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Quantitative Voronovskaya Type Theorems For A General Sequence of Linear Positive Operators

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Abstract. The present paper deal with the obtaining quantitative form of the results presented Butzer & Karsli [1]. That is, we prove quantitative simultaneous results by general sequence of positive linear operators which are valid for unbounded functions with polynomial growth. We present some applications of the general results by considering particular sequences of positive linear operators.

1. Introduction

In this paper we obtain estimates in terms of weighted modulus of continuity that allow us to give quantitative Voronovskaya type theorems in simultaneous approximation for a general sequence of linear positive operators including many classical approximation operators. With respect to quantitative Voronovskaya theorem, we cited the paper by H. Gonska and G. Tachev [4]. The authors proved quantitative Voronovskaya theorem for Bernstein operators in terms of the least concave majorant of the first order modulus of continuity. In [5] for simultaneous approximation general asymptotic and Voronovskaya theorems are obtained such that these results generalize the Voronovskaya type theorems obtained by Floater [3]. We also mention that Voronovskaya type theorems for derivatives of the Bernstein-Chlodovsky and Szász-Mirakyan operators were proved by P.L. Butzer and H. Karslı [1]. Our results are different from above in the sense that the class of operators which are considered are larger and the proved results are in quantitative form although mentioned above are also qualitative. More precisely, our results consist of Voronovskaya-type asymptotic formulas for the derivatives of a general sequence of operators and an error estimate is presented in terms of the weighted modulus of continuity of the derivatives of function approximated.

Denote by $C[0,\infty)$ set of continuous functions $f:[0,\infty)\to\mathbb{R}$. Set weight function

$$\rho(x)=\rho_m(x)=(1+x)^{-m},\ \ m\in\mathbb{N}.$$

We consider the space $C_\rho[0,\infty)=\left\{f\in C[0,\infty):\left\|f\right\|_\rho<\infty\right\}$, with

$$\left\|f\right\|_\rho=\sup_{x\in[0,\infty)}\rho(x)\left|f(x)\right|. \tag{1}$$

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Set $e_i(t) = t^i$ ($t \in \mathbb{R}, i = 0, 1, \ldots$). $C^k[0,\infty)$ is a set of function which have continuous k-th order derivatives on $[0,\infty)$. Our sequence of linear positive operators are in form $L_n : C_\rho[0,\infty) \to C^1[0,\infty)$, for which we use the notations $(L_n f)(x)$ and $L_n(f;x)$ interchangeably, has the following properties

(P1) $L_n(e_0) = e_0$

(P2) If $f \in C[0,\infty)$, then for every $x \geq 0$

$$L_n((t-x)f(t);x) = \alpha_n \varphi(x) L_n'(f;x),$$

where $\varphi(x) = ax^2 + x$ and (α_n) is a nonincreasing sequence of positive numbers converging to 0. Note that the number $a \in \mathbb{N}_0$ is fixed for sequence (L_n). (See [2]).

To estimate the rate of approximation we will use following modulus of continuity defined as

$$\omega_\psi(f;h) = \sup\left\{\left|f(x) - f(y)\right| : x \geq 0,\ y \geq 0,\ \left|x-y\right| \leq h\psi\left(\frac{x+y}{2}\right)\right\},\ h > 0,$$

where $\psi(x) = \frac{\sqrt{x}}{1+x^m}$, $x \in [0,\infty)$, $m \in \mathbb{N}$, $m \geq 2$ (see [6]).

Let us denote by $W_\psi[0,\infty)$ the space of all real valued function f such that the function $f \circ e_2$ is uniformly continuous on $[0,1]$ and the function $f \circ e_\nu$, $\nu = \frac{2}{2m+1}$ is uniformly continuous on $[1,\infty)$, where $e_\nu(x) = x^\nu$, $x \geq 0$. It is shown in [6] that if $f \in W_\psi[0,\infty)$ then

$$\lim_{h\to 0} \omega_\psi(f;h) = 0.$$

Also the modulus of continuity $\omega_\psi(f;h)$ has following property:

$$\left|f(t) - f(x)\right| \leq \left(1 + \sqrt{2}\frac{|t-x|}{h}\left(\frac{1 + \left(x + \frac{|x+t|}{2}\right)^m}{\sqrt{x}}\right)\right)\omega_\psi(f;h),$$

for $t, x \in [0,\infty)$ (see [7]).

For the function $f \in C^r[0,\infty)$, the space of r times continuously differentiable functions, the remainder in Taylor's formula at the point $x \in [0,\infty)$ is given by

$$R_r(f;t,x) = f(t) - \sum\nolimits_{k=0}^{r} \frac{f^{(k)}(x)}{k!}(t-x)^k.$$

It can be also represented by

$$R_r(f;t,x) = \frac{(t-x)^r}{r!}\left(f^{(r)}(\xi) - f^{(r)}(x)\right),$$

where ξ is a number, lying between t and x. It is shown in [7, inequality (2.4)] that

$$\left|R_r(f;t,x)\right| \leq \frac{|t-x|^r}{r!}\left(1 + \sqrt{2}\frac{|t-x|}{h}\left(\frac{1 + \left(x + \frac{|x-t|}{2}\right)^m}{\sqrt{x}}\right)\right)\omega_\psi\left(f^{(r)};h\right). \tag{2}$$

2. Auxiliary Results

We would like to mention some results which will be useful for the purposes of this paper:

Corollary 2.1. *(See Corollary 2 in [2]) For every $n \in \mathbb{N}$, $f \in C[0,\infty)$ and $x > 0$, we have*

$$(\alpha_n \varphi(x))^2 L_n''(f;x) = L_n\left(\left[(t-x)^2 - \alpha_n \varphi'(x)(t-x) - \alpha_n \varphi(x)\right]f(t);x\right). \tag{3}$$

Now we establish a recurrence formula for the moments. Then we describe recursively the moments of L_n and the images of the monomials under L_n.

