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FIXED POINT THEOREM FOR MULTIVALUED MAPPINGS SATISFYING AN IMPLICIT RELATION

DURAN TURKOGLU AND ISHAK ALTUN

Abstract. In this paper, we give a common fixed point theorem for multivalued mappings with Hausdorff metric.

1. Introduction

Throughout this paper, X stands for a metric space with the metric d whereas $CB(X)$ denotes the family of all nonempty closed bounded subsets of X . Let

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\},$$

where $A, B \in CB(X)$ and $d(x, A) = \inf\{d(x, y) : y \in A\}$. The function H is a metric on $CB(X)$ and is called Hausdorff metric. It is well known that if X is a complete metric space, then so is the metric space $(CB(X), H)$. Let $A, B \in CB(X)$ and $k > 1$. In the sequel the following well known fact will be used [4]: for each $a \in A$, there is $b \in B$ such that $d(a, b) \leq kH(A, B)$.

Kaneko and Sessa extend the definition of compatibility to include multivalued mappings in the following way.

Definition 1. ([3]) The mappings $f : X \rightarrow X$ and $S : X \rightarrow CB(X)$ are compatible if $fSx \in CB(X)$ for all $x \in X$ and

$$\lim_{n \rightarrow \infty} H(Sfx_n, fSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = A \in CB(X)$ and $\lim_{n \rightarrow \infty} fx_n = t \in A$.

Now, we consider the following conditions.

Condition 1. The mappings $f : X \rightarrow X$ and $S : X \rightarrow CB(X)$ are said to be satisfy Condition 1 iff $fSx \in CB(X)$ for all $x \in X$ and

$$\lim_{n \rightarrow \infty} H(Sfx_n, fSx_n) \leq \lim_{n \rightarrow \infty} H(Sfx_n, Sx_n)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = A \in CB(X)$ and $\lim_{n \rightarrow \infty} fx_n = t \in A$.

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Condition 2. The mappings $f : X \rightarrow X$ and $S : X \rightarrow CB(X)$ are said to be satisfy Condition 2 iff for all $x \in X$ and

$$\lim_{n \rightarrow \infty} d(ffx_n, fx_n) \leq \lim_{n \rightarrow \infty} H(Sfx_n, Sx_n)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = A \in CB(X)$ and $\lim_{n \rightarrow \infty} fx_n = t \in A$.

Remark 1. Note that, two compatible maps f and S are satisfy Condition 1.

The following example shows that f and S are satisfy Condition 1 and Condition 2 but they are not compatible.

Example 1. Let $X = [0, \infty)$ be endowed with the Euclidean metric d . Let $fx = \frac{x^2 + 2}{2}$ and $Sx = [x^2 + 1, x^2 + 2]$ for each $x \geq 0$. It is clear that f and S are continuous. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n \rightarrow \infty} fx_n = t, \lim_{n \rightarrow \infty} Sx_n = A \in CB(X),$$

then $t \in A$ if and only if $t = 1$. Indeed, if $fx_n \rightarrow t$, then $t \geq 1$ since $fx = \frac{x^2 + 2}{2} \geq 1$. Again, if $fx_n \rightarrow t$, then $x_n^2 \rightarrow 2t - 2$ and so $Sx_n \rightarrow A = [2t - 1, 2t]$.

Now, if $t \in A$, then $t \in [2t - 1, 2t]$, that is, $2t - 1 \leq t$ and so $t \leq 1$. Now, since $t \geq 1$ and $t \leq 1$, then $t = 1$. On the contrary, if $t = 1$, then $x_n \rightarrow 0$ since $fx_n \rightarrow t = 1$. Thus $Sx_n \rightarrow [1, 2] = A$ and so $t \in A$. Therefore, we have

$$0 \neq \lim_{n \rightarrow \infty} H(Sfx_n, fSx_n) = \frac{1}{2} \leq 1 = \lim_{n \rightarrow \infty} H(Sfx_n, Sx_n).$$

Thus f and S are satisfy Condition 1 but they are not compatible. On the other hand

$$\lim_{n \rightarrow \infty} d(ffx_n, fx_n) = \frac{1}{2} \leq 1 = \lim_{n \rightarrow \infty} H(Sfx_n, Sx_n).$$

Thus f and S are satisfy Condition 2.

2. Implicit relation

Implicit relations on metric spaces have been used in many articles (see [2] [5] [6] [8]).

