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Heat Transfer to a Power-Law Fluid in Arbitrary Cross-Sectional Ducts

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Abstract

Numerical solutions for laminar heat transfer of a non-Newtonian fluid in the thermal entrance region for triangular, square, sinusoidal, etc. ducts are presented for constant wall temperature. The continuity equation and parabolic forms of the energy and momentum equations in Cartesian coordinates are transformed by the elliptic grid generation technique into new non-orthogonal coordinates with the boundary of the duct coinciding with the coordinate surface. The effects of axial heat conduction, viscous dissipation and thermal energy sources within the fluid are neglected. The transformed equations are solved by the finite difference technique. As an application of the method, flow and heat transfer results are presented for ducts with triangular, square, sinusoidal and four-cusped cross sections and square cross sections with four indented corners. The results are compared with the results of previous works.

Key Words: Heat Transfer, Laminar Flow, Non-Newtonian, Elliptic ducts

Introduction

In food, polymer, petrochemical, rubber, paint and biological industries, fluids with non-Newtonian behavior are encountered. The investigation of heat transfer problems non-Newtonian fluids' heating and cooling in heat exchangers can have economic benefits. The most common heat exchangers employed in these industries use circular and rectangular ducts, which are easy to maintain. The other arbitrarily shaped cross-section ducts are not preferred, because of the difficulty of manufacturing and cleaning. It is important to have knowledge of the characteristics of the forced convective heat transfer in steady laminar non-Newtonian flow through ducts with arbitrarily shaped cross-sections in order to exercise proper control over the performance of the heat exchanger and to economize the process. There is insufficient research on laminar non-Newtonian fluid flow through irregular boundary ducts as compared to that on regular boundary ducts, because of the difficulty in describing non-Newtonian fluid behavior in irregular boundary ducts. Although studies of non-Newtonian fluid flow in circular, rectangular, triangular, trapezoidal and pentagonal ducts are available in the literature, sinusoidal ducts, four-cusped ducts and square ducts with four indented corners have not been studied previously.

Laminar flow solutions for Newtonian fluids were compiled by Shah and London (1971) and Porter (1971) in an exhaustive manner. While Porter considered a very general problem, the report by Shah and London is much more exhaustive in a limited area.

Shah (1975) solved the fully developed problem for Newtonian laminar heat transfer by using the Golub method Montgomery and Wilbulswas (1966) solved the thermal entry length problem for rectangular ducts by using the explicit finite difference method. Entrance region heat transfer in rhombic ducts has been studied by Asako and Faghri (1988) for a Newtonian fluid by algebraic coordinate transformation. Hydrodynamically developed channel flow and heat transfer to power-law fluids have been studied by Ashok and Sastri (1977) for a square duct under three thermal boundary conditions. Entrance region non-isothermal flow and heat transfer to power-law fluids have been studied by Lawal (1989) with rectangular coordinates transformed into new orthogonal coordinates and the finite difference technique for arbitrary cross-section ducts. The fully developed laminar flow of power-law non-Newtonian fluid in a rectangular duct has been studied by Syrjala (1995) by the finite element method. Laminar heat transfer in the entrance region of a circular duct and parallel plates has been studied by Nguyen (1992) by ADI and QUICK methods.

In this study, a computer algorithm was developed for both non-Newtonian and Newtonian fluid flow through the duct geometries mentioned above. Numerical results are presented for a square duct, a sinusoidal duct, a triangular duct, a four-cusped duct, a rhombic duct, and a square duct with four indented corners. Figures 1a-1d show the special duct geometries discussed in this study.

