

Generalized contractions on partial metric spaces

Ishak Altun*, Ferhan Sola, Hakan Simsek

Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey

ARTICLE INFO

Article history:

Received 13 May 2010

Received in revised form 10 August 2010

Accepted 18 August 2010

Keywords:

Fixed point

Generalized contraction

Partial metric space

ABSTRACT

In the present paper, we give some fixed point theorems for generalized contractive type mappings on partial metric space. Also, a homotopy result is given.

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1. Introduction and preliminaries

Fixed point theory became one of the most interesting area of research in the last fifty years. Many authors studied contractive type mappings on a complete metric space X which are generalizations of the following well-known Banach contraction principles: Let (X, d) be a complete metric space and let $F : X \rightarrow X$ be a mapping. If there exists $\lambda \in [0, 1)$ such that

$$d(Fx, Fy) \leq \lambda d(x, y) \quad (1.1)$$

for all $x, y \in X$, then F has a unique fixed point in X . Kannan [8] introduced the contractive condition: there exists $\lambda \in (0, 1/2)$ such that

$$d(Fx, Fy) \leq \lambda [d(x, Fx) + d(y, Fy)] \quad (1.2)$$

for all $x, y \in X$, and proved a fixed point theorem using (1.2) instead of (1.1). The conditions (1.1) and (1.2) are independent, as it was shown by two examples in [9]. In 1971, Reich [14] generalized Banach and Kannan fixed point theorems using the contractive condition: for all $x, y \in X$,

$$d(Fx, Fy) \leq \alpha d(x, y) + \beta d(x, Fx) + \gamma d(y, Fy),$$

where α, β, γ are nonnegative reals with $\alpha + \beta + \gamma < 1$. Again in 1971, Ćirić [4] generalized such theorems using the contractive condition: for each $x, y \in X$,

$$d(Fx, Fy) \leq \alpha(x, y)d(x, y) + \beta(x, y)d(x, Fx) + \gamma(x, y)d(y, Fy) + \delta(x, y)[d(x, Fy) + d(y, Fx)], \quad (1.3)$$

where $\alpha, \beta, \gamma, \delta$ are functions from X^2 into $[0, 1)$ such that

$$\lambda = \sup \{ \alpha(x, y) + \beta(x, y) + \gamma(x, y) + 2\delta(x, y) : x, y \in X \} < 1. \quad (1.4)$$

* Corresponding author.

E-mail address: ishakaltun@yahoo.com (I. Altun).

Mappings which satisfying (1.3) and (1.4) Ćirić [4] called generalized contractions. As observed in Ćirić [3], a self-mapping F on a metric space (X, d) is generalized contraction if and only if F satisfies the following condition:

$$d(Fx, Fy) \leq \lambda m(x, y),$$

where

$$m(x, y) = \max \left\{ d(x, y), d(x, Fx), d(y, Fy), \frac{1}{2} [d(x, Fy) + d(y, Fx)] \right\},$$

$\lambda \in (0, 1)$ and $x, y \in X$. Other generalization of the contractive condition (1.1) is nonlinear type [2], that is, for all $x, y \in X$,

$$d(Fx, Fy) \leq \phi(d(x, y)),$$

where $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ($\mathbb{R}^+$ is nonnegative reals) is upper semicontinuous function from the right (i.e. $\alpha_n \rightarrow \alpha$ implies $\limsup_{n \rightarrow \infty} \phi(\alpha_n) \leq \phi(\alpha)$) such that $\phi(t) < t$ for each $t > 0$. Again Matkowski [10] used different type nonlinear contractive condition such that: for all $x, y \in X$,

$$d(Fx, Fy) \leq \phi(d(x, y)),$$

where $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is nondecreasing and satisfies $\lim_{n \rightarrow \infty} \phi^n(t) = 0$ for all $t > 0$. After than in [1] and some other papers, the following condition, which is known as generalized nonlinear contractive condition, was used: for all $x, y \in X$,

$$d(Fx, Fy) \leq \phi(m(x, y)),$$

where $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous, nondecreasing function such that $\phi(t) < t$ for each $t > 0$.

The aim of this paper is to prove some generalized versions of the result of Matthews using different type condition in partial metric spaces. First, we recall some definitions of partial metric space and some properties of theirs [11–13,15,16].