Let $\mathcal{F}$ be the set of all continuous functions $F : R_+^6 \rightarrow R_+$ satisfying the following conditions:

$F_1 : F(t_1, \dots, t_6)$ is decreasing in $t_2, \dots, t_6$.

F_2 : there exist an increasing function $f : R_+ \rightarrow R_+$, $f(0) = 0$ and $k > 1$ with $f(s) < \frac{s}{k}$ such that the inequality

$(F_a) : u \leq kt$ and $F(t, v, v, u, u + v, 0) \leq 0$ or

$(F_b) : u \leq kt$ and $F(t, v, u, v, 0, u + v) \leq 0$ implies $t \leq f(v)$.

$F_3 : F(u, u, u, 0, u, u) > 0$ and $F(u, u, 0, u, u, u) > 0, \forall u > 0$.

Remark 2. Note that, if we choose $f(s) = hs$ with $hk < 1$ in F_2 , we obtain implicit relation of Popa [6]

Example 2. $F(t_1, \dots, t_6) = t_1 - m \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$, where $m \in (0, 1)$.

F_1 : Obviously.

F_2 : Let $u > 0$, $u \leq kt$ and $F(t, v, v, u, u + v, 0) = t - m \max\{u, v\} \leq 0$, where $1 < k < \frac{1}{m}$. If $u \geq v$ then $u \leq kt \leq km u < u$, a contradiction. Thus $u > v$ and $t \leq mv$. Similarly, $u > 0$, $u \leq kt$ and $F(t, v, u, v, 0, u + v) \leq 0$ imply $t \leq mv$. If $u = 0$, then $u \leq v$ and $t \leq mv$. Thus F_2 is satisfying with $f(s) = ms$.

F_3 : $F(u, u, u, 0, u, u) = F(u, u, 0, u, u, u) = u - mu > 0, \forall u > 0$.

Example 3. $F(t_1, \dots, t_6) = t_1 - \phi(\max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\})$, where $\phi : R_+ \rightarrow R_+$ increasing, $\phi(0) = 0$ and $k > 1$ with $\phi(s) < \frac{s}{k}$ for $s > 0$.

F_1 : Obviously.

F_2 : Let $u > 0$, $u \leq kt$ and $F(t, v, v, u, u + v, 0) = t - \phi(\max\{u, v\}) \leq 0$. If $u \geq v$, then $u \leq kt \leq k\phi(u) < u$, a contradiction. Thus $u < v$ and $t \leq \phi(v)$. Similarly $u > 0$, $u \leq kt$ and $F(t, v, u, v, 0, u + v) \leq 0$ imply $t \leq \phi(v)$. If $u = 0$, then $u \leq \phi(v)$. Thus F_2 is satisfying with $f = \phi$.

F_3 : $F(u, u, u, 0, u, u) = F(u, u, 0, u, u, u) = u - \phi(u) > 0, \forall u > 0$.

3. Main Result

We need the following lemma for the proof of our main theorem.

Lemma 1. [(7)] *For any $t > 0, \gamma(t) < t$ if and only if $\lim_{n \rightarrow \infty} \gamma^n(t) = 0$, where γ^n denotes the composition of γ n -times with itself.*

Now we give our main theorem.

Theorem 1. *Let (X, d) be a complete metric space. Let $f, g : X \rightarrow X$ and $S, T : X \rightarrow CB(X)$ be a continuous mapping such that f and S as well as g and T satisfy Condition 1 and Condition 2. Assume $T(X) \subseteq f(X)$, $S(X) \subseteq g(X)$ and for all $x, y \in X$*

$$F(H(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)) \leq 0, \quad (3.1)$$

where $F \in \mathcal{F}$. Then f, g, S and T have a common fixed point.

Proof. Let x_0 be arbitrary point in X . We shall construct two sequences $\{x_n\}$ and $\{y_n\}$ of elements in X and sequence $\{A_n\}$ of elements in $CB(X)$. Since $S(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $y_1 = gx_1 \in Sx_0$. Then there exists an element $y_2 = fx_2 \in Tx_1 = A_1$, because $T(X) \subseteq f(X)$, such that

$$d(y_1, y_2) = d(gx_1, fx_2) \leq kH(Sx_0, Tx_1).$$

Since $S(X) \subseteq g(X)$, we may choose $x_3 \in X$ such that $y_3 = gx_3 \in Sx_2 = A_2$ and

$$d(y_2, y_3) \leq kH(Tx_1, Sx_2).$$

By induction we produce the sequences $\{x_n\}$, $\{y_n\}$ and $\{A_n\}$ such that