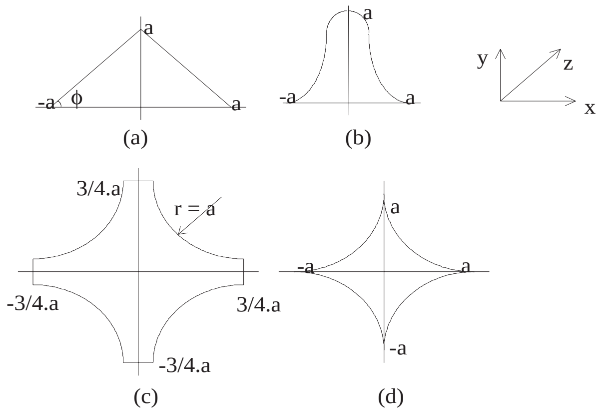

Figure 1. Ducts of arbitrary cross-sections a) triangular; b) sinusoidal; c) square duct with indented four-corners; d) four-cusped duct.

Governing Equations

In this study, steady, fully developed, laminar and purely viscous non-Newtonian fluid flow and heat transfer in power-law fluids in horizontal ducts of arbitrary cross section are studied. The duct configurations and coordinate system are shown in Figure 2. Both the velocity and temperature profiles are assumed to be uniform at the entrance, and hydrodynamically developed and thermally developing laminar flow is analyzed for a non-Newtonian fluid flow through a duct of arbitrary but constant cross section.

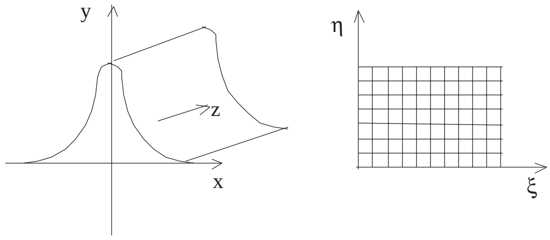

Figure 2. The transformation duct geometry from physical (x-y) plain to the computational plain $(\xi - \eta)$.

The numerical solution technique takes advantage of the marginal ellipticity of the physical problem by neglecting the axial diffusion terms in the equations of conservation of momentum and energy. The resulting equations are parabolic, and a two-dimensional computational mesh can be constructed at each of the cross sections, which are stacked together to form a three-dimensional domain. The strategy for dealing with the arbitrary shape of the duct cross section consists of transforming the physical domain into a rectangular duct using a coordinate transformation technique.

For the hydrodynamically developed and thermally developing flow, there is only one nonzero component of velocity (u), and the constitutive equations of motion reduce to a single nonlinear partial differential equation of the form

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = \frac{dp}{dz} \tag{1}$$

where (u) is the velocity component in the flow direction, (p) is pressure and (μ) is local viscosity coefficient at a point in the channel (Ashok, 1977). The power-law model is used in this work to describe the non-Newtonian behavior of the fluid, and consequently the expression for the local dimensionless viscosity (equations 2-4) at a point in the channel is given by

$$\mu = m\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]^{(n-1)/2} \tag{2}$$

where

$$\frac{\partial u}{\partial x} = \xi_x\frac{\partial u}{\partial \xi} + \eta_x\frac{\partial u}{\partial \eta} \tag{3}$$

$$\frac{\partial u}{\partial y} = \xi_y \frac{\partial u}{\partial \xi} + \eta_y \frac{\partial u}{\partial \eta} \quad (4)$$

with (n) being the power-law index and (m) the consistency factor.

The dimensionless momentum equation (5) in transformed coordinates can be written as

$$J^{n+1} = (\alpha U_{\xi\xi} - 2\beta U_{\xi\eta} + \gamma U_{\eta\eta}).S \quad (5)$$

The function (S) is a variable viscosity and is given by equation (6).

$$S = \left[\left(\frac{\partial\xi}{\partial X}\frac{\partial U}{\partial \xi} + \frac{\partial\eta}{\partial X}\frac{\partial U}{\partial \eta} \right)^2 + \left(\frac{\partial\xi}{\partial Y}\frac{\partial U}{\partial \xi} + \frac{\partial\eta}{\partial Y}\frac{\partial U}{\partial \eta} \right)^2 \right]^{(n-1)/2} \quad (6)$$

