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Ibrahim Canak; Firat Ozsarac

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Tauberian theorems for the weighted mean method of summability of integrals

Ibrahim Canak[1,a)] and Firat Ozsarac[2,b)]

[1]*Department of Mathematics, Ege University, Izmir, Turkey.*
[2]*Department of Mathematics, Kırıkkale University, Kırıkkale, Turkey.*

[a)]Corresponding author: ibrahim.canak@ege.edu.tr
[b)]firatozsarac@kku.edu.tr

Abstract. Let q be a positive weight function on $\mathbf{R}_+ := [0,\infty)$ which is integrable in Lebesgue's sense over every finite interval $(0,x)$ for $0 < x < \infty$, in symbol: $q \in L^1_{loc}(\mathbf{R}_+)$ such that $Q(x) := \int_0^x q(t)dt \neq 0$ for each $x > 0$, $Q(0) = 0$ and $Q(x) \to \infty$ as $x \to \infty$. Given a real or complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$, we define $s(x) := \int_0^x f(t)dt$ and

$$\tau_q^{(0)}(x) := s(x), \tau_q^{(m)}(x) := \frac{1}{Q(x)}\int_0^x \tau_q^{(m-1)}(t)q(t)dt \ \ (x > 0, m = 1, 2, ...),$$

provided that $Q(x) > 0$.

We say that $\int_0^\infty f(x)dx$ is summable to L by the m-th iteration of weighted mean method determined by the function $q(x)$, or for short, $(\overline{N}, q, m)$ integrable to a finite number L if

$$\lim_{x\to\infty} \tau_q^{(m)}(x) = L.$$

In this case, we write $s(x) \to L(\overline{N}, q, m)$.

It is known that if the limit $\lim_{x\to\infty} s(x) = L$ exists, then $\lim_{x\to\infty} \tau_q^{(m)}(x) = L$ also exists. However, the converse of this implication is not always true. Some suitable conditions together with the existence of the limit $\lim_{x\to\infty} \tau_q^{(m)}(x)$, which is so called Tauberian conditions, may imply convergence of $\lim_{x\to\infty} s(x)$.

In this paper, one- and two-sided Tauberian conditions in terms of the generating function and its generalizations for $(\overline{N}, q, m)$ summable integrals of real- or complex-valued functions have been obtained. Some classical type Tauberian theorems given for Cesàro summability $(C, 1)$ and weighted mean method of summability $(\overline{N}, q)$ have been extended and generalized.
Keywords: Tauberian conditions, Tauberian theorems, weighted mean method of summability, slow decrease and oscillation.
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INTRODUCTION

Let q be a positive weight function on $\mathbf{R}_+ := [0,\infty)$ which is integrable in Lebesgue's sense over every finite interval $(0,x)$ for $0 < x < \infty$, in symbol: $q \in L^1_{loc}(\mathbf{R}_+)$ such that $Q(x) := \int_0^x q(t)dt \neq 0$ for each $x > 0$, $Q(0) = 0$ and $Q(x) \to \infty$ as $x \to \infty$. Given a real or complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$, we define $s(x) := \int_0^x f(t)dt$ and

$$\tau_q^{(0)}(x) := s(x), \tau_q^{(m)}(x) := \frac{1}{Q(x)}\int_0^x \tau_q^{(m-1)}(t)q(t)dt \ \ (x > 0, m = 1, 2, ...),$$

provided that $Q(x) > 0$.

For each integer $m \geq 0$, we define $v_q^{(m)}(x)$ by

$$v_q^{(m)}(x) = \begin{cases} \frac{Q(x)}{q(x)} f(x) & , m = 0 \\ \frac{1}{Q(x)}\int_0^x f(t)Q(t)dt & , m = 1 \\ \frac{1}{Q(x)}\int_0^x v_q^{(m-1)}(t)q(t)dt & , m \geq 2. \end{cases}$$

The identity

$$\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) = v_q^{(m)}(x) \tag{1}$$

is known as the weighted Kronecker identity for the weighted mean method of summability.