A partial metric on a nonempty set X is a function $p: X \times X \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$:

$$(p_1) \quad x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y),$$

$$(p_2) \quad p(x, x) \leq p(x, y),$$

$$(p_3) \quad p(x, y) = p(y, x),$$

$$(p_4) \quad p(x, y) \leq p(x, z) + p(z, y) - p(z, z).$$

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X . It is clear that, if $p(x, y) = 0$, then from (p_1) and (p_2) $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. Other examples of partial metric spaces which are interesting from a computational point of view may be found in [5,11].

There are some generalizations of partial metrics. For example, O'Neill [13] generalized it a bit further by admitting negative distances. The partial metric of O'Neill sense is called dualistic partial metric. Also, Heckmann [7] generalized it by omitting small self-distance axiom $p(x, x) \leq p(x, y)$. The partial metric of Heckmann sense is called weak partial metric. The inequality $2p(x, y) \geq p(x, x) + p(y, y)$ is satisfied for all x, y in a weak partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family open p -balls $\{B_p(x, \varepsilon): x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X: p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X , then the function $p^s: X \times X \rightarrow \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on X .

Let (X, p) be a partial metric space. Then:

A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n)$.

A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$.

A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p , to a point $x \in X$ such that $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$.

It is easy to see that, every closed subset of a complete partial metric space is complete.

Lemma 1. ([11,12]) Let (X, p) be a partial metric space.

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s) .

(b) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore, $\lim_{n \rightarrow \infty} p^s(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m).$$

Lemma 2. Let (X, p) be a partial metric space, $A \subset X$ and $x_0 \in X$. Define $p(x_0, A) = \inf\{p(x_0, x) : x \in A\}$. Then $a \in \bar{A} \Leftrightarrow p(a, A) = p(a, a)$.

Proof.

$$\begin{aligned}
 a \in \bar{A} &\Leftrightarrow B_p(a, \varepsilon) \cap A \neq \emptyset \quad \text{for all } \varepsilon > 0 \\
 &\Leftrightarrow p(a, x) < \varepsilon + p(a, a) \quad \text{for all } \varepsilon > 0 \text{ and some } x \in A \\
 &\Leftrightarrow p(a, x) - p(a, a) < \varepsilon \quad \text{for all } \varepsilon > 0 \text{ and some } x \in A \\
 &\Leftrightarrow \inf\{p(a, x) - p(a, a) : x \in A\} = 0 \\
 &\Leftrightarrow \inf\{p(a, x) : x \in A\} - p(a, a) = 0 \\
 &\Leftrightarrow \inf\{p(a, x) : x \in A\} = p(a, a) \\
 &\Leftrightarrow p(a, A) = p(a, a). \quad \square
 \end{aligned}$$

After the definition of the concept of partial metric space, Matthews [11] obtained a Banach type fixed point theorem on complete partial metric spaces. In Section 2, we give some generalized versions of the fixed point theorem of Matthews. In Section 3, a homotopy result is presented.

2. Main result

We begin the generalized nonlinear contractive type fixed point theorem.

Theorem 1. Let (X, p) be a complete partial metric space and let $F : X \rightarrow X$ be a map such that

$$p(Fx, Fy) \leq \phi \left(\max \left\{ p(x, y), p(x, Fx), p(y, Fy), \frac{1}{2} [p(x, Fy) + p(y, Fx)] \right\} \right) \quad (2.1)$$

for all $x, y \in X$, where $\phi : [0, \infty) \rightarrow [0, \infty)$ is continuous, nondecreasing function such that $\phi(t) < t$ for each $t > 0$. Then F has a unique fixed point.

Proof. From the conditions on ϕ , it is clear that $\lim_{n \rightarrow \infty} \phi^n(t) = 0$ for $t > 0$. Let $x_0 \in X$ be an arbitrary point. Define a sequence $\{x_n\}$ in X by $x_n = Fx_{n-1}$ for $n = 1, 2, \dots$. Now if $x_{n_0} = x_{n_0+1}$ for some $n_0 = 0, 1, 2, \dots$, then it is clear that x_{n_0} is a fixed point of F . Now assume $x_n \neq x_{n+1}$ for all n . Then we have from (2.1)

$$\begin{aligned}
 p(x_{n+1}, x_n) &= p(Fx_n, Fx_{n-1}) \\
 &\leq \phi \left(\max \left\{ p(x_n, x_{n-1}), p(x_n, Fx_n), p(x_{n-1}, Fx_{n-1}), \frac{1}{2} [p(x_n, Fx_{n-1}) + p(x_{n-1}, Fx_n)] \right\} \right) \\
 &\leq \phi \left(\max \left\{ p(x_n, x_{n-1}), p(x_n, x_{n+1}), \frac{1}{2} [p(x_{n-1}, x_n) + p(x_n, x_{n+1})] \right\} \right) \\
 &= \phi(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\}),
 \end{aligned} \quad (2.2)$$