Proposition 2.2. *(See Proposition 2 in [2]) For the central moments of order $m \in \mathbb{N}_0$*

$$\mu_{n,m}^{L}(x) := L_n\left((t-x)^m ; x\right)$$

one has

$$\mu_{n,0}^{L}(x) = 1,\ \ \mu_{n,1}^{L}(x) = 0$$

and for $m \geq 1$

$$\mu_{n,m+1}^{L}(x) = \alpha_n \varphi(x)\left(\left[\mu_{n,m}^{L}(x)\right]' + m\mu_{n,m-1}^{L}(x)\right). \tag{4}$$

Using (4), the following was obtained:

$$\begin{aligned}
\mu_{n,2}^{L} &= \alpha_n\varphi,\ \mu_{n,3}^{L} = \alpha_n^2\varphi\varphi', \\
\frac{\mu_{n,4}^{L}}{\alpha_n^2\varphi^2} &= 3 + 2a\alpha_n + \alpha_n\frac{\left(\varphi'\right)^2}{\varphi}, \\
\frac{\mu_{n,5}^{L}}{\alpha_n^2\varphi^2\varphi'} &= 2\alpha_n\left(5 + 4a\alpha_n\right) + \alpha_n^2\frac{\left(\varphi'\right)^2}{\varphi}, \\
\frac{\mu_{n,6}^{L}}{\alpha_n^3\varphi^3} &= \left(15 + 30a\alpha_n + 16a^2\alpha_n^2\right) + \alpha_n\left(25 + 22a\alpha_n\right)\frac{\varphi'}{\varphi} + \alpha_n^2\frac{\left(\varphi'\right)^4}{\varphi^2}
\end{aligned}$$

and, in general, it can be proved that

$$\frac{\mu_{n,2m}^{L}}{\alpha_n^m\varphi^m} = \sum\nolimits_{k=0}^{m-1}\omega_{n,m,k}\left(\frac{\left(\varphi'\right)^2}{\varphi}\right)^k,$$

where $\omega_{n,m,k}$ are positive bounded constants as $n \to \infty$.(See [2]).

3. Main Results

Theorem 3.1. *Assume that the conditions $(P1)$ and $(P2)$ hold. If $f \in C^3[0,\infty) \cap C_\rho[0,\infty)$ and $f''' \in W_\psi[0,\infty)$ then we have for $x > 0$ that*

$$\left|\alpha_n^{-1}\left[L_n'(f;x) - f'(x)\right] - \frac{\varphi'(x)}{2}f''(x) - \frac{\mu_{n,4}^{L}(x)}{6\alpha_n^2\varphi(x)}f'''(x)\right|$$

$$\leq \frac{1}{6\alpha_n^2\varphi(x)}\left[\mu_{n,4}^{L}(x) + \sqrt{\frac{\mu_{n,10}^{L}(x)}{\mu_{n,2}^{L}(x)}}\sqrt{L_n\left(\left(1 + \left(x + \frac{|x-t|}{2}\right)^m\right)^2 ; x\right)}\right]\omega_\psi\left(f''' ; \sqrt{\frac{2\mu_{n,2}^{L}(x)}{x}}\right).$$

Proof. Substituting now Taylor's formula into $(P2)$, namely

$$f(t) = \sum\nolimits_{k=0}^{3}(t-x)^k\frac{f^{(k)}(x)}{k!} + R_3(f;t,x),$$

there follows

$$\begin{aligned}\alpha_n\varphi(x)L_n'(f;x) &= L_n\left((t-x)\left(\sum\nolimits_{k=0}^{3}(t-x)^k\frac{f^{(k)}(x)}{k!}+R_3(f;t,x)\right);x\right)\\ &= \sum\nolimits_{k=1}^{3}\frac{f^{(k)}(x)}{k!}\mu_{n,k+1}^L(x)+L_n((t-x)R_3(f;t,x);x)\end{aligned}$$

and

$$\begin{aligned}\alpha_n^{-1}\left[L_n'(f;x)-f'(x)\right] &= \frac{f'(x)}{\alpha_n^2\varphi(x)}\left(\mu_{n,2}^L(x)-\alpha_n\varphi(x)\right)\\ &\quad+\frac{f''(x)}{2\alpha_n^2\varphi(x)}\mu_{n,3}^L(x)+\frac{f'''(x)}{6\alpha_n^2\varphi(x)}\mu_{n,4}^L(x)\\ &\quad+\frac{1}{\alpha_n^2\varphi(x)}L_n((t-x)R_3(f;t,x);x).\end{aligned}$$