The energy equation for constant property flow, neglecting axial conduction and viscous dissipation in Cartesian coordinates, is

$$\left[\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y} \right) \right] = u\frac{dT}{dz}. \quad (7)$$

Energy equation (7) can be written in dimensionless form:

$$\left[\frac{\partial}{\partial X}\left(\frac{\partial \theta}{\partial X} \right) + \frac{\partial}{\partial Y}\left(\frac{\partial \theta}{\partial Y} \right) \right] = \frac{U}{U_m}\frac{d\theta}{dZ}. \quad (8)$$

where

$$\left[\frac{\partial}{\partial X}\left(\frac{\partial \theta}{\partial X} \right) + \frac{\partial}{\partial Y}\left(\frac{\partial \theta}{\partial Y} \right) \right] = \frac{(\alpha T_{\xi\xi} - 2\beta T_{\xi\eta} + \gamma T_{\eta\eta})}{J^2} \quad (9)$$

With the transformation of the physical domain, the boundaries now coincide with coordinate surfaces. The initial and boundary conditions of constant wall temperature are given as follows:

θ (0,η,ξ)=1
θ (z,0,ξ)=0
θ (z,η,0)=0

Velocity in entrance and boundary conditions is given as follows:

U=1.0 Z=0
U=0.0 ξ=1 and ξ=I for all η
U=0.0 η=1 and η=J for all ξ

Numerical Solution Method

The transformed energy, momentum and continuity equations are solved in a rectangular computational domain using finite difference formulation. The momentum and continuity equations are solved only in transverse directions at the first cross section in the axial direction. The energy equation is linearized by setting the unknown to its value at the previous axial step. The indices i, j and k indicate positions in the ξ, η and z directions respectively. The origin is designated by i=j=k=1, which is at the bottom left corner of the computational plain. The direction number (ξ, η and Z) of mesh spaces are is taken as equal to 1/NI, 1/NJ and 1/NK respectively. The dimensionless transverse step sizes, $\Delta\xi$ and $\Delta\eta$, are taken as equal but axial mesh space ΔZ was not. The axial step size ΔZ was taken as 5x10^{-5}. Starting with this value, subsequent step sizes are gradually increased using the relation ΔZ_{n+1}=(1.1)x(ΔZ_n) until ΔZ>1.0x10^{-3}.

The following finite difference representation (equations 10-16) are written with the indices given above.

$$U_{\eta\eta} = \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{(\Delta\eta)^2} \quad (10)$$

$$U_{\xi\xi} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\Delta\xi)^2} \quad (11)$$

$$U_{\xi\eta} = \frac{U_{i+1,j+1} - U_{i+1,j-1} - U_{i-1,j+1} + U_{i-1,j-1}}{(4.\Delta\xi.\Delta\eta)} \quad (12)$$

$$\theta_{\xi\eta} = \frac{\theta_{i+1,j+1,k} - \theta_{i+1,j-1,k} - \theta_{i-1,j+1,k} + \theta_{i-1,j-1,k}}{(4.\Delta\xi.\Delta\eta)} \quad (13)$$

$$\theta_{\xi\xi} = \frac{\theta_{i+1,j,k} - 2\theta_{i,j,k} + \theta_{i-1,j,k}}{(\Delta\xi)^2} \quad (14)$$

$$\theta_{\eta\eta} = \frac{\theta_{i,j+1} - 2\theta_{i,j} + \theta_{i,j-1}}{(\Delta\eta)^2} \quad (15)$$

$$\theta z = \frac{\theta_{i,j,k+1} - \theta_{i,j,k}}{\Delta Z} \quad (16)$$

The flow behavior index, n, is required to obtain velocity (U) and temperature (θ) distribution. This index number was started at 0.1 and increased by increments of 0.1 until it was equal to 1.0. The velocity and temperature values were computed for the selected index number. The velocity values were computed only for the first step (k=1) but temperature values were computed at every step (k=1 thru NK+1) in the axial direction for all index numbers.