$$y_{2n+1} = gx_{2n+1} \in Sx_{2n} = A_{2n}, \quad (3.2)$$

$$y_{2n+2} = f x_{2n+2} \in T x_{2n+1} = A_{2n+1}, \quad (3.3)$$

$$d(y_{2n+1}, y_{2n}) \leq kH(Sx_{2n}, Tx_{2n-1}), \quad (3.4)$$

$$d(y_{2n+1}, y_{2n+2}) \leq kH(Sx_{2n}, Tx_{2n+1}) \quad (3.5)$$

for every $n \in N$. Letting $x = x_{2n}, y = x_{2n+1}$ in (3.1), we have successively

$$\begin{aligned} &F(H(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, gx_{2n+1}), d(fx_{2n}, Sx_{2n}), \\ &d(gx_{2n+1}, Tx_{2n+1}), d(fx_{2n}, Tx_{2n+1}), d(gx_{2n+1}, Sx_{2n})) \leq 0, \end{aligned}$$

and so

$$\begin{aligned} &F(H(Sx_{2n}, Tx_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), \\ &d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+2}), d(y_{2n+1}, y_{2n+1})) \leq 0. \end{aligned}$$

Thus

$$\begin{aligned} &F(H(Sx_{2n}, Tx_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), \\ &d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}), 0) \leq 0. \end{aligned} \quad (3.6)$$

From (3.5), (3.6) and (F_a) , there exist an increasing function $f : R_+ \rightarrow R_+, f(0) = 0$ and $k > 1$ with $f(s) < \frac{s}{k}$, we have

$$H(Sx_{2n}, Tx_{2n+1}) = H(A_{2n}, A_{2n+1}) \leq f(d(y_{2n}, y_{2n+1})) \quad (3.7)$$

and so

$$d(y_{2n+1}, y_{2n+2}) \leq kf(d(y_{2n}, y_{2n+1})). \quad (3.8)$$

Similarly we obtain

$$H(Sx_{2n}, Tx_{2n-1}) = H(A_{2n}, A_{2n-1}) \leq f(d(y_{2n-1}, y_{2n})) \quad (3.9)$$

and so

$$d(y_{2n}, y_{2n+1}) \leq kf(d(y_{2n-1}, y_{2n})). \quad (3.10)$$

Since $kf(s) < s$ it follows from (3.8), (3.10) and Lemma 1 that $\{y_n\}$ is a Cauchy sequence. Hence there exists $z \in X$ such that $y_n \rightarrow z$. Therefore, $gx_{2n+1} \rightarrow z$ and $fx_{2n} \rightarrow z$. Also from (3.7) and (3.9) and the fact that $\{y_n\}$ is Cauchy sequence it follows that $\{A_k\}$ is Cauchy sequence in the complete metric space $(CB(X), H)$. Thus $A_k \rightarrow A \in CB(X)$. This implies $Tx_{2n+1} \rightarrow A$ and $Sx_{2n} \rightarrow A$ and therefore $z \in A$, because

$$d(z, A) = \lim_{n \rightarrow \infty} d(y_n, A) \leq \lim_{n \rightarrow \infty} H(A_{n-1}, A_n) = 0.$$

Since A is closed, $t \in A$ and f and S are satisfying Condition 1 and Condition 2 implies that

$$\lim_{n \rightarrow \infty} H(fSx_{2n}, Sfx_{2n}) \leq \lim_{n \rightarrow \infty} H(Sx_{2n}, Sfx_{2n})$$

and

$$\lim_{n \rightarrow \infty} d(ffx_{2n}, fx_{2n}) \leq \lim_{n \rightarrow \infty} H(Sx_{2n}, Sfx_{2n}).$$

This along with the continuity of f and S imply that

$$H(fA, Sz) \leq H(A, Sz) \quad (3.11)$$

and

$$d(fz, z) \leq H(A, Sz). \quad (3.12)$$

Now

$$\begin{aligned} d(fz, Sz) &\leq d(fz, fgx_{2n+1}) + d(fgx_{2n+1}, Sz) \\ &\leq d(fz, fgx_{2n+1}) + H(fSx_{2n}, Sz) \\ &\leq d(fz, fgx_{2n+1}) + H(fSx_{2n}, Sfx_{2n}) + H(Sfx_{2n}, Sz) \end{aligned}$$

and letting $n \rightarrow \infty$ we have

$$d(fz, Sz) \leq \lim_{n \rightarrow \infty} H(fSx_{2n}, Sfx_{2n}) \leq H(A, Sz).$$