since

$$p(x_n, x_n) + p(x_{n-1}, x_{n+1}) \leq p(x_{n-1}, x_n) + p(x_n, x_{n+1})$$

and ϕ is nondecreasing. Now if

$$\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\} = p(x_n, x_{n+1})$$

for some n , then from (2.2) we have

$$p(x_{n+1}, x_n) \leq \phi(p(x_n, x_{n+1})) < p(x_{n+1}, x_n)$$

which is a contradiction since $p(x_n, x_{n+1}) > 0$. Thus

$$\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\} = p(x_n, x_{n-1})$$

for all n . Then from (2.2) we have

$$p(x_{n+1}, x_n) \leq \phi(p(x_n, x_{n-1}))$$

and hence

$$p(x_{n+1}, x_n) \leq \phi^n(p(x_1, x_0)). \quad (2.3)$$

On the other hand, since

$$\max\{p(x_n, x_n), p(x_{n+1}, x_{n+1})\} \leq p(x_n, x_{n+1}),$$

then from (2.3) we have

$$\max\{p(x_n, x_n), p(x_{n+1}, x_{n+1})\} \leq \phi^n(p(x_1, x_0)). \quad (2.4)$$

Therefore

$$\begin{aligned} p^s(x_n, x_{n+1}) &= 2p(x_n, x_{n+1}) - p(x_n, x_n) - p(x_{n+1}, x_{n+1}) \\ &\leq 2p(x_n, x_{n+1}) + p(x_n, x_n) + p(x_{n+1}, x_{n+1}) \\ &\leq 4\phi^n(p(x_1, x_0)). \end{aligned}$$

This shows that $\lim_{n \rightarrow \infty} p^s(x_n, x_{n+1}) = 0$. Now we have

$$\begin{aligned} p^s(x_{n+k}, x_n) &\leq p^s(x_{n+k}, x_{n+k-1}) + \cdots + p^s(x_{n+1}, x_n) \\ &\leq 4\phi^{n+k-1}(p(x_1, x_0)) + \cdots + 4\phi^n(p(x_1, x_0)). \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s) . Since (X, p) is complete then from Lemma 1 (X, p^s) is complete and so the sequence $\{x_n\}$ is converges in the metric space (X, p^s) , say $\lim_{n \rightarrow \infty} p^s(x_n, x) = 0$. Again from Lemma 1, we have

$$p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m). \quad (2.5)$$

Moreover since $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s) , we have $\lim_{n, m \rightarrow \infty} p^s(x_n, x_m) = 0$ and from (2.4) we have $\lim_{n \rightarrow \infty} p(x_n, x_n) = 0$, thus from the definition p^s we have $\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0$. Therefore from (2.5) we have $p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0$. Now we show that $p(x, Fx) = 0$. Assume this is not true, then from (2.1) we obtain

$$\begin{aligned} p(x, Fx) &\leq p(x, Fx_n) + p(Fx_n, Fx) - p(Fx_n, Fx_n) \\ &\leq p(x, x_{n+1}) + p(Fx_n, Fx) \\ &\leq p(x, x_{n+1}) + \phi\left(\max\left\{p(x, x_n), p(x, Fx), p(x_n, x_{n+1}), \frac{1}{2}[p(x, x_{n+1}) + p(x_n, Fx)]\right\}\right) \\ &\leq p(x, x_{n+1}) + \phi\left(\max\left\{p(x, x_n), p(x, Fx), p(x_n, x_{n+1}), \frac{1}{2}[p(x, x_{n+1}) + p(x_n, x) + p(x, Fx) - p(x, x)]\right\}\right) \\ &= p(x, x_{n+1}) + \phi\left(\max\left\{p(x, x_n), p(x, Fx), p(x_n, x_{n+1}), \frac{1}{2}[p(x, x_{n+1}) + p(x_n, x) + p(x, Fx)]\right\}\right) \end{aligned}$$

using the continuity of ϕ and letting $n \rightarrow \infty$, we have

$$p(x, Fx) \leq \phi(p(x, Fx)),$$

which is a contradiction. Thus $p(x, Fx) = 0$ and so $x = Fx$. Now let z is another fixed point of F , that is $x \neq z$ then from (2.1), since $p(x, x) = 0$, we have

