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A New Class of Leonardo Hybrid Numbers and Some Remarks on Leonardo Quaternions over Finite Fields

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Abstract: In this paper, we present a new class of Leonardo hybrid numbers that incorporate quantum integers into their components. This advancement presents a broader generalization of the q -Leonardo hybrid numbers. We explore some fundamental properties associated with these numbers. Moreover, we study special Leonardo quaternions over finite fields. In particular, we determine the Leonardo quaternions that are zero divisors or invertible elements in the quaternion algebra over the finite field $\mathbb{Z}_p$ for special values of prime integer p .

Keywords: hybrid numbers; quaternions; Fibonacci numbers; Leonardo numbers; quantum integer; zero divisor; finite fields

MSC: 11B37; 11B39; 11R52; 16G30



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1. Introduction

The study of second-order linear recurrence relations has various applications in mathematics, computer science, physics, engineering, and other fields where the behavior of sequences is of interest. For some recent studies, see [1–5]. The general second-order linear recurrence $\{W_n\}$ is defined by

$$W_n = sW_{n-1} + tW_{n-2}, \quad n \geq 2$$

with arbitrary initial values W_0, W_1 and nonzero integers s, t . The Binet formula of the sequence $\{W_n\}$ is

$$W_n = \frac{(W_1 - W_0\delta)\gamma^n - (W_1 - W_0\gamma)\delta^n}{\gamma - \delta},$$

where γ, δ are the roots of the characteristic polynomial $x^2 - sx - t$. Many well-known second-order linear recurrences arise as a special case of the sequence $\{W_n\}$. The well-known Fibonacci sequence $\{F_n\}$ arises by taking $s = t = 1$ and $W_0 = 0, W_1 = 1$ in the sequence $\{W_n\}$. In addition, if we take $s = t = 1$ with initial values $W_0 = 2, W_1 = 1$ in the sequence $\{W_n\}$, it gives the classical Lucas sequence $\{L_n\}$. Thus, the Binet formulas of the Fibonacci and Lucas sequences are $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$, respectively, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. For further information regarding the general second-order linear recurrences, we refer to [6–9].

There are several nonhomogenous extensions of the Fibonacci recurrence relation. One among them is the Leonardo sequence $\{\mathcal{L}_n\}$, which is defined by the following nonhomogenous recurrence relation:

$$\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2} + 1, \quad n \geq 2 \quad (1)$$

with initial values $\mathcal{L}_0 = \mathcal{L}_1 = 1$. The Leonardo sequence finds applications in various fields, including mathematics, computer science, and cryptography. It appears in problems involving tiling, divisibility, and combinatorial counting, among others. The sequence has been the subject of mathematical research and exploration, leading to the discovery of intriguing patterns and connections with other sequences and mathematical concepts. In 1981, Dijkstra [10] used these numbers as an integral part of his smoothsort algorithm. For the properties of Leonardo numbers and related studies, we refer to [11–16], and for the history of Leonardo sequences, see [A001595] in the On-Line Encyclopedia of Integer Sequences [17].

Recently, Kuhapatanakul and Chobsorn [18] have defined the generalized Leonardo sequence $\{\mathcal{L}_{k,n}\}$ by the following nonhomogenous relation:

$$\mathcal{L}_{k,n} = \mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k, \quad n \geq 2 \quad (2)$$

with initial values $\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1$. The parameter k is a fixed positive integer. It is clear to see that when $k = 1$, it reduces to the classical Leonardo sequence $\{\mathcal{L}_n\}$. Shattuck [19] gave a combinatorial interpretation for the generalized Leonardo sequence in terms of colored linear tilings. In particular, the generalized Leonardo number $\mathcal{L}_{k,n}$ counts the number of all linear colored tilings of length n by using squares, dominoes, and the k -tiles where a square covers a single cell, a domino covers two consecutive cells, and a k -tile is a rectangular piece coming in one of k colors, which must occur as the first piece in a tiling, if it occurs at all, and has an arbitrary length greater than or equal to two. For more on linear tilings, we refer to [19,20].