The temperature variable (θ) is known on the $\theta_{i,j,k}$ plane, while the variable $\theta_{i,j,k+1}$ is to be determined in the axial direction. The overrelaxation technique, which is an iterative procedure, was used in computing velocity and temperature values. This procedure requires initial estimates of the variables at each node. Therefore, the results from the preceding axial position (k) are substituted as initial estimates for the variables at the (k+1)th position.

Once the fully developed velocity and developing temperature distributions are obtained, the bulk mean temperature θ_b, the local Nusselt number Nu_T, and the mean Nusselt number Nu_m, are computed by employing the following equations (17-19) at the axial position.

$$\theta_b = \frac{1}{Z.U_m}\int U.\theta.dZ \tag{17}$$

$$Nu(Z) = \frac{d\theta_b}{dZ}\frac{1}{-4.\theta_b} \tag{18}$$

$$Nu_m = \frac{1}{Z}\int {}_{k=1}^{k=n} Nu(Z)_k.dZ_k \tag{19}$$

Results and Discussion

Reliable data for laminar fully developed flow in square and rhombic ducts are available in the literature. Several test calculations were carried out in order to verify the performance of the solution procedure. Firstly the flow of Newtonian fluid was considered because numerical results are already available in previous studies. The numerical results were presented for comparison. In addition, numerical comparison of the elliptic grid generation technique and other methods are also included. Excellent agreement was found between this work and the others. The mean Nusselt numbers for developing flow of a power-law fluid in square duct agreed with the constant property results of Ashok and Sastri (1977), despite the differences in the numerical solution techniques. And also there was excellent agreement with the results of Asako and Faghri (1988) for rhombic ducts with various angles Uzun and Unsal, (1997).

The limiting Nusselt number Nu_T for Newtonian fluids for a square duct is presented in Table 1. The results of other works are also included. The limiting Nusselt numbers (Nu_T) for $0.5\leq n\leq 1.0$ for hydrodynamically and thermally developed laminar flow of non-Newtonian fluid in triangular and square ducts are presented in Table 3. It is clear from Tables 1-3 that the differences between the Nu_T values given in this study and those in previous investigations are less than 1-5% . The actual values of Nu_m and Nu_T selected at axial locations for different values of the power law index (n=1, n=0.5) are presented in Tables 4-6.

Three-dimensional fully developed velocity profiles for four-cusped and triangular ducts are presented in Figures 3a and 3b, respectively. As expected triangular dimensionless velocitiy values (U and U_{max}) are greater than those of the four-cusped duct. Representative contours of axial fully developed velocity U are presented in Figures 4a-c for triangular, four-cusped and trapezoidal ducts, respectively. Figures 5a and 5b depict the effect of the power law index (n) on the axial velocity profiles at selected axial location for triangular and four-cusped ducts respectively. As expected on physical grounds, pseudoplastic (n<1) fluids are characterized by steeper velocity gradients near the wall and flatter profiles close to the core of the duct.

Table 1. Nu_T for Newtonian fluid for a square duct.

Investigations	Nu_T
Shah and London (1971)	2.976
Montgomery and Wilbulswas (1966)	2.650
Ashok and Sastri (1977)	2.975
Asako and Faghri (1988)	2.976
Present Study	2.971

Table 2. Nusselts numbers for fully developed velocity and temperature profiles in triangular and square ducts for pseudoplastic fluids.

Power law index, n	Nu_T Present Study		Nu_T	
	Triangular	Square	Square Ashok (1977)	Triangular Shah (1975)
1.0	2.353	2.971	2.975	2.34
0.9	2.373	2.997	2.997	-
0.8	2.400	3.034	3.030	-
0.7	2.435	3.037	3.070	-
0.6	2.481	3.135	3.120	-
0.5	2.543	3.208	3.184	-

Table 3. Nu_T and Nu_m Nusselt numbers (Nu_T and Nu_m) for fully developed velocity and temperature profiles in a four-cusped duct and a square duct with four indented corners.