$$\begin{aligned} p(x, z) &= p(Fx, Fz) \\ &\leq \phi\left(\max\left\{p(x, z), p(x, Fx), p(z, Fz), \frac{1}{2}[p(x, Fz) + p(z, Fx)]\right\}\right) \\ &= \phi\left(\max\left\{p(x, z), p(x, x), p(z, z), \frac{1}{2}[p(x, z) + p(z, x)]\right\}\right) \\ &= \phi(\max\{p(x, z), p(z, z)\}) \\ &= \phi(p(x, z)), \end{aligned}$$

which is a contradiction. Thus $x = z$. $\square$

Example 1. Let $X = [0, \infty)$ and $p(x, y) = \max\{x, y\}$, then it is clear that (X, p) is a complete partial metric space. Let $T : X \rightarrow X$, $Fx = \frac{x^2}{1+x}$ for all $x \in X$ and $\phi : [0, \infty) \rightarrow [0, \infty)$, $\phi(t) = \frac{t^2}{1+t}$. Then for all $x, y \in X$ with $x \geq y$ we have

$$\begin{aligned} p(Fx, Fy) &= \max\left\{\frac{x^2}{1+x}, \frac{y^2}{1+y}\right\} \\ &= \frac{x^2}{1+x} \\ &= \phi(p(x, y)) \\ &\leq \phi\left(\max\left\{p(x, y), p(x, Fx), p(y, Fy), \frac{1}{2}[p(x, Fy) + p(y, Fx)]\right\}\right). \end{aligned}$$

This shows that all conditions of Theorem 1 are satisfied and so F has a fixed point in X . But we cannot apply the result of Matthews (see Corollary 2 below) to this example, because there is no $\alpha \in [0, 1)$ such that $p(Fx, Fy) \leq \alpha p(x, y)$ for all $x, y \in X$.

If we take $\phi(t) = \lambda t$ for $\lambda \in [0, 1)$ in Theorem 1, we have the following corollary.

Corollary 1. Let (X, p) be a complete partial metric space and let $F : X \rightarrow X$ be a map such that

$$p(Fx, Fy) \leq \lambda \max\left\{p(x, y), p(x, Fx), p(y, Fy), \frac{1}{2}[p(x, Fy) + p(y, Fx)]\right\}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then F has a unique fixed point.

Now we state Hardy and Rogers type [6] fixed point theorem.

Theorem 2. Let (X, p) be a complete partial metric space and let $F : X \rightarrow X$ be a map such that

$$p(Fx, Fy) \leq ap(x, y) + bp(x, Fx) + cp(y, Fy) + dp(x, Fy) + ep(y, Fx) \quad (2.6)$$

for all $x, y \in X$, where $a, b, c, d, e \geq 0$ and, if $d \geq e$, then $a + b + c + d + e < 1$, if $d < e$, then $a + b + c + d + 2e < 1$. Then F has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Define a sequence $\{x_n\}$ in X by $x_n = Fx_{n-1}$ for $n = 1, 2, \dots$. Now if $x_{n_0} = x_{n_0+1}$ for some $n_0 = 0, 1, 2, \dots$, then it is clear that x_{n_0} is a fixed point of F . Now assume $x_n \neq x_{n+1}$ for all n . Then we have from (2.6)

$$\begin{aligned} p(x_{n+1}, x_n) &= p(Fx_n, Fx_{n-1}) \\ &\leq ap(x_n, x_{n-1}) + bp(x_n, Fx_n) + cp(x_{n-1}, Fx_{n-1}) + dp(x_n, Fx_{n-1}) + ep(x_{n-1}, Fx_n) \\ &= ap(x_n, x_{n-1}) + bp(x_n, x_{n+1}) + cp(x_{n-1}, x_n) + dp(x_n, x_n) + ep(x_{n-1}, x_{n+1}) \\ &\leq (a + c + e)p(x_n, x_{n-1}) + (b + e)p(x_n, x_{n+1}) + (d - e)p(x_n, x_n). \end{aligned} \quad (2.7)$$