On the other hand, quaternions can be viewed as an extension of complex numbers and have been explored in various fields, including computer science, physics, differential geometry, and quantum physics, as extensively documented by researchers. Let F be a field with characteristic not 2. The generalized quaternion algebra over a field F is defined as:

$$Q_F(a, b) = \{x + yi + zj + tk \mid x, y, z, t \in F, i^2 = a, j^2 = b, ij = -ji = k\} \quad (3)$$

where a, b are nonzero invertible elements of field F . It is clear to see that the algebra $Q_{\mathbb{R}}(-1, -1)$ reduces to the real quaternion algebra. We recall that a generalized quaternion algebra is a division algebra if and only if a quaternion with a norm of zero is necessarily the zero quaternion. In other words, for $X = x + yi + zj + tk \in Q_F(a, b)$, the norm of X , denoted as $N(X)$ and defined as $N(X) = x^2 - ay^2 - bz^2 + abt^2$, equals zero if and only if $X = 0$. Otherwise, the algebra is called a split algebra. It is known that real quaternion algebra is a division algebra and the quaternion algebra over finite field $\mathbb{Z}_p$, denoted as $Q_{\mathbb{Z}_p}(-1, -1)$, is a split algebra, where p is an odd prime integer, see [21]. In [22], the author studied special elements in quaternion algebras over finite fields. For more information related to quaternion algebras, we refer to [22] and the references therein.

Similar to real quaternions, hybrid number multiplication is also noncommutative. The concept of hybrid numbers was introduced by Ozdemir [23] as a generalization of complex, hyperbolic, and dual numbers. They are defined as

$$\mathbb{H} = \{x + yi + z\epsilon + th \mid x, y, z, t \in \mathbb{R}, i^2 = -1, \epsilon^2 = 0, h^2 = 1, ih = -hi = \epsilon + i\}. \quad (4)$$

The addition and the subtraction of two hybrid numbers are defined component-wise, and the product of two hybrid numbers is defined according to the rule specified in (4). For more details on the hybrid numbers, we refer to [23].

Many authors have extensively researched different types of quaternions and hybrid numbers, where their components are derived from terms found in special integer sequences. In particular, Leonardo hybrid quaternions were studied in [24], Leonardo sedenions were studied in [25], Szynal [26] studied Horadam hybrid numbers, which generalize the classical Fibonacci hybrid numbers and the classical Lucas hybrid numbers. Polynomial versions of Fibonacci and Lucas hybrid numbers were studied in [27]. Tan and

Ait-Amrane [28] introduced the bi-periodic Horadam hybrid numbers. Alp and Kocer [29] bring together hybrid numbers and Leonardo numbers and defined the hybrid Leonardo numbers as:

$$\mathbb{H}\mathcal{L}_n = \mathcal{L}_n + \mathcal{L}_{n+1}i + \mathcal{L}_{n+2}\epsilon + \mathcal{L}_{n+3}h \quad (5)$$

where $\mathcal{L}_n$ is the n th Leonardo number. We note that throughout this paper, we call the sequence $\{\mathbb{H}\mathcal{L}_n\}$ as the Leonardo hybrid sequence. By considering the relation between Leonardo numbers and Fibonacci numbers, $\mathcal{L}_n = 2F_{n+1} - 1$, Ozimamoglu [30] expressed the coefficient of Leonardo hybrid numbers in terms of q -integers as

$$\begin{aligned} \mathbb{H}\mathcal{L}_n(\alpha; q) &= \left(2\alpha^n[n+1]_q - 1\right) + \left(2\alpha^{n+1}[n+2]_q - 1\right)i \\ &\quad + \left(2\alpha^{n+2}[n+3]_q - 1\right)\epsilon + \left(2\alpha^{n+3}[n+4]_q - 1\right)h \end{aligned} \quad (6)$$