It is clear from (1) that

$$\frac{Q(x)}{q(x)}\frac{d}{dx}\tau_q^{(m)}(x) = v_q^{(m)}(x)$$

for each integer $m \geq 0$ (see [14]). Here, we call $v_q^{(m)}(x)$ the generator of $\tau_q^{(m-1)}(x)$ for each integer $m \geq 1$.

We say that $\int_0^\infty f(x)dx$ is summable to L by the m-th iteration of weighted mean method determined by the function $q(x)$, or for short, $(\overline{N}, q, m)$ summable to a finite number L if

$$\lim_{x\to\infty} \tau_q^{(m)}(x) = L. \tag{2}$$

It is obvious that $(\overline{N}, q, m)$ summability reduces to the ordinary convergence for $m = 0$ and $(\overline{N}, q, 1)$ is the $(\overline{N}, q)$ method. If $q(x) = 1$ on $\mathbf{R}_+$, then $(\overline{N}, q, m)$ method is the Hölder method of order m and $(\overline{N}, q, 1)$ method is the Cesàro summability method $(C, 1)$.

It is well known that condition $Q(x) \to \infty$ as $x \to \infty$ is a necessary and sufficient condition that the existence of the integral

$$\int_0^\infty s(x)dx = L \tag{3}$$

implies (2). That is, the $(\overline{N}, q, m)$ summability method is regular, where m is a nonnegative integer. However, the converse of this implication is not always true. Notice that some suitable condition on $s(x)$ together with (2) may imply (3). Such a condition is called a Tauberian condition and resulting theorem is said to be a Tauberian theorem.

Móricz [8] and Fekete and Móricz [6] obtained one-sided and two-sided Tauberian conditions for the weighted mean method $(\overline{N}, q)$ of integrals. Following these works, Totur and Okur [13] proved one-sided boundedness of $v_q^{(0)}(x)$ is a Tauberian condition for the weighted mean method of summability $(\overline{N}, q)$ of integrals. From the fact that condition $v_q^{(0)}(x) \geq -C$ implies slow decreasing of $s(x)$, Totur and Okur [13] generalized their first result and proved that slow decrease of $s(x)$ is also a Tauberian condition for $(\overline{N}, q)$ method. For a detailed study and some interesting results related to Tauberian theorems for the weighted mean method of summability, we refer the reader to Borwein and Kratz [1], Çanak and Totur [2], Çanak and Totur [3], Çanak and Totur [4], Özsaraç and Çanak [9], Sezer and Çanak [10], Tietz and Zeller [11] and Totur and Çanak [12], etc.

In this paper, one- and two-sided Tauberian conditions in terms of the generating function and its generalizations for summable integrals by m-th iteration of weighted means of real- or complex-valued functions have been obtained, respectively. Some classical type Tauberian theorems given for Cesàro summability $(C, 1)$ and weighted mean method of summability $(\overline{N}, q)$ have been extended and generalized.

AN AUXILIARY RESULT

The following two representations of $s(x) - \tau_q^{(1)}(x)$ will be needed in the proofs of our main results.

Lemma 1 *([13])*

(i) For $\rho > 1$ and sufficiently large x,

$$\begin{aligned} s(x) - \tau_q^{(1)}(x) &= \frac{Q(\rho x)}{Q(\rho x) - Q(x)}\left(\tau_q^{(1)}(\rho x) - \tau_q^{(1)}(x)\right) \\ &- \frac{1}{Q(\rho x) - Q(x)}\int_x^{\rho x}(s(t) - s(x))\,q(t)\,dt. \end{aligned}$$

(ii) For $0 < \rho < 1$ and sufficiently large x,

$$s(x) - \tau_q^{(1)}(x) = \frac{Q(\rho x)}{Q(x) - Q(\rho x)}\left(\tau_q^{(1)}(x) - \tau_q^{(1)}(\rho x)\right)$$

$$+ \quad \frac{1}{Q(x)-Q(\rho x)} \int_{\rho x}^{x} (s(x)-s(t))\, q(t)\, dt.$$

MAIN RESULTS

For the main results of the paper, we need the following definitions and notations.