Now if $d \geq e$, then from (2.7) we have

$$p(x_{n+1}, x_n) \leq \max\left\{\frac{a+c+d}{1-b-e}, \frac{a+c+e}{1-b-d}\right\} p(x_n, x_{n-1}) \quad (2.8)$$

for all n . If $d < e$, then from (2.7) by omitting the term $-ed(x_n, x_n)$, we have

$$p(x_{n+1}, x_n) \leq \max\left\{\frac{a+c+d+e}{1-b-e}, \frac{a+c+e}{1-b-d-e}\right\} p(x_n, x_{n-1}). \quad (2.9)$$

Hence from (2.8) and (2.9) we have

$$p(x_{n+1}, x_n) \leq \lambda^n p(x_1, x_0),$$

where

$$\lambda = \begin{cases} \max\left\{\frac{a+c+d}{1-b-e}, \frac{a+c+e}{1-b-d}\right\}, & d \geq e, \\ \max\left\{\frac{a+c+d+e}{1-b-e}, \frac{a+c+e}{1-b-d-e}\right\}, & d < e. \end{cases}$$

It is clear that $\lambda \in [0, 1)$. On the other hand, since

$$\max\{p(x_n, x_n), p(x_{n+1}, x_{n+1})\} \leq p(x_n, x_{n+1}),$$

then we have

$$\max\{p(x_n, x_n), p(x_{n+1}, x_{n+1})\} \leq \lambda^n p(x_1, x_0). \quad (2.10)$$

Therefore

$$\begin{aligned} p^s(x_n, x_{n+1}) &= 2p(x_n, x_{n+1}) - p(x_n, x_n) - p(x_{n+1}, x_{n+1}) \\ &\leq 2p(x_n, x_{n+1}) + p(x_n, x_n) + p(x_{n+1}, x_{n+1}) \\ &\leq 4\lambda^n p(x_1, x_0). \end{aligned}$$

This shows that $\lim_{n \rightarrow \infty} p^s(x_n, x_{n+1}) = 0$. Now we have

$$\begin{aligned} p^s(x_{n+k}, x_n) &\leq p^s(x_{n+k}, x_{n+k-1}) + \cdots + p^s(x_{n+1}, x_n) \\ &\leq 4\lambda^{n+k-1} p(x_1, x_0) + \cdots + 4\lambda^n p(x_1, x_0) \\ &= 4 \frac{\lambda^n (1 - \lambda^k)}{1 - \lambda} p(x_1, x_0) \\ &\leq 4 \frac{\lambda^n}{1 - \lambda} p(x_1, x_0). \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s) . Since (X, p) is complete then from Lemma 1, the sequence $\{x_n\}$ is converges in the metric space (X, p^s) , say $\lim_{n \rightarrow \infty} p^s(x_n, x) = 0$. Again from Lemma 1, we have

$$p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m). \quad (2.11)$$

Moreover since $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s) , we have $\lim_{n, m \rightarrow \infty} p^s(x_n, x_m) = 0$ and from (2.10) we have $\lim_{n \rightarrow \infty} p(x_n, x_n) = 0$, thus from the definition p^s we have $\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0$. Therefore from (2.11) we have $p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0$. Now we show that $p(x, Fx) = 0$. Assume this is not true, then from (2.6) we obtain

$$\begin{aligned} p(x, Fx) &\leq p(x, Fx_n) + p(Fx_n, Fx) - p(Fx_n, Fx_n) \\ &\leq p(x, x_{n+1}) + p(Fx_n, Fx) \\ &\leq p(x, x_{n+1}) + ap(x, x_n) + bp(x, Fx) + cp(x_n, x_{n+1}), dp(x, x_{n+1}) + ep(x_n, Fx) \\ &\leq p(x, x_{n+1}) + ap(x, x_n) + bp(x, Fx) + cp(x_n, x_{n+1}), dp(x, x_{n+1}) + ep(x_n, x) + ep(x, Fx) \end{aligned}$$

letting $n \rightarrow \infty$, we have

$$p(x, Fx) \leq (b + e)p(x, Fx),$$

which is a contradiction. Thus $p(x, Fx) = 0$ and so $x = Fx$. The uniqueness of fixed point follows from (2.6) easily. $\square$

We can have the following corollaries from Theorem 2.

Corollary 2 (Banach type). Let (X, p) be a complete partial metric space and let $F : X \rightarrow X$ be a map such that

$$p(Fx, Fy) \leq \alpha p(x, y)$$

for all $x, y \in X$, where $0 \leq \alpha < 1$. Then F has a unique fixed point.