and called them as q -Leonardo hybrid numbers. It is clear to see that by taking $q = \frac{-1}{\alpha^2}$, the q -Leonardo hybrid numbers reduce to the classical Leonardo hybrid numbers in (5). To express the coefficients of the Leonardo hybrid numbers in terms of quantum integers, it is essential to recall the definition of quantum integers. Quantum integers are mathematical objects that generalize the concept of integers and emerge in the field of quantum physics, particularly in the study of quantum groups and their representations. For positive integer n , the quantum integer (q -integer) n is defined by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1} \quad (7)$$

where q is a complex number with $q \neq 1$. For details on the theory of quantum calculus, we refer to the book by Kac and Cheung [31]. We also note that in [32], the authors established the concept of Fibonacci quaternions with coefficients from quantum integers. By using a similar approach, Kizilates [33] defined the Fibonacci and Lucas hybrid numbers with quantum integers as

$$\mathbb{H}F_n(\gamma; q) = \gamma^{n-1}[n]_q + \gamma^n[n+1]_q i + \gamma^{n+1}[n+2]_q \epsilon + \gamma^{n+2}[n+3]_q h \quad (8)$$

and

$$\mathbb{H}L_n(\gamma; q) = \gamma^n \frac{[2n]_q}{[n]_q} + \gamma^{n+1} \frac{[2n+2]_q}{[n+1]_q} i + \gamma^{n+2} \frac{[2n+4]_q}{[n+2]_q} \epsilon + \gamma^{n+3} \frac{[2n+6]_q}{[n+3]_q} h, \quad (9)$$

respectively.

In this paper, we introduce the concept of generalized Leonardo hybrid numbers, which generalize the classical Leonardo hybrid numbers. Different from the papers [29,30], we define a new family of hybrid numbers that reflects the generalized Leonardo numbers $\mathcal{L}_{k,n}$. Additionally, we define a new class of Leonardo hybrid numbers, called q -generalized Leonardo hybrid numbers, that incorporate quantum integers into their components. We derive several fundamental properties of these numbers including recurrence relations, the exponential generating function, the Binet formula, Vajda's identity, and summation formulas. We should note that this new class of Leonardo hybrid numbers is even more general than the one studied in [29,30]. Considering the extensive usage of quantum integers in physics, quantum hybrid numbers are expected to garner wider interest and find various applications. The major innovation point of the paper is that we study certain special Leonardo quaternions in quaternion algebras over finite fields. In particular, we consider the quaternion algebra $\mathbb{Q}_{\mathbb{Z}_p}(-1, -1)$, and we determine the Leonardo quaternions, which are zero divisors or invertible elements in the quaternion algebra over finite field $\mathbb{Z}_p$ for special values of prime integer p .

To conclude this section, we recall some basic identities associated with the generalized Leonardo numbers, which can be found in [18]:

$$\mathcal{L}_{k,n} = (k+1)F_{n+1} - k, \quad (10)$$

$$\mathcal{L}_{k,n} = (k+1)(L_n - F_{n-1}) - k, \quad (11)$$

$$\sum_{j=0}^n \mathcal{L}_{k,j} = \mathcal{L}_{k,n+2} - k(n+1) - 1. \quad (12)$$

2. q -Generalized Leonardo Hybrid Numbers

In this section, we give the definitions of generalized Leonardo hybrid numbers and q -generalized Leonardo hybrid numbers. We explore the fundamental properties of these sequences. Throughout this section, we adopt the notation $I := 1 + i + \epsilon + h$. Since we take $\alpha = \frac{1+\sqrt{5}}{2}$, we use the notations $\mathbb{H}F_n^q$ and $\mathbb{H}L_n^q$ instead of $\mathbb{H}F_n(\alpha; q)$ and $\mathbb{H}L_n(\alpha; q)$ in (8) and (9), respectively.