Power law index, n	Four-cusped duct		Square duct with four indented corners	
	Nu_T	Nu_m	Nu_T	Nu_m
1.0	1.0747	1.1363	2.0635	2.1236
0.9	1.0848	1.1467	2.0812	2.1414
0.8	1.0981	1.1595	2.1052	2.1650
0.7	1.1140	1.1754	2.1348	2.1952
0.6	1.1340	1.1956	2.1724	2.2332
0.5	1.1586	1.2219	2.2207	2.2820

Table 4. Nu_T and Nu_m versus axial direction for a pseudoplastic fluid in sinusoidal ducts.

Z	n=1.0		n=0.5	
	Nu_T	Nu_m	Nu_T	Nu_m
0.6050E-05	18.495	18.495	21.523	21.523
0.1121E-03	13.697	15.304	15.390	17.495
0.1101E-02	7.808	10.215	8.2932	11.164
0.1252E-01	3.384	4.933	3.5991	5.2432
0.1024E+00	2.151	2.704	2.3229	2.8987
0.5024E+00	2.102	2.228	2.2740	2.4045
0.9774E+00	2.102	2.167	2.2739	2.3411

Table 5. Nu_T and Nu_m versus axial direction for a pseudoplastic fluid in triangular ducts.

Z	n=1.0		n=0.5	
	Nu_T	Nu_m	Nu_T	Nu_m
0.6050E-05	26.44	26.44	26.11	26.11
0.1121E-03	18.53	21.56	18.87	21.64
0.1101E-02	8.641	12.34	9.060	12.77
0.1252E-01	3.739	5.513	3.966	5.796
0.1024E+00	2.412	3.021	2.603	3.228
0.5024E+00	2.535	2.494	2.543	2.687
0.9774E+00	2.535	2.524	2.543	2.617

Table 6. Nu_T and Nu_m versus axial direction for a pseudoplastic fluid (n=0.5) in a four-cusped duct and a square duct with four indented corners.

Z	Four-cusped duct		Square duct with four indented corners	
	Nu_T	Nu_m	Nu_T	Nu_m
0.6050E-05	12.725	12.725	22.339	22.339
0.1121E-03	9.2064	10.381	15.416	17.729
0.1101E-02	5.2554	6.8633	8.0849	11.045
0.1252E-01	2.2847	3.3975	3.4795	5.1965
0.1024E+00	1.2505	1.7155	2.2599	2.8064
0.5195E+00	1.1569	1.2777	2.2204	2.3389
0.9995E+00	1.1586	1.2219	2.2207	2.2820

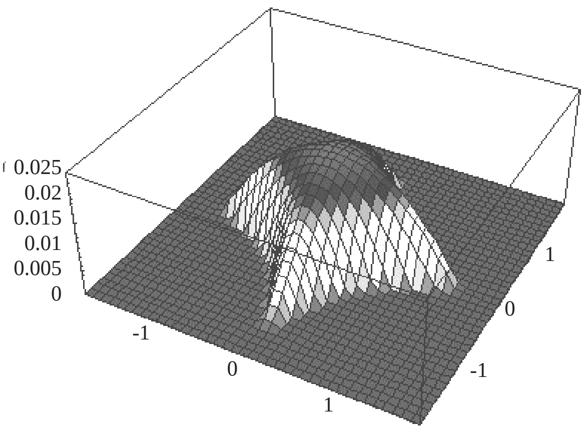

Figure 3a. Three-dimensional fully developed velocity profiles in a four-cusped duct (n=0.5).

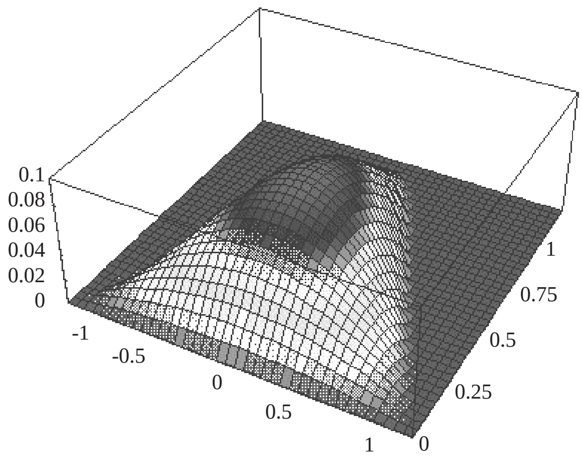

Figure 3b. Three dimensional fully developed velocity profiles in triangular duct (n=1.0).