Definition 1 *([7]) A positive function Q is called regularly varying of index $\alpha > 0$ if*

$$\lim_{x\to\infty} \frac{Q(\rho x)}{Q(x)} = \rho^{\alpha}, \quad \rho > 0. \tag{4}$$

It easily follows from Definition 1 that for all $\rho > 1$ and sufficiently large x,

$$\frac{\rho^{\alpha}}{2(\rho^{\alpha}-1)} \le \frac{Q(\rho x)}{Q(\rho x)-Q(x)} \le \frac{3\rho^{\alpha}}{2(\rho^{\alpha}-1)} \tag{5}$$

and for all $0 < \rho < 1$ and sufficiently large x,

$$\frac{\rho^{\alpha}}{2(1-\rho^{\alpha})} \le \frac{Q(\rho x)}{Q(x)-Q(\rho x)} \le \frac{3\rho^{\alpha}}{2(1-\rho^{\alpha})}. \tag{6}$$

We note that if (4) holds, then the following equivalent conditions are clearly satisfied (see [5]):

$$\liminf_{x\to\infty} \frac{Q(x)}{Q(\rho x)} < 1, \quad \text{for every } \rho > 1 \tag{7}$$

and

$$\liminf_{x\to\infty} \frac{Q(\rho x)}{Q(x)} < 1, \quad \text{for every } 0 < \rho < 1. \tag{8}$$

First, we consider real-valued functions and prove the following Tauberian theorems.

Theorem 1 *Let (4) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, m\right)$ summable to L and $v_q^{(m-1)}(x)$ is one-sided bounded, then $s(x)$ is $\left(\overline{N}, q, m-1\right)$ summable to L.*

Corollary 1 *([13]) Let (4) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, 1\right)$ summable to L and $v_q^{(0)}(x)$ is one-sided bounded, then $s(x)$ converges to L.*

Theorem 2 *Let (7) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, m\right)$ summable to L and $\tau_q^{(m-1)}(x)$ is slowly decreasing, then $s(x)$ is $\left(\overline{N}, q, m-1\right)$ summable to L.*

Corollary 2 *([13]) Let (7) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, 1\right)$ summable to L and slowly decreasing, then $s(x)$ converges to L.*

A real-valued function $s(x)$ defined on $\mathbf{R}_+$ is said to be slowly decreasing if

$$\lim_{\rho\to 1^+} \liminf_{x\to\infty} \min_{x\le t\le \rho x} (s(t)-s(x)) \ge 0. \tag{9}$$

Note that condition (9) can be equivalently reformulated as follows:

$$\lim_{\rho\to 1^-} \liminf_{x\to\infty} \min_{\rho x\le t\le x} (s(x)-s(t)) \ge 0. \tag{10}$$

Second, we consider complex-valued functions and prove the following Tauberian theorems.

Theorem 3 *Let (4) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, m\right)$ summable to L and $v_q^{(m-1)}(x)$ is bounded, then $s(x)$ is $\left(\overline{N}, q, m-1\right)$ summable to L.*

Corollary 3 *Let (4) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, 1\right)$ summable to L and $v_q^{(0)}(x)$ is bounded, then $s(x)$ converges to L.*

Theorem 4 *Let (7) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, m\right)$ summable to L and $\tau_q^{(m-1)}(x)$ is slowly oscillating, then $s(x)$ is $\left(\overline{N}, q, m-1\right)$ summable to L.*

Corollary 4 *Let (7) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, 1\right)$ summable to L and slowly oscillating, then $s(x)$ converges to L.*

A complex-valued function $s(x)$ defined on $\mathbf{R}_+$ is said to be slowly oscillating if

$$\lim_{\rho \to 1^+} \limsup_{x \to \infty} \max_{x \le t \le \rho x} |s(t) - s(x)| = 0. \tag{11}$$

Note that condition (11) can be equivalently reformulated as follows:

$$\lim_{\rho \to 1^-} \limsup_{x \to \infty} \max_{\rho x \le t \le x} |s(x) - s(t)| = 0. \tag{12}$$

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