Definition 1. The n th generalized Leonardo hybrid number is defined by

$$\mathbb{H}\mathcal{L}_{k,n} = \mathcal{L}_{k,n} + \mathcal{L}_{k,n+1}i + \mathcal{L}_{k,n+2}\epsilon + \mathcal{L}_{k,n+3}h,$$

where i, ϵ , and h satisfy the multiplication rules in (4).

In the following theorem, we establish certain relations concerning the generalized Leonardo hybrid numbers that yield analogous results to those in Equations (1), (11) and (12), respectively.

Theorem 1. For generalized Leonardo hybrid numbers, we have the following

$$(i) \mathbb{H}\mathcal{L}_{k,n} = \mathbb{H}\mathcal{L}_{k,n-1} + \mathbb{H}\mathcal{L}_{k,n-2} + kI, \quad n \geq 2,$$

$$(ii) \mathbb{H}\mathcal{L}_{k,n} = (k+1)(\mathbb{H}L_n - \mathbb{H}F_{n-1}) - kI,$$

$$(iii) \sum_{j=0}^n \mathbb{H}\mathcal{L}_{k,j} = \mathbb{H}\mathcal{L}_{k,n+2} - k(n+1)I - I - (k+1)(i + 2\epsilon + 4h).$$

Proof. (i) For $n \geq 2$, by using Definition 1 and Equation (2), the generalized Leonardo hybrid numbers satisfy the following recurrence relation

$$\begin{aligned} \mathbb{H}\mathcal{L}_{k,n} &= \mathcal{L}_{k,n} + \mathcal{L}_{k,n+1}i + \mathcal{L}_{k,n+2}\epsilon + \mathcal{L}_{k,n+3}h \\ &= (\mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k) + (\mathcal{L}_{k,n} + \mathcal{L}_{k,n-1} + k)i \\ &\quad + (\mathcal{L}_{k,n+1} + \mathcal{L}_{k,n} + k)\epsilon + (\mathcal{L}_{k,n+2} + \mathcal{L}_{k,n-1} + k)h \\ &= (\mathcal{L}_{k,n-1} + \mathcal{L}_{k,n}i + \mathcal{L}_{k,n+1}\epsilon + \mathcal{L}_{k,n+2}h) \\ &\quad + (\mathcal{L}_{k,n-2} + \mathcal{L}_{k,n-1}i + \mathcal{L}_{k,n}\epsilon + \mathcal{L}_{k,n-1}h) \\ &\quad + k(1 + i + \epsilon + h) \\ &= \mathbb{H}\mathcal{L}_{k,n-1} + \mathbb{H}\mathcal{L}_{k,n-2} + kI. \end{aligned}$$

(ii) By using Definition 1 and Equation (11), the generalized Leonardo hybrid numbers can be expressed in terms of Fibonacci hybrid numbers and Lucas hybrid numbers as:

$$\begin{aligned}
 \mathbb{H}\mathcal{L}_{k,n} &= \mathcal{L}_{k,n} + \mathcal{L}_{k,n+1}i + \mathcal{L}_{k,n+2}\epsilon + \mathcal{L}_{k,n+3}h \\
 &= ((k+1)(L_n - F_{n-1}) - k) + ((k+1)(L_{n+1} - F_n) - k)i \\
 &\quad + ((k+1)(L_{n+2} - F_{n+1}) - k)\epsilon + ((k+1)(L_{n+3} - F_{n+2}) - k)h \\
 &= (k+1)((L_n + L_{n+1}i + L_{n+2}\epsilon + L_{n+3}h) - (F_{n-1} + F_n i + F_{n+1}\epsilon + F_{n+2}h)) \\
 &\quad - k(1 + i + \epsilon + h) \\
 &= (k+1)(\mathbb{H}L_n - \mathbb{H}F_{n-1}) - kI.
 \end{aligned}$$

