

On Tensor Product Surfaces of Lorentzian Planar Curves with Pointwise 1-Type Gauss Map

Mehmet Yıldırım

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ABSTRACT

In this article, we study the tensor product surfaces of two Lorentzian planar, non-null curves to have pointwise 1-type Gauss map.

Keywords: Tensor product immersion, Lorentzian curves, pointwise 1-type Gauss map.

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1. Introduction

B. Y. Chen initiated the study of the *tensor product immersion* of two immersions of a given Riemannian manifold [7]. This concept originated from the investigation of the quadratic representation of submanifold. Inspired by Chen's definition, F. Decruyenaere, F. Dillen, L. Verstraelen and L. Vrancken studied in [9] the tensor product of two immersions of, in general, different manifolds. Under some conditions, this realizes an immersion of the product manifold.

Let M and N be two differentiable manifolds and assume that

$$f : M \rightarrow \mathbb{E}^m,$$

and

$$h : N \rightarrow \mathbb{E}^n$$

are two immersions. Then the direct sum and tensor product maps are defined respectively by

$$f \oplus h : M \times N \rightarrow \mathbb{E}^{m+n}$$

$$(p, q) \rightarrow f(p) \oplus h(q) = (f^1(p), \dots, f^m(p), h^1(q), \dots, h^n(q))$$

and

$$f \otimes h : M \times N \rightarrow \mathbb{E}^{mn}$$

$$(p, q) \rightarrow f(p) \otimes h(q) = (f^1(p)h^1(q), \dots, f^1(p)h^n(q), \dots, f^m(p)h^1(q), \dots, f^m(p)h^n(q)) \quad (1.1)$$

Necessary and sufficient conditions for $f \otimes h$ to be an immersion were obtained in [10]. It is also proved there that the pairing $(\oplus, \otimes)$ determines a structure of a semiring on the set of classes of differentiable manifolds transversally immersed in Euclidean spaces, modulo orthogonal transformations. Some semirings were studied in [9].

Let M be a submanifold of a Euclidean space. We denote by G the Gauss map of M , which is defined as in [5]. In addition the laplacian of G is denoted by ΔG .

If a submanifold M of a Euclidean space has 1- type Gauss map G , then $\Delta G = \lambda(G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C . There are some surfaces, such as the helicoid, catenoid, right cones in $\mathbb{E}^3$ and also some hypersurfaces, the Laplacian of their Gauss map take the form

$$\Delta G = f(G + C) \quad (1.2)$$

for some smooth function f on M and some constant vector C . A submanifold with pointwise 1-type Gauss map is said to be of the *first kind* if the vector C in (1.2) is the zero vector. Otherwise, a submanifold with pointwise 1-type Gauss map is said to be of the *second kind*.

Surfaces in Euclidean spaces and in pseudo-Euclidean spaces with pointwise 1-type Gauss map were recently studied in [1], [5], [8], [11], [13], [14], [15], [17] and [18].

In [2], Arslan et al. investigated a tensor product surface with pointwise 1-type Gauss map in Euclidean 4-space $\mathbb{E}^4$. In addition tensor product immersions with harmonic Gauss map and tensor product immersions of two plane curves with pointwise 1-type Gauss map in Euclidean 4-space $\mathbb{E}^4$ are studied. Also tensor product surfaces of a Lorentzian space curve and a Lorentzian plane curve were studied in [12].

In this article, we investigate a tensor product surface M which is obtained from two curves. One of them is a Lorentzian circle and the other is a unit speed curve in $\mathbb{E}_1^2$. First, we obtain necessary and sufficient conditions for being of M with a harmonic Gauss map according to casual characters of the tangent vectors. Further we investigate tensor product immersions of two Lorentzian plane curves mentioned above with pointwise 1-type Gauss map of first kind in pseudo-Euclidean 4-space $\mathbb{E}_2^4$.

We remark that the notions related with pseudo-Riemannian geometry are taken from [16].

2. Preliminaries

In the present section we give some definitions about Riemannian submanifolds from [4] and [6]. Let $\iota : M \rightarrow \mathbb{E}^n$ be an immersion from an m -dimensional connected Riemannian manifold M into an n -dimensional Euclidean space $\mathbb{E}^n$. We denote by g the metric tensor of $\mathbb{E}^n$ as well as induced metric on M . Let $\bar{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^n$ and ∇ the induced connection on M . Then the Gaussian and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$\bar{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi,$$

where X, Y are vector fields tangent to M and ξ normal to M , h is the second fundamental form, $\nabla^\perp$ is linear connection induced in the normal bundle $T^\perp M$, called normal connection and A_ξ is the shape operator in the direction of ξ that is related with h by,

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The covariant differentiation $\bar{\nabla}h$ of the second fundamental form h on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^\perp M$ of M is defined by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for any vector fields X, Y and Z tangent to M . Then we have the Codazzi equation

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z) \quad (2.1)$$

We denote by R the curvature tensor associated with ∇ ;

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X, Y]} Z,$$

and denote by $R^\perp$ the curvature tensor associated with $\nabla^\perp$;

$$R^\perp(X, Y)\eta = \nabla_Y^\perp \nabla_X^\perp \eta - \nabla_X^\perp \nabla_Y^\perp \eta - \nabla_{[X, Y]}^\perp \eta$$

[6].