Further, in view of inequality (2) with $r=3$, we have

$$\left|R_3(f;t,x)\right|\le\frac{|t-x|^3}{3!}\left(1+\sqrt{2}\frac{|t-x|}{h}.\frac{1+\left(x+\frac{|x+t|}{2}\right)^m}{\sqrt{x}}\right)\omega_\psi\left(f''';h\right).$$

We apply the operator L_n to both sides of the last representation and estimate as follows:

$$\begin{aligned}&\left|\alpha_n^{-1}\left[L_n'(f;x)-f'(x)\right]-\frac{\varphi'(x)}{2}f''(x)-\frac{\mu_{n,4}^L(x)}{6\alpha_n^2\varphi(x)}f'''(x)\right|\\ &\le\frac{1}{6\alpha_n^2\varphi(x)}\left[\mu_{n,4}^L(x)+\frac{\sqrt{2}}{h\sqrt{x}}L_n\left(|t-x|^5\left(1+\left(x+\frac{|x-t|}{2}\right)^m\right);x\right)\right]\omega_\psi\left(f''';h\right).\end{aligned}\tag{5}$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\left|\alpha_n^{-1}\left[L_n'(f;x)-f'(x)\right]-\frac{\varphi'(x)}{2}f''(x)-\frac{\mu_{n,4}^L(x)}{6\alpha_n^2\varphi(x)}f'''(x)\right|\\ \le\;&\frac{1}{6\alpha_n^2\varphi(x)}\left[\mu_{n,4}^L(x)+\frac{\sqrt{2}}{h\sqrt{x}}\sqrt{\mu_{n,10}^L(x)}\right.\\ &\left.\sqrt{L_n\left(\left(1+\left(x+\frac{|x-t|}{2}\right)^m\right)^2;x\right)}\right]\omega_\psi\left(f''';h\right).\end{aligned}$$

Choosing $h=\sqrt{\frac{2\mu_{n,2}^L(x)}{x}}$, we have

$$\begin{aligned}&\left|\alpha_n^{-1}\left[L_n'(f;x)-f'(x)\right]-\frac{\varphi'(x)}{2}f''(x)-\frac{\mu_{n,4}^L(x)}{6\alpha_n^2\varphi(x)}f'''(x)\right|\\ &\le\frac{1}{6\alpha_n^2\varphi(x)}\left[\mu_{n,4}^L(x)+\sqrt{\frac{\mu_{n,10}^L(x)}{\mu_{n,2}^L(x)}}\sqrt{L_n\left(\left(1+\left(x+\frac{|x-t|}{2}\right)^m\right)^2;x\right)}\right]\omega_\psi\left(f''';\sqrt{\frac{2\mu_{n,2}^L(x)}{x}}\right).\end{aligned}$$

□

Now we proceed with the evaluation of following term using Cauchy-Schwarz inequality:

$$\begin{aligned}
& L_n\left(|t-x|^{r+1}\left(1+\left(x+\frac{|x-t|}{2}\right)^m\right);x\right) \\
&= M_{n,r+1}^L(x)+\sum_{k=0}^m \binom{m}{k} x^{m-k}\frac{1}{2^k}M_{n,k+r+1}^L(x) \\
&\leq \sqrt{\mu_{n,4}^L(x)}\left[\sqrt{\mu_{n,2(r-1)}^L(x)}+\sum_{k=0}^m \binom{m}{k} x^{m-k}\frac{1}{2^k}\sqrt{\mu_{n,2(k+r-1)}^L(x)}\right] \\
&:= \sqrt{\mu_{n,4}^L(x)}C_{n,r,m}^L(x), \qquad (6)
\end{aligned}$$

where $M_{n,r}^L(x)=L_n\left(|t-x|^r;x\right)$ are the moments of order r. Substituting $h=\sqrt{\frac{\mu_{n,6}^L(x)}{\mu_{n,4}^L(x)}}$ in (5), for $r=4$ we have following result on quantitative Voronovskaya theorem in simultaneous approximation.

Theorem 3.2. *Assume that the conditions $(P1)$ and $(P2)$ hold. If $f\in C^3[0,\infty)\cap C_\rho[0,\infty)$ and $f'''\in W_\psi[0,\infty)$ then we have for $x>0$ that*

$$\begin{aligned}
&\left|\alpha_n^{-1}\left[L_n'(f;x)-f'(x)\right]-\frac{\varphi'(x)}{2}f''(x)-\frac{\mu_{n,4}^L(x)}{6\alpha_n^2\varphi(x)}f'''(x)\right| \\
\leq\ & \frac{1}{6}\frac{\mu_{n,4}^L(x)}{\alpha_n^2\varphi(x)}\left[1+\frac{1}{\sqrt{x}}\frac{C_{n,4,m}^L(x)}{\sqrt{\mu_{n,6}^L(x)}}\right]\omega_\psi\left(f''';\sqrt{\frac{2\mu_{n,6}^L(x)}{\mu_{n,4}^L(x)}}\right),
\end{aligned}$$

where $C_{n,4,m}^L(x)$ defined as in (6).

