 = 0$, $\varphi = 0$, $k_1(s) = s$ and $\varepsilon_0 = 1$, $\lambda = 1$ in theorem 3.2, we find the parametric equation of the spacelike general helix with a spacelike principal normal and a lightlike slope axis as follows*

$$\gamma_1(s) = \left(\frac{-s^5}{40}, \frac{-s^5}{40} + s, \frac{s^3}{6} \right).$$

It is easily seen that the second curvature of $\gamma_1(s)$ is $k_2(s) = s$ and its constant lightlike axis is $U = (1, 1, 0)$.

Example 5.2. *If we take $c_1 = c_2 = c_3 = 0$, $\sigma(s) = \cos s$ and $\varphi = 0$ in theorem 3.5, we find the parametric equation of pseudo null helix with a lightlike slope axis $U = (1, 1, 0)$ as follows*

$$\gamma_2(s) = (\sin s, \sin s, s).$$

It is easily seen that the second curvature of $\gamma_2(s)$ is $k_2(s) = \cot s$.
If we take $c_1 = c_2 = c_3 = 0$, $\sigma(s) = s^2$ and $\varphi = \pi/4$ in theorem 3.5, we find the parametric equation of pseudo null helix with a lightlike slope axis $U = (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ as follows

$$\gamma_3(s) = \left(\frac{s^3}{3}, \frac{1}{\sqrt{2}} \left(\frac{s^3}{3} - s \right), \frac{1}{\sqrt{2}} \left(\frac{s^3}{3} + s \right) \right).$$

It is easily seen that $k_2 = 1/s$ and $k_2' + k_2^2 = 0$ which implies that $\gamma_3(s)$ is a biharmonic pseudo null helix with a lightlike slope axis.

Example 5.3. *Putting $c_0 = c_1 = c_2 = c_3 = 0$, $\varphi = 0$, $k_1(s) = s$ and $\varepsilon_0 = 1$, $\lambda = 1$ in theorem 3.7, we find the parametric equation of the timelike general helix with a lightlike slope axis as follows*

$$\gamma_4(s) = \left(\frac{-s^5}{40} - s, \frac{-s^5}{40}, \frac{s^3}{6} \right).$$

It is easily seen that the second curvature of $\gamma_4(s)$ is $k_2(s) = s$ and its constant lightlike axis is $U = (1, 1, 0)$.

Example 5.4. *Taking $\varphi = \frac{\pi}{2}$, $c = \frac{2}{3}$, $u_1 = 1$ and $c_1 = c_2 = c_3 = 0$ in theorem 3.9, we find the parametric equation of the null helix with lightlike slope axis as follows*

$$\gamma_5(s) = \left(-\frac{s^3}{4} - \frac{s}{3}, \frac{s^2}{2}, -\frac{s^3}{4} + \frac{s}{3} \right).$$

It is easily seen that the second curvature of $\gamma_5(s)$ is $k_2(s) = 0$ and its constant lightlike axis is $U = (1, 0, 1)$.

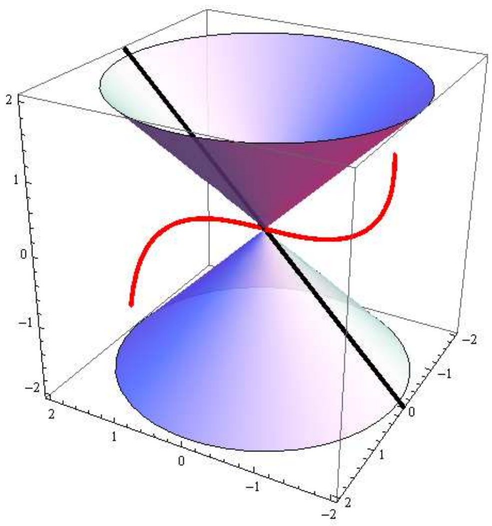

Figure 1: The graphic is for the curve $\gamma_1(s)$.

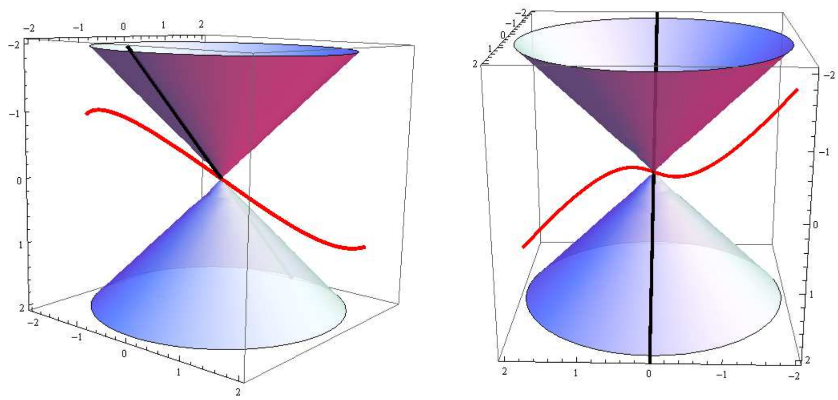

Figure 2: The graphic on the left is for the curve $\gamma_2(s)$ and the graphic on the right is for the curve $\gamma_3(s)$.

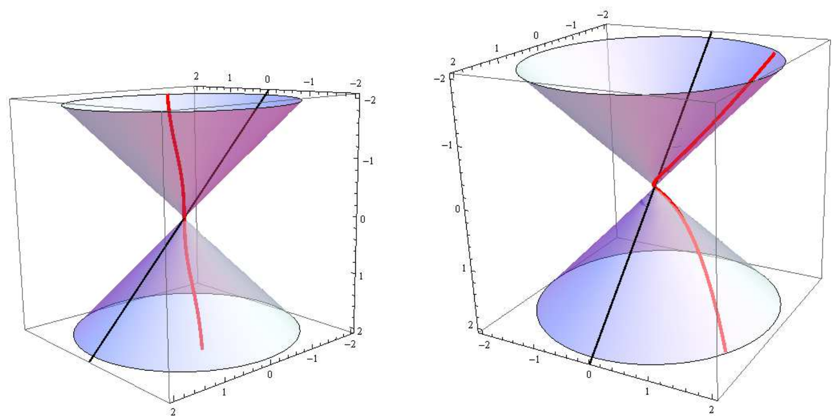

Figure 3: The graphic on the left is for the curve $\gamma_4(s)$ and the graphic on the right is for the curve $\gamma_5(s)$.

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