Corollary 3 (Kannan type). Let (X, p) be a complete partial metric space and let $F : X \rightarrow X$ be a map such that

$$p(Fx, Fy) \leq \beta p(x, Fx) + \gamma p(y, Fy)$$

for all $x, y \in X$, where $\beta, \gamma \geq 0$ and $\beta + \gamma < 1$. Then F has a unique fixed point.

Corollary 4 (Reich type). Let (X, p) be a complete partial metric space and let $F : X \rightarrow X$ be a map such that

$$p(Fx, Fy) \leq \alpha p(x, y) + \beta p(x, Fx) + \gamma p(y, Fy)$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma < 1$. Then F has a unique fixed point.

3. A homotopy result

Now we state a homotopy result.

Theorem 3. Let (X, p) be a complete partial metric space, U an open subset of X . Suppose that $H : \bar{U} \times [0, 1] \rightarrow X$ with the following properties:

1. $x \neq H(x, \lambda)$ for every $x \in \partial U$ and $\lambda \in [0, 1]$ (here ∂U denotes the boundary of U in X).
2. For all $x, y \in \bar{U}$ and $\lambda \in [0, 1]$, $L \in [0, 1]$, such that

$$p(H(x, \lambda), H(y, \lambda)) \leq Lp(x, y).$$

3. There exists $M \geq 0$, such that

$$p(H(x, \lambda), H(x, \mu)) \leq M|\lambda - \mu|$$

for every $x \in \bar{U}$ and $\lambda, \mu \in [0, 1]$.

If $H(\cdot, 0)$ has a fixed point in U , then $H(\cdot, 1)$ has a fixed point in U .

Proof. Consider the set

$$A = \{\lambda \in [0, 1] : x = H(x, \lambda) \text{ for some } x \in U\}$$

Since $H(\cdot, 0)$ has a fixed point in U , then A is nonempty, that is, $0 \in A$. We will show that A is both open and closed in $[0, 1]$ and hence by connectedness we have that $A = [0, 1]$. As a result, $H(\cdot, 1)$ has a fixed point in U . We first show that A is closed in $[0, 1]$. To see this let $\{\lambda_n\}_{n=1}^\infty \subseteq A$ with $\lambda_n \rightarrow \lambda \in [0, 1]$ as $n \rightarrow \infty$. We must show that $\lambda \in A$. Since $\lambda_n \in A$ for $n = 1, 2, 3, \dots$, there exists $x_n \in U$ with $x_n = H(x_n, \lambda_n)$. Also for $n, m \in \{1, 2, 3, \dots\}$ we have

$$\begin{aligned} p(x_n, x_m) &= p(H(x_n, \lambda_n), H(x_m, \lambda_m)) \\ &\leq p(H(x_n, \lambda_n), H(x_n, \lambda_m)) + p(H(x_n, \lambda_m), H(x_m, \lambda_m)) - p(H(x_n, \lambda_m), H(x_n, \lambda_m)) \\ &\leq p(H(x_n, \lambda_n), H(x_n, \lambda_m)) + p(H(x_n, \lambda_m), H(x_m, \lambda_m)) \\ &\leq M|\lambda_n - \lambda_m| + Lp(x_n, x_m), \end{aligned}$$

that is,

$$p(x_n, x_m) \leq \left(\frac{M}{1-L} \right) |\lambda_n - \lambda_m|.$$

Since $\{\lambda_n\}_{n=1}^\infty$ is a Cauchy sequence we have that $\lim_{n,m \rightarrow \infty} p(x_n, x_m) = 0$, that is, $\{x_n\}$ is a Cauchy sequence in X . Since X is complete there exists $x \in \bar{U}$ with $p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n) = \lim_{n,m \rightarrow \infty} p(x_n, x_m) = 0$. Since

$$\begin{aligned} p(x_n, H(x, \lambda)) &= p(H(x_n, \lambda_n), H(x, \lambda)) \\ &\leq p(H(x_n, \lambda_n), H(x_n, \lambda)) + p(H(x_n, \lambda), H(x, \lambda)) - p(H(x_n, \lambda), H(x_n, \lambda)) \\ &\leq p(H(x_n, \lambda_n), H(x_n, \lambda)) + p(H(x_n, \lambda), H(x, \lambda)) \\ &\leq M|\lambda_n - \lambda| + Lp(x_n, x), \end{aligned}$$

we have $\lim_{n \rightarrow \infty} p(x_n, H(x, \lambda)) = 0$ and so

$$\lim_{n \rightarrow \infty} p(x_n, H(x, \lambda)) = p(x, H(x, \lambda)) = 0.$$

Thus $\lambda \in A$ and A is closed in $[0, 1]$.