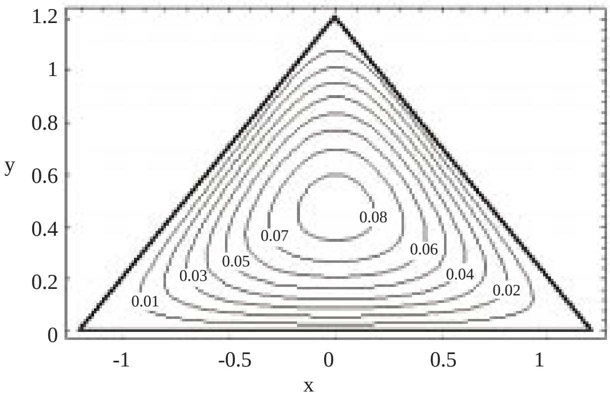

Figure 4a. Axial velocity contours in a triangular duct (n=1.0).

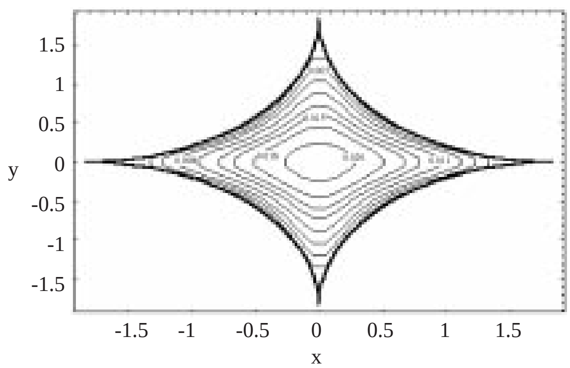

Figure 4b. Axial velocity contours in a four-cusped duct (n=0.5).

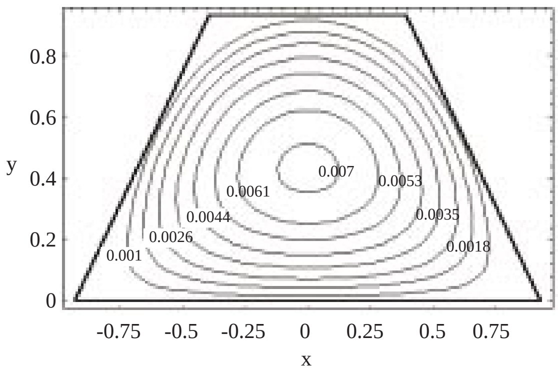

Figure 4c. Axial velocity contours in a trapezoidal duct (60° , n=0.5).

Results for Nu_T and Nu_m as a function of the Graetz number are plotted in Figures 6a and 6b for triangular and trapezoidal ducts. The broken lines in these figures indicate the local peripheral average Nusselt number Nu_T. The fully developed Nusselt number values are also plotted in these figures. As expected, Nu_T and Nu_m decrease with (z) and approach the fully developed values. The bulk temperature values are also plotted in these figures. An extremely good agreement is obtained for fully developed values for triangular and square ducts.