Remark 3.3. *It is clear that, for $3\leq n\leq m$*

$$\frac{\mu_{n,2n}^L}{\mu_{n,6}^L}=\frac{\mu_{n,2n}^L}{\alpha_n^n\varphi^n}\frac{\alpha_n^3\varphi 3}{\mu_{n,6}^L(x)}\alpha_n^{n-3}\varphi^{n-3}.$$

From Proposition 2.2, it is easy to verify that for fixed x and m, the ratio $\frac{\mu_{n,2n}^L(x)}{\mu_{n,6}^L(x)}$ is bounded when $n\to\infty$, thus $\frac{C_{n,4,m}^L(x)}{\sqrt{\mu_{n,6}^L(x)}}$ is also bounded for the operator L_n. Thus the whole expressions in front of the $\omega_\psi(f;\cdot)$ in Theorem 3.1 and Theorem 3.2 are bounded for fixed x, when $n\to\infty$.

Theorem 3.4. *Assume that the conditions $(P1)$ and $(P2)$ hold. If $f\in C^4[0,\infty)\cap C_\rho[0,\infty)$ and $f^{(4)}\in W_\psi[0,\infty)$ then we have for $x>0$ that*

$$\left|\alpha_n^{-1}\left[L_n''(f;x)-f''(x)\right]-f''(x)a-\frac{f'''(x)}{6}C_n^1(x)-\frac{f^{(4)}(x)}{24}C_n^2(x)\right|$$

$$\begin{aligned}
\leq\ & \frac{1}{24}\left\{\frac{\mu_{n,6}^L(x)}{\alpha_n^3\varphi^2(x)}+\varphi'(x)\frac{M_{n,5}^L(x)}{\alpha_n^2\varphi^2(x)}+\frac{\mu_{n,4}^L(x)}{\alpha_n^2\varphi(x)}\right. \\
&+\left(\sqrt{\frac{\mu_{n,14}^L(x)}{\alpha_n^7\varphi^5(x)}}+\varphi'(x)\sqrt{\frac{\mu_{n,12}^L(x)}{\alpha_n^5\varphi^5(x)}}+\sqrt{\frac{\mu_{n,10}^L(x)}{\alpha_n^5\varphi^3(x)}}\right) \\
&\left.\times\sqrt{L_n\left(\left(1+\left(x+\frac{|x-t|}{2}\right)^m\right)^2;x\right)}\right\}\omega_\psi\left(f^{(4)};\sqrt{\frac{2\mu_{n,2}^L(x)}{x}}\right),
\end{aligned}$$

where

$$C_n^1(x) = 6(1 + a\alpha_n)\varphi'(x) \tag{7}$$

and

$$C_n^2(x) = 2(1 + a\alpha_n)\left((6 + 8a\alpha_n)\varphi(x) + 7\alpha_n\left(\varphi'(x)\right)^2\right). \tag{8}$$

Proof. Substituting now Taylor's formula into (3), namely

$$f(t) = \sum\nolimits_{k=0}^{4} (t-x)^k \frac{f^{(k)}(x)}{k!} + R_4(f;t,x),$$

where there follows

$$\begin{aligned}
(\alpha_n\varphi(x))^2 L_n''(f;x) &= L_n\left(\left[(t-x)^2 - \alpha_n\varphi'(x)(t-x) - \alpha_n\varphi(x)\right] f(t);x\right) \\
&= L_n\left((t-x)^2\left(\sum\nolimits_{k=0}^{4}(t-x)^k\frac{f^{(k)}(x)}{k!} + R_4(f;t,x)\right);x\right) \\
&\quad -\alpha_n\varphi'(x)L_n\left((t-x)\left(\sum\nolimits_{k=0}^{4}(t-x)^k\frac{f^{(k)}(x)}{k!} + R_4(f;t,x)\right);x\right) \\
&\quad -\alpha_n\varphi(x)L_n\left(\sum\nolimits_{k=0}^{4}(t-x)^k\frac{f^{(k)}(x)}{k!} + R_4(f;t,x);x\right)
\end{aligned}$$

and consequently we have

$$\begin{aligned}
(\alpha_n\varphi(x))^2 L_n''(f;x) &= \sum\nolimits_{k=0}^{4}\frac{f^{(k)}(x)}{k!}\mu_{n,k+2}^L(x) + L_n\left((t-x)^2 R_4(f;t,x);x\right) \\
&\quad -\alpha_n\varphi'(x)\left(\sum\nolimits_{k=0}^{4}\frac{f^{(k)}(x)}{k!}\mu_{n,k+1}^L(x) + L_n((t-x)R_4(f;t,x);x)\right) \\
&\quad -\alpha_n\varphi(x)\left(\sum\nolimits_{k=0}^{4}\frac{f^{(k)}(x)}{k!}\mu_{n,k}^L(x) + L_n(R_4(f;t,x);x)\right).
\end{aligned}$$