Next we show that A is an open in $[0, 1]$. Let $\lambda_0 \in A$. Then there exists $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$. Since U is open, then there exists $r > 0$ such that $B_p(x_0, r) \subseteq U$. Now, let $\delta = p(x_0, \partial U) = \inf\{p(x_0, x) : x \in \partial U\}$. Then from Lemma 2, $r = \delta - p(x_0, x_0) > 0$. Fix $\varepsilon > 0$ with

$$\varepsilon < \frac{(1-L)\delta}{M}.$$

Let $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, then for $x \in \overline{B_p(x_0, r)} = \{x \in X : p(x, x_0) \leq r + p(x_0, x_0)\}$,

$$\begin{aligned} p(H(x, \lambda), x_0) &= p(H(x, \lambda), H(x_0, \lambda_0)) \\ &\leq p(H(x, \lambda), H(x, \lambda_0)) + p(H(x, \lambda_0), H(x_0, \lambda_0)) - p(H(x, \lambda_0), H(x, \lambda_0)) \\ &\leq p(H(x, \lambda), H(x, \lambda_0)) + p(H(x, \lambda_0), H(x_0, \lambda_0)) \\ &\leq M|\lambda - \lambda_0| + Lp(x, x_0) \\ &\leq (1 - L)\delta + L(r + p(x_0, x_0)) \\ &= r + p(x_0, x_0). \end{aligned}$$

Thus for each fixed $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$,

$$H(\cdot, \lambda) : \overline{B_p(x_0, r)} \rightarrow \overline{B_p(x_0, r)}.$$

We can now apply Corollary 2 to deduce that $H(\cdot, \lambda)$ has a fixed point in $\overline{U}$. But this fixed point must be in U since (1) holds. Thus $\lambda \in A$ for any $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ and therefore A is open in $[0, 1]$. $\square$

Acknowledgements

The authors thank the referees for their appreciation, valuable comments and suggestions.

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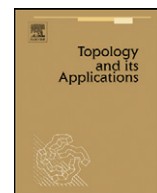
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Update

Topology and its Applications

Volume 158, Issue 13, 15 August 2011, Page 1738–1740

DOI: <https://doi.org/10.1016/j.topol.2011.05.023>



Corrigendum

Corrigendum to “Generalized contractions on partial metric spaces” [Topology Appl. 157 (2010) 2778–2785]

I. Altun^{a,*}, K. Sadarangani^b

^a Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey

^b Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain

ARTICLE INFO

Article history:

Received 21 February 2011

Accepted 13 May 2011

MSC:

54H25

47H10

Keywords:

Fixed point

Generalized contraction

Partial metric space

ABSTRACT

We correct the proof of Theorem 1 in the paper in the title.

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The following correction for the paper [1] should be noted.

In the proof of Theorem 1, the authors prove that

$$p^s(x_n, x_{n+1}) \leq 4\phi^n(p(x_1, x_0))$$

and from this inequality they obtain

$$\lim_{n \rightarrow \infty} p^s(x_n, x_{n+1}) = 0. \quad (0.1)$$

In order to prove that $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s) they use the following inequality

$$\begin{aligned} p^s(x_{n+k}, x_n) &\leq p^s(x_{n+k}, x_{n+k-1}) + \cdots + p^s(x_{n+1}, x_n) \\ &\leq 4\phi^{n+k-1}(p(x_1, x_0)) + \cdots + 4\phi^n(p(x_1, x_0)) \end{aligned} \quad (0.2)$$

and (0.1).

This argument is false as it is proved with the following example. Consider $(\mathbb{R}, d)$, where d is the usual metric in $\mathbb{R}$ and $x_n = \sum_{i=1}^n \frac{1}{i}$. Obviously,

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = \lim_{n \rightarrow \infty} |x_{n+1} - x_n| = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$$

On the other hand, the sequence $\{x_n\}$ is not a Cauchy sequence because it is not convergent.

DOI of original article: [10.1016/j.topol.2010.08.017](https://doi.org/10.1016/j.topol.2010.08.017).

* Corresponding author.

E-mail addresses: ishakaltun@yahoo.com (I. Altun), ksadaran@dma.ulpgc.es (K. Sadarangani).

The argument is correct when (X, d) is an ultrametric space.

In order to obtain the conclusion of Theorem 1 we must impose some conditions to the function $\phi : [0, \infty) \rightarrow [0, \infty)$.