The equations Gauss and Ricci are given by

$$\langle R(X, Y)Z, W \rangle = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle \quad (2.2)$$

$$\langle \bar{R}(X, Y)\eta, \xi \rangle - \langle R^\perp(X, Y)\eta, \xi \rangle = \langle [A_\eta, A_\xi]X, Y \rangle \quad (2.3)$$

for any vector fields X, Y, Z, W tangent to M and ξ, η normal vector fields to M .

Let $G(m, n)$ denote the Grassmannian manifold consisting of all oriented m -planes through origin of $\mathbb{E}^n$. The Gauss map $G : M \rightarrow G(m, n)$ of M is a smooth map which carries a point $p \in M$ into the oriented m -plane through the origin of $\mathbb{E}^n$ obtained by the parallel translation of the tangent space of M at p in $\mathbb{E}^n$.

Since $G(m, n)$ is canonically embedded in $\Lambda^m \mathbb{E}^n = \mathbb{E}^N$, $N = \binom{n}{m}$, the notion of the type of the Gauss map is naturally defined. If $\{e_1, e_2, \dots, e_m\}$ is an oriented orthonormal normal frame on M , then the Gauss map $G : M \rightarrow G(m, n) \subset \mathbb{E}^N$ is given by $G(p) = (e_1 \wedge e_2 \wedge \dots \wedge e_m)(p)$. The inner product on $\Lambda^m \mathbb{E}^n$ is defined by $w_1 = u_1 \wedge u_2 \wedge \dots \wedge u_m$, $w_2 = v_1 \wedge v_2 \wedge \dots \wedge v_m$

$$\langle w_1, w_2 \rangle = \det \langle u_i, v_j \rangle. \quad (2.4)$$

For $n = 4$, an orthonormal basis of $\Lambda^2 \mathbb{E}^4$ with respect to this inner product is the set

$$\{e_i \wedge e_j \mid 1 \leq i < j \leq 4\}.$$

For any real function f on M , the Laplacian of f is defined by

$$\Delta f = - \sum_i \varepsilon_i (\bar{\nabla}_{e_i} \bar{\nabla}_{e_i} f - \bar{\nabla}_{\nabla_{e_i} e_i} f) \quad (2.5)$$

3. Tensor product surfaces with finite type Gauss map

In the following section, we will consider the tensor product immersions which is obtained from two Lorentzian planar curves (for geometry of tensor product surfaces of Lorentzian planar curves see [3]).

Let $c_1 : \mathbb{R} \rightarrow \mathbb{E}_1^2$ and $c_2 : \mathbb{R} \rightarrow \mathbb{E}_1^2$ be two Lorentzian curves. Put $c_1(t) = (\alpha_1(t), \alpha_2(t))$ and $c_2(s) = (\beta_1(s), \beta_2(s))$.

Then by considering (1.1) their tensor product surface is given by

$$x = c_1 \otimes c_2 : \mathbb{R}^2 \rightarrow \mathbb{E}_2^4$$

$$x(t, s) = (\alpha_1(t)\beta_1(s), \alpha_1(t)\beta_2(s), \alpha_2(t)\beta_1(s), \alpha_2(t)\beta_2(s)). \quad (3.1)$$

The metric tensor on $\mathbb{E}_1^2$ and $\mathbb{E}_2^4$ is given by

$$g = -dx_1^2 + dx_2^2$$

and from [12],

$$g = dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2, \quad (3.2)$$

respectively.

If we take c_1 as a Lorentzian unit circle $c_1(t) = (\sinh t, \cosh t)$ and $c_2(s) = (\alpha(s), \beta(s))$ is a spacelike or timelike curve with unit speed then from (3.1) the surface patch becomes

$$M : x(t, s) = (\alpha(s) \sinh t, \beta(s) \sinh t, \alpha(s) \cosh t, \beta(s) \cosh t) \quad (3.3)$$

An orthonormal frame tangent to M with respect to (3.2) is given by

$$\begin{aligned} e_1 &= \frac{1}{\|c_2\|} \frac{\partial x}{\partial t} \\ &= \frac{1}{\|c_2\|} (\alpha(s) \cosh t, \beta(s) \cosh t, \alpha(s) \sinh t, \beta(s) \sinh t), \\ e_2 &= \frac{\partial x}{\partial s} \\ &= (\alpha'(s) \sinh t, \beta'(s) \sinh t, \alpha'(s) \cosh t, \beta'(s) \cosh t). \end{aligned}$$