Above equality can be arranged as following

$$\begin{aligned}
(\alpha_n\varphi(x))^2 L_n''(f;x) &= f(x)\left(\mu_{n,2}^L(x) - \alpha_n\varphi'(x)\mu_{n,1}^L(x) - \alpha_n\varphi(x)\mu_{n,0}^L(x)\right) \\
&\quad +f'(x)\left(\mu_{n,3}^L(x) - \alpha_n\varphi'(x)\mu_{n,2}^L(x) - \alpha_n\varphi(x)\mu_{n,1}^L(x)\right) \\
&\quad +\frac{f''(x)}{2}\left(\mu_{n,4}^L(x) - \alpha_n\varphi'(x)\mu_{n,3}^L(x) - \alpha_n\varphi(x)\mu_{n,2}^L(x)\right) \\
&\quad +\frac{f'''(x)}{6}\left(\mu_{n,5}^L(x) - \alpha_n\varphi'(x)\mu_{n,4}^L(x) - \alpha_n\varphi(x)\mu_{n,3}^L(x)\right) \\
&\quad +\frac{f^{(4)}(x)}{24}\left(\mu_{n,6}^L(x) - \alpha_n\varphi'(x)\mu_{n,5}^L(x) - \alpha_n\varphi(x)\mu_{n,4}^L(x)\right) \\
&\quad +L_n\left((t-x)^2 R_4(f;t,x);x\right) - \alpha_n\varphi'(x)L_n((t-x)R_4(f;t,x);x) \\
&\quad -\alpha_n\varphi(x)L_n(R_4(f;t,x);x).
\end{aligned}$$

From Proposition 2.2, we obtain

$$
\begin{aligned}
&\alpha_n^{-1}\left[L_n''(f;x)-f''(x)\right]\\
&= f''(x)\,a+\frac{f'''(x)}{6}C_n^1(x)+\frac{f^{(4)}(x)}{24}C_n^2(x)\\
&+\frac{1}{\alpha_n^3\varphi^2(x)}\left\{L_n\left((t-x)^2R_4(f;t,x);x\right)-\alpha_n\varphi'(x)L_n\left((t-x)R_4(f;t,x);x\right)\right.\\
&\left.-\alpha_n\varphi(x)L_n\left(R_4(f;t,x);x\right)\right\}, \qquad (9)
\end{aligned}
$$

where $C_n^1(x)$ and $C_n^2(x)$ defined as in (7) and (8). We evaluate each of the representations in (9).

Firstly, in view of inequality (2) with $r=4$, we have

$$
\left|R_4(f;t,x)\right|\le\frac{(t-x)^4}{4!}\left(1+\sqrt{2}\frac{|t-x|}{h}.\frac{1+\left(x+\frac{|t-x|}{2}\right)^m}{\sqrt{x}}\right)\omega_\psi\left(f^{(4)};h\right).
$$

Using Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&L_n\left((t-x)^2R_4(f;t,x);x\right)\\
&\le\frac{1}{24}\left[\mu_{n,6}^L(x)+\frac{\sqrt{2}}{h\sqrt{x}}L_n\left(|t-x|^7\left(1+\left(x+\frac{|t-x|}{2}\right)^m\right)\right)\right]\omega_\psi\left(f^{(4)};h\right)\\
&\le\frac{1}{24}\left[\mu_{n,6}^L(x)+\frac{\sqrt{2}}{h\sqrt{x}}\sqrt{\mu_{n,14}^L(x)}\times\right.\\
&\left.\sqrt{L_n\left(\left(1+\left(x+\frac{|t-x|}{2}\right)^m\right)^2;x\right)}\right]\omega_\psi\left(f^{(4)};h\right), \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
&L_n\left((t-x)R_4(f;t,x);x\right)\\
&\le\frac{1}{24}\left[M_{n,5}^L(x)+\frac{\sqrt{2}}{h\sqrt{x}}L_n\left(|t-x|^6\left(1+\left(x+\frac{|t-x|}{2}\right)^m\right)\right)\right]\omega_\psi\left(f^{(4)};h\right)\\
&\le\frac{1}{24}\left[M_{n,5}^L(x)+\frac{\sqrt{2}}{h\sqrt{x}}\sqrt{\mu_{n,12}^L(x)}\times\right.\\
&\left.\sqrt{L_n\left(1+\left(x+\frac{|t-x|}{2}\right)^m;x\right)}\right]\omega_\psi\left(f^{(4)};h\right) \qquad (11)
\end{aligned}
$$

and

$$
\begin{aligned}
L_n\left(R_4(f;t,x);x\right) &\le \frac{1}{24}\left[\mu_{n,4}^L(x)+\frac{\sqrt{2}}{h\sqrt{x}}L_n\left(|t-x|^5\left(1+\frac{|t-x|}{2}\right)^m\right)\right]\omega_\psi\left(f^{(4)};h\right)\\
&\le \frac{1}{24}\left[\mu_{n,4}^L(x)+\frac{\sqrt{2}}{h\sqrt{x}}\sqrt{\mu_{n,10}^L(x)}\times\right.\\
&\qquad\left.\sqrt{L_n\left(\left(1+\left(x+\frac{|t-x|}{2}\right)^m\right)^2;x\right)}\right]\omega_\psi\left(f^{(4)};h\right). \qquad (12)
\end{aligned}
$$