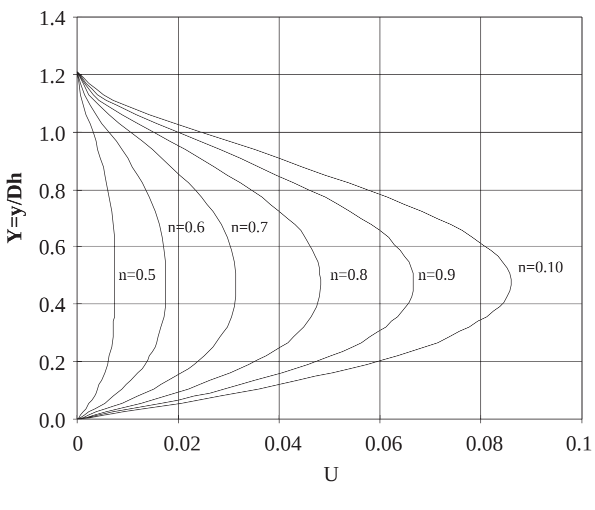

Figure 5a. Centerplane velocity profiles for non-Newtonian fluids in a triangular duct.

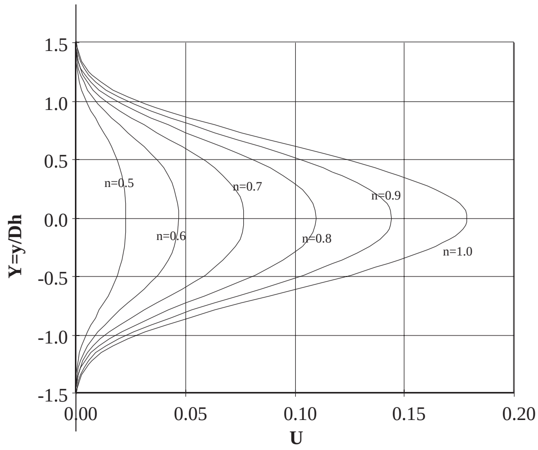

Figure 5b. Centerplane velocity profiles for non-Newtonian fluids in a four-cusped duct.

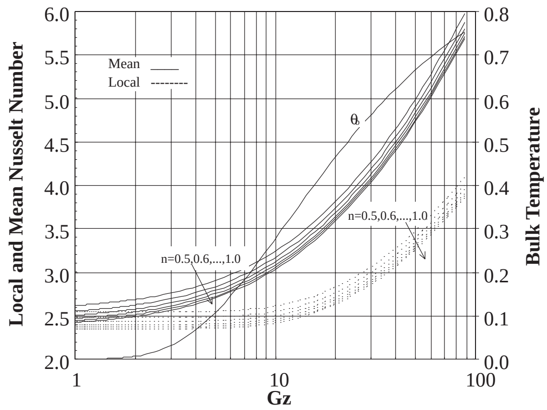

Figure 6a. Nusselt numbers and bulk temperature for power-law fluids in a triangular duct.

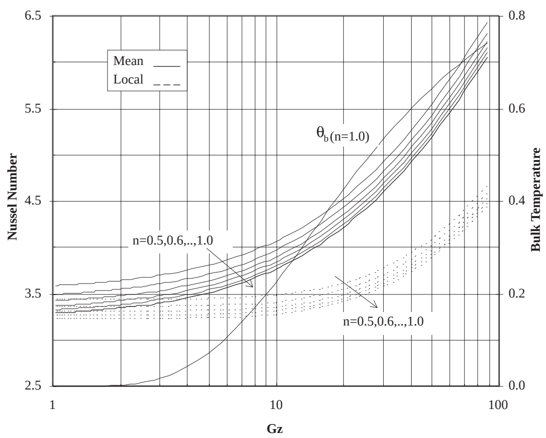

Figure 6b. Nusselt numbers and bulk temperature for power-law fluids in a trapezoidal duct.

Conclusions

The hydrodynamically fully developed and thermally developing laminar flow of non-Newtonian fluid in arbitrary cross-sectional ducts was analyzed in this study. In order to eliminate the disadvantage of the non-uniform mesh and to improve the numerical accuracy, the elliptic grid generation technique was used. The partial differential equations in Cartesian coordinates were transformed into the computational $\xi-\eta$ domain. Then, the transformed equations were solved by means of the finite difference method using the overrelaxation iterative procedure. Square ducts, triangular ducts, sinusoidal ducts, rhombic ducts, square ducts with four indented corners and four-cusped ducts were investigated in this study.