(iii) By using Definition 1 and Equation (12), we have a sum formula for generalized Leonardo hybrid numbers as:

$$\begin{aligned}
 \sum_{j=0}^n \mathbb{H}\mathcal{L}_{k,j} &= \sum_{j=0}^n \mathcal{L}_{k,j} + \sum_{j=0}^n \mathcal{L}_{k,j+1}i + \sum_{j=0}^n \mathcal{L}_{k,j+2}\epsilon + \sum_{j=0}^n \mathcal{L}_{k,j+3}h \\
 &= (\mathcal{L}_{k,n+2} - k(n+1) - 1) + (\mathcal{L}_{k,n+3} - k(n+2) - 1 - \mathcal{L}_{k,0})i \\
 &\quad + (\mathcal{L}_{k,n+4} - k(n+3) - 1 - \mathcal{L}_{k,0} - \mathcal{L}_{k,1})\epsilon \\
 &\quad + (\mathcal{L}_{k,n+5} - k(n+4) - 1 - \mathcal{L}_{k,0} - \mathcal{L}_{k,1} - \mathcal{L}_{k,2})h \\
 &= \mathbb{H}\mathcal{L}_{k,n+2} - k(n+1)I - I - (k+1)(i + 2\epsilon + 4h).
 \end{aligned}$$

□

Remark 1. If we take $k = 1$ in Theorem 1 (i), we obtain the recurrence relation for the classical Leonardo hybrid numbers

$$\mathbb{H}\mathcal{L}_n = \mathbb{H}\mathcal{L}_{n-1} + \mathbb{H}\mathcal{L}_{n-2} + I, \quad n \geq 2,$$

which was given in [29]. If we take $k = 1$ in Theorem 1 (iii), we obtain

$$\sum_{j=0}^n \mathbb{H}\mathcal{L}_j = \mathbb{H}\mathcal{L}_{n+2} - (n+2)I - (2i + 4\epsilon + 8h),$$

which can be found in ([29] Theorem 2.5).

Definition 2. The n th q -generalized Leonardo hybrid number is defined by

$$\begin{aligned}
 \mathbb{H}\mathcal{L}_{k,n}^q &= ((k+1)\alpha^n[n+1]_q - k) + ((k+1)\alpha^{n+1}[n+2]_q - k)i \\
 &\quad + ((k+1)\alpha^{n+2}[n+3]_q - k)\epsilon + ((k+1)\alpha^{n+3}[n+4]_q - k)h,
 \end{aligned}$$

where i, ϵ , and h satisfy the multiplication rules in (4).

Some special cases of q -generalized Leonardo hybrid numbers can be given as follows:

1. If we take $q = \frac{-1}{\alpha^2}$ in Definition 2, we obtain the generalized Leonardo hybrid numbers $\mathbb{H}\mathcal{L}_{k,n}$ in Definition 1.
2. If we take $k = 1$ in Definition 2, we obtain the q -Leonardo hybrid numbers $\mathbb{H}\mathcal{L}_n(\alpha; q)$ in (6).
3. If we take $k = 1$ and $q = \frac{-1}{\alpha^2}$ in Definition 2, we obtain the classical Leonardo hybrid numbers $\mathbb{H}\mathcal{L}_n$ in (5).

The following result gives a relation between q -generalized Leonardo hybrid numbers and q -Fibonacci hybrid numbers.

Theorem 2. For $n > 0$, we have

$$\mathbb{H}\mathcal{L}_{k,n}^q = (k+1)\mathbb{H}F_{n+1}^q - kI,$$

where $\mathbb{H}F_n^q$ is the n th q -Fibonacci hybrid number.

Proof. By using the definitions of q -Fibonacci hybrid numbers and q -generalized Leonardo hybrid numbers, we obtain the desired result. $\square$

Remark 2. If we take $k = 1$ in Theorem 2, we obtain the identity in ([30] Corollary 3.1).