We can arrange above inequality as in following:

$$\begin{aligned}
&\Big| L_n\left((t-x)^2 R_4(f;t,x);x\right) - \alpha_n\varphi'(x) L_n((t-x)R_4(f;t,x);x) \\
&-\alpha_n\varphi(x) L_n(R_4(f;t,x);x)\Big| \\
\leq\;& \frac{1}{24}\Big\{\mu_{n,6}^L(x) + \alpha_n\varphi'(x) M_{n,5}^L(x)(x) + \alpha_n\varphi(x)\mu_{n,4}^L(x) \\
&+\frac{\sqrt{2}}{h\sqrt{x}}\left(\sqrt{\mu_{n,14}^L(x)} + \alpha_n\varphi'(x)\sqrt{\mu_{n,12}^L(x)} + \alpha_n\varphi(x)\sqrt{\mu_{n,10}^L(x)}\right) \\
&\times\sqrt{L_n\left(\left(1+\left(x+\frac{|t-x|}{2}\right)^m\right)^2;x\right)}\Bigg\}\omega_\psi\left(f^{(4)};h\right). \qquad (13)
\end{aligned}$$

Substituting (13) into (9) and choosing $h=\sqrt{\frac{2\mu_{n,2}^L(x)}{x}}$, desired result is obtained. □

Using (6) for $r=6,5$ and 4, respectively, we have

$$L_n\left(|t-x|^7\left(1+\left(x+\frac{|t-x|}{2}\right)^m\right);x\right) = \sqrt{\mu_{n,4}^L(x)}C_{n,6,m}^L(x),$$

$$L_n\left(|t-x|^6\left(1+\left(x+\frac{|t-x|}{2}\right)^m\right);x\right) = \sqrt{\mu_{n,4}^L(x)}C_{n,5,m}^L(x)$$

and

$$L_n\left(|t-x|^5\left(1+\left(x+\frac{|t-x|}{2}\right)^m\right);x\right) = \sqrt{\mu_{n,4}^L(x)}C_{n,4,m}^L(x).$$

Also by Cauchy Schwarz inequality we can write

$$\mu_{n,6}^L(x) \leq \sqrt{\mu_{n,4}^L(x)}\sqrt{\mu_{n,8}^L(x)}$$

and

$$M_{n,5}^L(x) \leq \sqrt{\mu_{n,4}^L(x)}\sqrt{\mu_{n,6}^L(x)}.$$

Using these inequalities in (10)-(12), we have

$$L_n\left((t-x)^2 R_4(f;t,x);x\right) \leq \frac{\sqrt{\mu_{n,4}^L(x)}}{24}\left[\sqrt{\mu_{n,8}^L(x)} + \frac{\sqrt{2}}{h\sqrt{x}}C_{n,6,m}^L(x)\right]\omega_\psi\left(f^{(4)};h\right),$$

$$L_n((t-x)R_4(f;t,x);x) \leq \frac{\sqrt{\mu_{n,4}^L(x)}}{24}\left[\sqrt{\mu_{n,6}^L(x)} + \frac{\sqrt{2}}{h\sqrt{x}}C_{n,5,m}^L(x)\right]\omega_\psi\left(f^{(4)};h\right)$$

and

$$L_n(R_4(f;t,x);x) \leq \frac{\sqrt{\mu_{n,4}^L(x)}}{24}\left[\sqrt{\mu_{n,4}^L(x)} + \frac{\sqrt{2}}{h\sqrt{x}}C_{n,4,m}^L(x)\right]\omega_\psi\left(f^{(4)};h\right).$$

Substituting $h=\sqrt{\frac{\mu_{n,6}^L(x)}{\mu_{n,4}^L(x)}}$ above equalities, from (9) we have following result on quantitative Voronovskaya theorem in simultaneous approximation.

Theorem 3.5. *Assume that the conditions* $(P1)$ *and* $(P2)$ *hold. If* $f \in C^4[0,\infty) \cap C_\rho[0,\infty)$ *and* $f^{(4)} \in W_\psi[0,\infty)$ *then we have for* $x > 0$ *that*

$$\left|\alpha_n^{-1}\left[L_n''(f;x) - f''(x)\right] + f''(x)a + \frac{f'''(x)}{6}C_n^1(x) + \frac{f^{(4)}(x)}{24}C_n^2(x)\right|$$

$$\begin{aligned}\leq\ & \frac{\mu_{n,4}^L(x)}{24\alpha_n^3\varphi^2(x)}\left\{\sqrt{\frac{\mu_{n,8}^L(x)}{\mu_{n,4}^L(x)}} + \alpha_n\varphi'(x)\sqrt{\frac{\mu_{n,6}^L(x)}{\mu_{n,4}^L(x)}} + \alpha_n\varphi(x)\right.\\ & \left. + \frac{\sqrt{2}}{\sqrt{x}}\frac{1}{\sqrt{\mu_{n,6}^L(x)}}\left(C_{n,6,m}^L(x) + \alpha_n\varphi'(x)C_{n,5,m}^L(x) + \alpha_n\varphi(x)C_{n,4,m}^L(x)\right)\right\}\\ & \times\omega_\psi\left(f^{(4)};\sqrt{\frac{\mu_{n,6}^L(x)}{\mu_{n,4}^L(x)}}\right),\end{aligned}$$

where $C_{n,r,m}^L(x)$ *defined as in (6).*

Remark 3.6. *From Proposition 2.2, it is easy to verify that for fixed* x *and* m*, all terms in front of modulus of continuity are bounded when* $n \to \infty$.