Also, the normal space of M is spanned by

$$n_1 = \frac{1}{\|c_2\|}(\beta(s) \cosh t, \alpha(s) \cosh t, \beta(s) \sinh t, \alpha(s) \sinh t),$$

$$n_2 = (\beta'(s) \sinh t, \alpha'(s) \sinh t, \beta'(s) \cosh t, \alpha'(s) \cosh t),$$

where

$$\mathbf{g}(e_1, e_1) = -\mathbf{g}(n_1, n_1) = -\frac{g(c_2(s), c_2(s))}{\|c_2\|^2} = \varepsilon_1,$$

$$\mathbf{g}(e_2, e_2) = -\mathbf{g}(n_2, n_2) = g(c_2'(s), c_2'(s)) = \varepsilon_2$$

and $\varepsilon_1 = \mp 1$, $\varepsilon_2 = \mp 1$.

By covariant differentiation with respect to e_1 and e_2 a straightforward calculation gives

$$\begin{aligned}\bar{\nabla}_{e_1} e_1 &= a\varepsilon_2 e_2 - b\varepsilon_2 n_2 \\ \bar{\nabla}_{e_1} e_2 &= -a\varepsilon_1 e_1 - b\varepsilon_1 n_1 \\ \bar{\nabla}_{e_1} n_1 &= -b\varepsilon_2 e_2 + a\varepsilon_2 n_2 \\ \bar{\nabla}_{e_1} n_2 &= -b\varepsilon_1 e_1 - a\varepsilon_1 n_1\end{aligned}\tag{3.4}$$

$$\begin{aligned}\bar{\nabla}_{e_2} e_1 &= -b\varepsilon_1 n_1 \\ \bar{\nabla}_{e_2} e_2 &= -c\varepsilon_2 n_2 \\ \bar{\nabla}_{e_2} n_1 &= -b\varepsilon_1 e_1 \\ \bar{\nabla}_{e_2} n_2 &= -c\varepsilon_2 e_2\end{aligned}\tag{3.5}$$

where a , b and c are Christoffel symbols and as in follows

$$a = a(s) = \frac{\beta\beta' - \alpha\alpha'}{\|c_2\|^2},\tag{3.6}$$

$$b = b(s) = \frac{\alpha'\beta - \alpha\beta'}{\|c_2\|^2},\tag{3.7}$$

$$c = c(s) = \alpha'\beta'' - \alpha''\beta' .\tag{3.8}$$

Corollary 3.1. *If $b = 0$ then c is also zero.*

By using Corollary 3.1 and the equalities (3.4) and (3.5) we obtain following corollary.

Corollary 3.2. *M is a totally geodesic surface in $\mathbb{E}_2^4$ if and only if $b = 0$.*

From (3.4) and (3.5), the induced covariant differentiation on M as in follows,

$$\left. \begin{aligned}\nabla_{e_1} e_1 &= a\varepsilon_2 e_2, \\ \nabla_{e_1} e_2 &= -a\varepsilon_1 e_1, \\ \nabla_{e_2} e_1 &= 0, \\ \nabla_{e_2} e_2 &= 0.\end{aligned}\right\}\tag{3.9}$$

$$\left. \begin{aligned}\nabla_{e_1}^\perp n_1 &= a\varepsilon_2 n_2, \\ \nabla_{e_1}^\perp n_2 &= -a\varepsilon_1 n_1,\end{aligned}\right\}\tag{3.10}$$

$$\left. \begin{aligned}\nabla_{e_2}^\perp n_1 &= 0, \\ &= 0, \\ \nabla_{e_2}^\perp n_2 &= 0,\end{aligned}\right\}\tag{3.11}$$

where the equalities (3.10) and (3.11) define the normal connection on M .

Lemma 3.1. *Let $x = c_1 \otimes c_2$ be a tensor product immersion of a circle c_1 and unit speed non-null curve c_2 in $\mathbb{E}_1^2$. Then we have,*

$$A_{n_1} = \begin{bmatrix} 0 & b\varepsilon_1 \\ b\varepsilon_2 & 0 \end{bmatrix}, \quad A_{n_2} = \begin{bmatrix} b\varepsilon_1 & 0 \\ 0 & c\varepsilon_2 \end{bmatrix}.$$

For our goal in this paper, we must calculate the Laplacian of Gauss map. From (2.5) we have following equality

$$\begin{aligned}\Delta G &= (2b^2\varepsilon_2 + b^2\varepsilon_2 + c^2\varepsilon_2)e_1\Lambda e_2 + (-2ab\varepsilon_2 + ac\varepsilon_1 - c')e_1\Lambda n_2 \\ &= +(-3ab\varepsilon_2 + b'\varepsilon_1\varepsilon_2)e_2\Lambda n_1 + (2b^2\varepsilon_2 + 2bc\varepsilon_1)n_1\Lambda n_2.\end{aligned}\quad (3.12)$$