Suppose that $\phi : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function and such that $\sum_{n=0}^{\infty} \phi^n(t)$ is a convergent series for any $t > 0$. These functions are known in the literature as (c)-comparison functions.

It is easily proved that if ϕ is a (c)-comparison function then $\phi(t) < t$ for any $t > 0$.

Then Theorem 1 can be replaced by the following theorem.

Theorem 1. Let (X, p) be a complete partial metric space and $F : X \rightarrow X$ be a map such that

$$p(Fx, Fy) \leq \phi \left(\max \left\{ p(x, y), p(x, Fx), p(y, Fy), \frac{1}{2}[p(x, Fy) + p(y, Fx)] \right\} \right)$$

for all $x, y \in X$ where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a (c)-comparison function. Then F has a unique fixed point.

Proof. Following the lines of the proof of Theorem 1 of [1], from (0.2) we obtain

$$p^s(x_{n+k}, x_n) \leq 4 \sum_{p=n}^{n+k-1} \phi^p(p(x_1, x_0))$$

and, as $\sum_{p=0}^{\infty} \phi^p(p(x_1, x_0))$ is convergent, from the last inequality, using Cauchy's criterium for convergent series, we obtain that $\{x_n\}$ is a Cauchy sequence. $\square$

On the other hand, the authors of [1] use the continuity of ϕ in order to prove that $p(x, Fx) = 0$.

More precisely, they obtain the following inequality

$$p(x, Fx) \leq p(x, x_{n+1}) + \phi \left(\max \left\{ p(x, x_n), p(x, Fx), p(x_n, x_{n+1}), \frac{1}{2}[p(x, x_{n+1}) + p(x_n, x) + p(x, Fx)] \right\} \right) \quad (0.3)$$

and letting $n \rightarrow \infty$ and using the continuity of ϕ

$$p(x, Fx) \leq \phi(p(x, Fx))$$

and from this fact the authors obtain $p(x, Fx) = 0$.

The following argument proves that the continuity of ϕ is not necessary in order to obtain the same conclusion.

As ϕ is a (c)-comparison function, $\phi(t) < t$ for $t > 0$.

Now, suppose that $p(x, Fx) > 0$, as $\lim_{n \rightarrow \infty} p(x_{n+1}, x_n) = 0$ and $\lim_{n \rightarrow \infty} p(x_n, x) = 0$, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$p(x_{n+1}, x_n) < \frac{1}{3}p(x, Fx) \quad (0.4)$$

and there exists $n_1 \in \mathbb{N}$ such that for $n > n_1$,

$$p(x_n, x) < \frac{1}{3}p(x, Fx). \quad (0.5)$$

If we take $n > \max\{n_0, n_1\}$ then, by (0.4), (0.5) and triangular inequality, we have

$$\begin{aligned} \frac{1}{2}[p(x_n, Fx) + p(x, Fx_n)] &\leq \frac{1}{2}[p(x_n, x) + p(x, Fx) - p(x, x) + p(x, Fx_n)] \\ &\leq \frac{1}{2} \left[\frac{1}{3}p(x, Fx) + p(x, Fx) + \frac{1}{3}p(x, Fx) \right] \\ &= \frac{5}{6}p(x, Fx). \end{aligned} \quad (0.6)$$

Now for $n > \max\{n_0, n_1\}$, by (0.4), (0.5) and (0.6), we have

$$\begin{aligned} p(x_{n+1}, Fx) &= p(Fx_n, Fx) \\ &\leq \phi \left(\max \left\{ p(x_n, x), p(x_n, Fx_n), p(x, Fx), \frac{1}{2}[p(x_n, Fx) + p(x, Fx_n)] \right\} \right) \\ &\leq \phi(p(x, Fx)). \end{aligned}$$

Letting $n \rightarrow \infty$ in the last inequality, we have $p(x, Fx) \leq \phi(p(x, Fx))$, which is a contradiction. Thus $p(x, Fx) = 0$. Therefore, Theorem 1 is an improvement of Theorem 1 of [1].

Finally, we present an example of a discontinuous (c)-comparison function.

Let $\phi : R_+ \rightarrow R_+$ be the function defined by

$$\phi(t) = \begin{cases} \frac{1}{4}t, & 0 \leq t < 1, \\ \frac{t}{t+1}, & 1 \leq t. \end{cases}$$

It is easily seen that ϕ is a (c)-comparison function and it is not continuous at $t_0 = 1$.

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