4. Examples

Starting from some examples verifying conditions $(P1)$ and $(P2)$, we give some applications of above theorems.

Example 4.1. *Szász-Mirakyan operators.*
The operators are defined as

$$S_n(f;x) = \sum_{k=0}^{\infty} e^{-nx} f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!},$$

where $x \in [0,\infty)$ *and* $n \in \mathbb{N}$*. Choosing* $\varphi(x) = x$ *and* $\alpha_n = 1/n$*, the operators satisfy* $(P1)$ *and* $(P2)$ *and the recurrence relation (4) given as following:*

$$\mu_{n,m+1}^S(x) = \frac{x}{n}\left(\left[\mu_{n,m}^S(x)\right]' + m\mu_{n,m-1}^S(x)\right).$$

It follows that

$$\mu_{n,m}^S(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right),\ n \to \infty.$$

In particular

$$\mu_{n,2}^S(x) = \frac{x}{n},\ \mu_{n,4}^S(x) = \frac{x}{n^3} + \frac{3x^2}{n^2},\ \mu_{n,6}^S(x) = \frac{x}{n^5} + \frac{25x^2}{n^4} + \frac{15x^3}{n^3}$$

and

$$\mu_{n,10}^S(x) = \frac{x}{n^9} + \frac{501x^2}{n^8} + \frac{6825x^3}{n^7} + \frac{9450x^4}{n^6} + \frac{945x^5}{n^5}.$$

It can be seen that

$$\begin{aligned}\left[1+\left(x+\frac{|t-x|}{2}\right)^m\right]^2 &= 1+2\left(x+\frac{|t-x|}{2}\right)^m+\left(x+\frac{|t-x|}{2}\right)^{2m}\\ &= 1+2\sum_{k=0}^{m}\binom{m}{k}x^k\left(\frac{|t-x|}{2}\right)^{m-k}\\ &\quad+\sum_{k=0}^{2m}\binom{2m}{k}x^k\left(\frac{|t-x|}{2}\right)^{2m-k}.\end{aligned}$$

Hence

$$\begin{aligned}S_n\left(\left[1+\left(x+\frac{|t-x|}{2}\right)^m\right]^2;x\right) &= 1+2\sum_{k=0}^{m}\binom{m}{k}x^kM^S_{n,m-k}(x)\frac{1}{2^{m-k}}\\ &\quad+2\sum_{k=0}^{2m}\binom{2m}{k}x^kM^S_{n,2m-k}(x)\frac{1}{2^{2m-k}}\\ &= A_{n,m}(x).\end{aligned} \tag{14}$$

It is verified that the term $A_{n,m}(x)$ is bounded for fixed x and m when $n\to\infty$.(See [7, eq. (3.2)*]).*

The applications of Theorem 3.1 and Theorem 3.2 for Szász-Mirakyan operators can be given as follows:

Theorem 4.2. *If $f\in C^3[0,\infty)\cap C_\rho[0,\infty)$ and $f'''\in W_\psi[0,\infty)$ then we have for $x>0$ that*

$$\begin{aligned}&\left|n\left[S_n'(f;x)-f'(x)\right]-\frac{1}{2}f''(x)-\frac{f'''(x)}{6}\left(3x+\frac{1}{n}\right)\right|\\ \leq\ &\frac{1}{6}\left[\left(3x+\frac{1}{n}\right)+\sqrt{B_{n,m}(x)}\sqrt{A_{n,m}(x)}\right]\omega_\psi\left(f''';\sqrt{\frac{2}{n}}\right),\end{aligned}$$

where $A_{n,m}(x)$ defined as in (14) and

$$B_{n,m}(x):=\frac{1}{n^4x^2}+\frac{501}{n^3x}+\frac{6825}{n^2}+\frac{9450x}{n}+945x^2.$$

Theorem 4.3. *If $f\in C^3[0,\infty)\cap C_\rho[0,\infty)$ and $f'''\in W_\psi[0,\infty)$ then we have for $x>0$ that*

$$\begin{aligned}&\left|n\left[S_n'(f;x)-f'(x)\right]-\frac{1}{2}f''(x)-\frac{f'''(x)}{6}\left(3x+\frac{1}{n}\right)\right|\\ \leq\ &\frac{1}{6}\left(3x+\frac{1}{n}\right)\left[1+\frac{1}{\sqrt{x}}\frac{C^S_{n,4,m}(x)}{\sqrt{\mu^S_{n,6}(x)}}\right]\\ &\times\omega_\psi\left(f''';\sqrt{\frac{1+5nx(5+3xn)}{n^2(1+3nx)}}\right),\end{aligned}$$

where

$$\frac{C^S_{n,4,m}(x)}{\sqrt{\mu^S_{n,6}(x)}}=1+\sum\nolimits_{k=0}^{m}\binom{m}{k}x^{m-k}\frac{1}{2^k}\sqrt{\frac{\mu^S_{n,2(k+3)}(x)}{\mu^S_{n,6}(x)}}.$$