Inspection of the numerical solutions shows that a non-Newtonian fluid with a flow behavior index of less than one gives a higher heat transfer coefficient than a Newtonian fluid. For example, the Nusselt number was found to be 2.543 for n=0.5 and 2.353 for n=1 in triangular ducts. Due to the reduction in friction power requirement and the increase in heat transfer rates, pseudoplastic fluids seem to be better working fluids in a heat exchange equipment than Newtonian fluids.

Nomenclature

A	duct cross-sectional area
D_h	hydraulic diameter, 4A/P
Gz	Graetz number (1/Z)
J	Jacobian $J = y_\eta x_\xi - x_\eta y_\xi$
Nu	Nusselt number, (h.D_h) / k
p	fluid static pressure
P	perimeter of cross section
Re	Reynolds number for the duct based on hydraulic diameter, (u.D_h)/ν
T	temperature (°C)
u	axial fluid velocity in duct
U	dimensionless axial fluid velocity in duct, [u/{(-dP/dz.μ)(D_h)^2}]
x,y,z	rectangular Cartesian coordinates
X,Y	dimensionless transverse coordinates, X=x / D_h, Y=y / D_h
Z	dimensionless axial coordinate, [Z=z /(D_h.Re.Pr)]
α, β, γ	dimensionless transform functions, ($\alpha = x_\eta x_\eta + y_\eta y_\eta$, $\beta = x_\xi x_\eta + y_\xi y_\eta$, $\gamma = x_\xi x_\xi + y_\xi y_\xi$)
η, ξ	dimensionless transformed coordinates, (η=η / D_h, ξ=ξ / D_h)
μ	dynamic viscosity
ν	kinematic viscosity
θ	dimensionless temperature $(T - T_w)/(T_i - T_w)$.

Subscripts

b	bulk mean
T	fully developed flow
i	initial
m	mean

References

Asako, Y. and Faghri, M. "Three-Dimensional Laminar Heat Transfer and Fluid Flow Characteristics in the entrance Region of a Rhombic Duct", J.Heat transfer, 110, 855-861, 1988

Ashok, R. C. and Sastri, V.M.K., "Laminar Forced Convection Heat Transfer of A Non-Newtonian Fluid in a Square Duct", Int.J.H.M.T., 20, 1315-1324, 1977.

Lawal, A., "Mixed Convection Heat Transfer to Power Law Fluids in Arbitrary Cross-Sectional Ducts", J.Heat Transfer, 111, 399-406, 1989.

Montgomery, S. R., and Wilbulswas, P., Laminar Flow Heat Transfer in Ducts of rectangular Cross-section, Proc. 3rd Int. Heat transfer Conf., New York, 1, 85-98, 1966.

Nguyen, T.V., "Laminar Heat Transfer for Thermally Developing Flow in Ducts", Int.J.H.M.T., 35, 1733-1741, 1992.

Porter, J. E., Heat transfer at Low Reynolds Number (Highly Viscous Liquids in Laminar Flow), Trns. Inst. Chem. Engrs, 49, 1-29, 1971

Shah, R. K., and London, A. L., Laminar flow forced convection heat transfer and flow friction in straight and curved ducts-a summary of analytical solutions, T.R., No.75, Stanford, 1971

Shah, R.K., "Laminar Flow Friction and Forced Convection Heat Transfer in Ducts of Arbitrary Geometry", Int.J.H.M.T., 18, 849-862, 1975.

Syrjala S., "Finite-Element Analysis of Fully Developed Laminar Flow of Power-Law Non-Newtonian Fluid in a Rectangular Duct", Int. Comm. H.M.T., 22, 549-557, 1995.

Uzun, İ. and Ünsal, M., "A Numerical Study of Laminar Heat Convection in Ducts of Irregular Cross-Sections", Int. Comm. H.M.T., 24, 835-848, 1997