Now using (3.1) we have

$$\begin{aligned} &F(H(Sz, Tx_{2n+1}), d(fz, gx_{2n+1}), d(fz, Sz), \\ &d(gx_{2n+1}, Tx_{2n+1}), d(fz, Tx_{2n+1}), d(gx_{2n+1}, Sz)) \leq 0, \end{aligned}$$

and

$$\begin{aligned} &F(H(Sz, Tx_{2n+1}), d(fz, gx_{2n+1}), d(fz, Sz), \\ &d(gx_{2n+1}, Tx_{2n+1}), d(fz, fx_{2n+2}) + d(fx_{2n+2}, Tx_{2n+1}), d(gx_{2n+1}, Sz)) \leq 0. \end{aligned}$$

Letting $n \rightarrow \infty$ we obtain

$$F(H(Sz, A), d(fz, z), d(fz, Sz), d(z, A), d(fz, z) + d(z, A), d(z, Sz)) \leq 0$$

and so

$$F(H(Sz, A), H(Sz, A), H(Sz, A), 0, H(Sz, A), H(Sz, A)) \leq 0$$

which is a contradiction to F_3 . Thus $H(Sz, A) = 0$ and so, from (3.11) and (3.12) we have $z = fz \in Sz$. Similarly we have $z = gz \in Tz$. Thus z is a common fixed point of this four mappings.

Corollary 1. *Let (X, d) be a complete metric space. Let $f, g : X \rightarrow X$ and $S, T : X \rightarrow CB(X)$ be a continuous mapping such that f and S as well as g and T satisfy Condition 1 and Condition 2. Assume $T(X) \subseteq f(X)$, $S(X) \subseteq g(X)$ and that $H(Sx, Ty) \leq rd(fx, gy)$ for all $x, y \in X$ where $0 < r < 1$. Then f, g, S and T have a common fixed point.*

Example 4. Let $X = [0, 1]$ be endowed with the Euclidean metric d . Let $fx = \frac{x}{2}$, $gx = \frac{x^2}{3}$, $Sx = [0, \frac{x}{4}]$ and $Tx = [0, \frac{x^2}{6}]$. It is clear that these mappings are continuous and $S(X) = [0, \frac{1}{4}] \subset$

$[0, \frac{1}{2}] = f(X)$ and $T(X) = [0, \frac{1}{6}] \subset [0, \frac{1}{3}] = g(X)$. Also f and S as well as g and T are satisfy Condition 1 and Condition 2. Indeed, let $\{x_n\}$ be a sequence in X such that

$$\lim_{n \rightarrow \infty} f x_n = t, \lim_{n \rightarrow \infty} S x_n = A \in CB(X),$$

then $t \in A$ if and only if $t = 0$. Therefore,

$$\lim_{n \rightarrow \infty} H(S f x_n, f S x_n) = \lim_{n \rightarrow \infty} d(f f x_n, f x_n) = 0 = \lim_{n \rightarrow \infty} H(S f x_n, S x_n).$$

Thus f and S are satisfy Condition 1 and Condition 2. Again, for all $x, y \in X$, we have

$$\begin{aligned} H(Sx, Ty) &= H([0, \frac{x}{4}], [0, \frac{y^2}{6}]) \\ &= \left| \frac{x}{4} - \frac{y^2}{6} \right| \\ &= \frac{1}{2} \left| \frac{x}{2} - \frac{y^2}{3} \right| \\ &= \frac{1}{2} d(fx, gy). \end{aligned}$$

Consequently, these mappings are satisfy all conditions of Corollary 1, then they have a common fixed point in X .

Theorem 2. [(1)] *Let (X, d) be a complete metric space. Let $f, g : X \rightarrow X$ and $S, T : X \rightarrow CB(X)$ be a continuous mapping such that f is compatible with S and g is compatible with T . Assume $T(X) \subseteq f(X)$, $S(X) \subseteq g(X)$ and that $H(Sx, Ty) \leq r d(fx, gy)$ for all $x, y \in X$ where $0 < r < 1$. Then there is a coincidence point for f and S , as g and T .*

Remark 3. In Theorem 2, Azam and Beg [1] find a coincidence point of mappings assumed compatibility. In Corollary 1, we find a common fixed point of mappings with Condition 1 and Condition 2 replaced by compatibility.

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