Example 4.4. *Baskakov operators.*
The operators are defined as

$$B_n(f;x) = \sum\nolimits_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right),$$

where $x \in [0,\infty)$ *and* $n \in \mathbb{N}$. *Choosing* $\varphi(x) = x(1+x)$ *and* $\alpha_n = 1/n$, *the operators satisfy* $(P1)$ *and* $(P2)$ *and the recurrence relation (4) given as following:*

$$\mu_{n,m+1}^B(x) = \frac{x(1+x)}{n}\left(\left[\mu_{n,m}^B(x)\right]' + m\mu_{n,m-1}^B(x)\right).$$

It follows that

$$\mu_{n,m}^B(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right),\ n\to\infty. \tag{15}$$

In particular

$$\begin{aligned}
\mu_{n,2}^B(x) &= \frac{x(1+x)}{n},\ \frac{\mu_{n,4}^B(x)}{\alpha_n^2\varphi(x)} = \frac{1+3(2+n)x(1+x)}{n} = O(1),\ n\to\infty. \\
\frac{\mu_{n,6}^B(x)}{\alpha_n^3\varphi^2(x)} &= \frac{1+5(6+5n)x+5\left(30+31n+3n^2\right)x^2+10\left(24+26n+3n^2\right)x^3}{n^2x(1+x)} \\
&\quad + \frac{5\left(24+26n+3n^2\right)x^4}{n^2x(1+x)} \\
&= O(1),\ n\to\infty.
\end{aligned}$$

Similarly for $m \geq 4$, *from (15) we can write*

$$\frac{\mu_{n,2m}^B}{\alpha_n^m\varphi^m} = O(1),\ n\to\infty.$$

We apply Theorem 3.4 and obtain the proof of our next result for Baskakov operators, which states that:

Theorem 4.5. *If* $f \in C^4[0,\infty) \cap C_\rho[0,\infty)$ *and* $f^{(4)} \in W_\psi[0,\infty)$ *then we have for* $x > 0$ *that*

$$\left| n\left[B_n''(f;x) - f''(x)\right] - f''(x) - f'''(x)\left(1+\frac{1}{n}\right)(2x+1) - \frac{f^{(4)}(x)}{24}C_n^2(x)\right|$$

$$\begin{aligned}
\leq\ & \frac{1}{24}\left\{O(1) + \varphi'(x)O\left(n^{-1}\right) + O(1)\right. \\
&+\left(\sqrt{O(1)} + (1+2x)\sqrt{\frac{x(1+x)}{n}}\sqrt{O(1)} + \sqrt{x(1+x)}\sqrt{O(1)}\right) \\
&\left.\times\sqrt{A_{n,m}(x)}\right\}\omega_\psi\left(f^{(4)};\sqrt{\frac{2(1+x)}{n}}\right),
\end{aligned}$$

where $A_{n,m}(x)$ *is given by (14) with* S_n *replaced by* B_n *and*

$$C_n^2(x) = 2\left(1+\frac{1}{n}\right)\left(\left(6+\frac{8}{n}\right)x(1+x) + \frac{7}{n}(2x+1)^2\right).$$

Since

$$\frac{\mu_{n,8}^{B}(x)}{\mu_{n,4}^{B}(x)} = o(1), \text{ and } \frac{\mu_{n,6}^{B}(x)}{\mu_{n,4}^{B}(x)} = o(1), n \to \infty,$$

from Theorem 3.5 we can state that:

Theorem 4.6. *If $f \in C^4[0,\infty) \cap C_\rho[0,\infty)$ and $f^{(4)} \in W_\psi[0,\infty)$ then we have for $x > 0$ that*

$$\begin{aligned}
&\left| n\left[B_n''(f;x) - f''(x)\right] - f''(x) - f'''(x)\left(1+\frac{1}{n}\right)(2x+1) - \frac{f^{(4)}(x)}{24}C_n^2(x)\right| \\
&\leq O(1)\left\{\sqrt{o(1)} + \frac{2x+1}{n}\sqrt{o(1)} + \frac{x(x+1)}{n}\right. \\
&\left. + \frac{\sqrt{2}}{\sqrt{x}}\frac{1}{\sqrt{\mu_{n,6}^{L}(x)}}\left(C_{n,6,m}^{L}(x) + \alpha_n\varphi'(x)C_{n,5,m}^{L}(x) + \alpha_n\varphi(x)C_{n,4,m}^{L}(x)\right)\right\} \\
&\times\omega_\psi\left(f^{(4)}; \sqrt{\frac{\mu_{n,6}^{L}(x)}{\mu_{n,4}^{L}(x)}}\right),
\end{aligned}$$

where $C_{n,r,m}^{L}(x)$ defined as in (6) and

$$\begin{aligned}
\frac{\mu_{n,6}^{L}(x)}{\mu_{n,4}^{L}(x)} &= \frac{1+5(6+5n)x+5\left(30+31n+3n^2\right)x^2+10\left(24+26n+3n^2\right)x^3}{n^2(1+3(2+n)\varphi(x))} \\
&+\frac{5\left(24+26n+3n^2\right)x^4}{n^2(1+3(2+n)\varphi(x))}.
\end{aligned}$$

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