We suppose that the Gauss map of M is harmonic, i.e., $\Delta G = 0$. From (3.12) we get

$$\left. \begin{aligned}2b^2\varepsilon_2 + b^2\varepsilon_2 + c^2\varepsilon_2 &= 0, \\ -2ab\varepsilon_2 + ac\varepsilon_1 - c' &= 0, \\ -3ab\varepsilon_2 + b'\varepsilon_1\varepsilon_2 &= 0, \\ 2b^2\varepsilon_2 + 2bc\varepsilon_1 &= 0.\end{aligned}\right\} \quad (3.13)$$

By using (3.13), we obtain that $\Delta G = 0$ if and only if $b = 0$. Thus, by considering Corollary 3.2 we get the following theorem.

Theorem 3.1. *Let $M \subset \mathbb{E}_2^4$ be a tensor product surface of a Lorentzian plane circle c_1 centered at the origin with a unit speed curve c_2 in $\mathbb{E}_1^2$. Then the Gauss map of M is harmonic if and only if M is a totally geodesic surface in $\mathbb{E}_2^4$.*

Let M has pointwise 1-type Gauss map, i.e., $\Delta G = f(G + C)$ and

$$C = \lambda_{12}e_1\Lambda e_2 + \lambda_{13}e_1\Lambda e_3 + \lambda_{14}e_1\Lambda e_4 + \lambda_{23}e_2\Lambda e_3 + \lambda_{24}e_2\Lambda e_4 + \lambda_{34}e_1\Lambda e_2, \quad (3.14)$$

where $e_3 = n_1$, $e_4 = n_2$. Because of the set $\{e_i\Lambda e_j \mid 1 \leq i < j \leq 4\}$ is an orthonormal basis of $\Lambda^2\mathbb{E}^4$, we have the followings,

$$\left. \begin{aligned}2b^2\varepsilon_2 + b^2\varepsilon_2 + c^2\varepsilon_2 &= f(1 + \lambda_{12}) \\ -2ab\varepsilon_2 + ac\varepsilon_1 - c' &= f\lambda_{14} \\ -3ab\varepsilon_2 + b'\varepsilon_1\varepsilon_2 &= f\lambda_{23} \\ 2b^2\varepsilon_2 + 2bc\varepsilon_1 &= f\lambda_{34}\end{aligned}\right\} \quad (3.15)$$

$$\lambda_{13} = \lambda_{24} = 0 \quad (3.16)$$

By considering (2.2) and (2.3), we see that Gauss and Ricci equations of M are identical and they are obtained as in follows,

$$a' - a^2\varepsilon_1 = b^2\varepsilon_1 - bc\varepsilon_2. \quad (3.17)$$

On the other hand, Codazzi equation of M is

$$b' = 2ab\varepsilon_1 - ac\varepsilon_2 \quad (3.18)$$

Thus we give the following theorem.

Theorem 3.2. *If M is a tensor product surface of a Lorentzian circle and a non-null unit speed curve in $\mathbb{E}_1^2$ then the Christoffel symbols of M satisfy the following Riccati equation*

$$(a + b)' = \varepsilon_1(a + b)^2 - c\varepsilon_2(a + b).$$

From (1.2), (3.15), (3.16), (3.17) and (3.18) we obtain following theorem.

Theorem 3.3. *Let $M \subset \mathbb{E}_2^4$ be a tensor product surface of a Lorentzian plane circle c_1 centered at the origin with a unit speed curve c_2 in $\mathbb{E}_1^2$. Then we have the followings:*

- i) If $\varepsilon_1 = \varepsilon_2$, M doesn't have pointwise 1-type Gauss map of first kind.
- ii) If $\varepsilon_1 = -\varepsilon_2 = 1$, M has pointwise 1-type Gauss map of first kind if and only if

$$\begin{aligned}b &= c = \lambda(\beta^2 - \alpha^2)^{-3/2} \\ a' - a^2 &= 2b^2\end{aligned}$$

- iii) If $\varepsilon_1 = -\varepsilon_2 = -1$, M has pointwise 1-type Gauss map of first kind if and only if

$$\begin{aligned}b &= c = \lambda(\alpha^2 - \beta^2)^{3/2} \\ a' + a^2 &= -2b^2\end{aligned}$$

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Affiliations

MEHMET YILDIRIM

ADDRESS: Kırıkkale University, Faculty of Sciences and Arts, Department of Mathematics 71450, Kırıkkale-Turkey.

E-MAIL: myildirim@kku.edu.tr