Next, we state the Binet formula for the q -generalized Leonardo hybrid numbers.

Theorem 3. The Binet formula for the q -generalized Leonardo hybrid numbers is

$$\mathbb{H}\mathcal{L}_{k,n}^q = (k+1) \left(\frac{\alpha^{n+1}\alpha^* - (\alpha q)^{n+1}\beta^*}{\alpha(1-q)} \right) - kI,$$

where $\alpha^* = 1 + \alpha i + \alpha^2 \epsilon + \alpha^3 h$ and $\beta^* = 1 + (\alpha q)i + (\alpha q)^2 \epsilon + (\alpha q)^3 h$.

Proof. From Theorem 2 and the Binet formula of q -Fibonacci hybrid numbers in ([33] Theorem 2), we obtain

$$\mathbb{H}\mathcal{L}_{k,n}^q = (k+1)\mathbb{H}F_{n+1}^q - kI = (k+1) \left(\frac{\alpha^{n+1}\alpha^* - (\alpha q)^{n+1}\beta^*}{\alpha(1-q)} \right) - kI.$$

$\square$

Theorem 4. The exponential generating function of the q -generalized Leonardo hybrid numbers is

$$\sum_{n=0}^{\infty} \mathbb{H}\mathcal{L}_{k,n}^q \frac{x^n}{n!} = (k+1) \left(\frac{\alpha^* e^{\alpha x} - q\beta^* e^{\alpha q x}}{1-q} \right) - kIe^x,$$

where $\alpha^* = 1 + \alpha i + \alpha^2 \epsilon + \alpha^3 h$ and $\beta^* = 1 + (\alpha q)i + (\alpha q)^2 \epsilon + (\alpha q)^3 h$.

Proof. From the Binet formula of q -generalized Leonardo hybrid numbers in Theorem 3, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{H}\mathcal{L}_{k,n}^q \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \left((k+1) \left(\frac{\alpha^{n+1}\alpha^* - (\alpha q)^{n+1}\beta^*}{\alpha(1-q)} \right) - kI \right) \frac{x^n}{n!} \\ &= (k+1) \sum_{n=0}^{\infty} \left(\frac{\alpha^{n+1}\alpha^* - (\alpha q)^{n+1}\beta^*}{\alpha(1-q)} \right) \frac{x^n}{n!} - kI \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= (k+1) \frac{\alpha^*}{1-q} \sum_{n=0}^{\infty} \frac{(\alpha x)^n}{n!} - (k+1) \frac{q\beta^*}{1-q} \sum_{n=0}^{\infty} \frac{(\alpha q x)^n}{n!} - kI \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= (k+1) \left(\frac{\alpha^* e^{\alpha x} - q\beta^* e^{\alpha q x}}{1-q} \right) - kIe^x. \end{aligned}$$

$\square$

We have the following summation formula for the q -generalized Leonardo hybrid numbers.

Theorem 5. For $n \geq 0$, we have

$$\sum_{j=0}^n \mathbb{H}\mathcal{L}_{k,j}^q = (k+1) \left(\frac{\mathbb{H}F_1^q - q\alpha^2 \mathbb{H}F_0^q - \mathbb{H}F_{n+2}^q + q\alpha^2 \mathbb{H}F_{n+1}^q}{(1-\alpha)(1-\alpha q)} \right) - kI(n+1).$$

Proof. First, we give a summation formula for the q -Fibonacci hybrid numbers.

$$\begin{aligned} \sum_{j=0}^n \mathbb{H}F_{j+1}^q &= \sum_{j=0}^n \frac{\alpha^{j+1}\alpha^* - (\alpha q)^{j+1}\beta^*}{\alpha(1-q)} \\ &= \frac{1}{1-q} \sum_{j=0}^n \alpha^j \alpha^* - q(\alpha q)^j \beta^* = \frac{\alpha^*}{1-q} \sum_{j=0}^n \alpha^j - \frac{q\beta^*}{1-q} \sum_{j=0}^n (\alpha q)^j \\ &= \frac{1}{1-q} \left(\frac{\alpha^*(1-\alpha^{n+1})}{1-\alpha} - \frac{q\beta^*(1-(\alpha q)^{n+1})}{1-\alpha q} \right) \\ &= \frac{1}{1-q} \left(\frac{\alpha^*(1-\alpha^{n+1})(1-\alpha q) - q\beta^*(1-(\alpha q)^{n+1})(1-\alpha)}{(1-\alpha)(1-\alpha q)} \right) \\ &= \frac{\alpha^*(1-\alpha q - \alpha^{n+1} + q\alpha^{n+2}) - q\beta^*(1-\alpha - (\alpha q)^{n+1} + \alpha^{n+2}q^{n+1})}{(1-q)(1-\alpha)(1-\alpha q)} \\ &= \frac{(\alpha^* - q\beta^*) - (\alpha^*\alpha q - \alpha q\beta^*) - (\alpha^*\alpha^{n+1} - q\beta^*(\alpha q)^{n+1}) + (\alpha^*q\alpha^{n+2} - \beta^*(\alpha q)^{n+2})}{(1-q)(1-\alpha)(1-\alpha q)} \\ &= \frac{1}{(1-\alpha)(1-\alpha q)} \times \\ &\quad \left(\frac{\alpha\alpha^* - \alpha q\beta^*}{\alpha(1-q)} - \frac{q\alpha^2(\alpha^* - \beta^*)}{\alpha(1-q)} - \frac{\alpha^{n+2}\alpha^* - (\alpha q)^{n+2}\beta^*}{\alpha(1-q)} + \frac{q\alpha^2(\alpha^{n+1}\alpha^* - (\alpha q)^{n+1}\beta^*)}{\alpha(1-q)} \right). \end{aligned}$$

By using the Binet formula of q -Fibonacci hybrid numbers in ([33] Theorem 2), we obtain

$$\sum_{j=0}^n \mathbb{H}F_{j+1}^q = \frac{\mathbb{H}F_1^q - q\alpha^2 \mathbb{H}F_0^q - \mathbb{H}F_{n+2}^q + q\alpha^2 \mathbb{H}F_{n+1}^q}{(1-\alpha)(1-\alpha q)}. \quad (13)$$

On the other hand, from Theorem 2, we have

$$\begin{aligned} \sum_{j=0}^n \mathbb{H}\mathcal{L}_{k,j}^q &= \sum_{j=0}^n \left((k+1)\mathbb{H}F_{j+1}^q - kI \right) \\ &= (k+1) \sum_{j=0}^n \mathbb{H}F_{j+1}^q - kI \sum_{j=0}^n 1 \\ &= (k+1) \sum_{j=0}^n \mathbb{H}F_{j+1}^q - kI(n+1). \end{aligned} \quad (14)$$

By using the sum Formula (13) in Equation (14), we obtain the desired result. $\square$

In the following theorem, we provide Vajda's identity for the q -generalized Leonardo hybrid numbers. As a corollary of this theorem, we express Catalan's identity, Cassini's identity, and d'Ocagne's identity in terms of q -integers. It should be noted that Vajda's identity for the classical Fibonacci numbers can be found in ([34] Identity (20a)). It is also

worth noting that setting $q = \frac{-1}{\alpha^2}$ where $\alpha = \frac{1+\sqrt{5}}{2}$ in the following identities yields the corresponding results for the generalized Leonardo hybrid numbers.

Theorem 6. For nonnegative integers n, r , and s , we have

$$\begin{aligned} \mathbb{H}\mathcal{L}_{k,n+r}^q \mathbb{H}\mathcal{L}_{k,n+s}^q - \mathbb{H}\mathcal{L}_{k,n}^q \mathbb{H}\mathcal{L}_{k,n+r+s}^q &= \frac{(k+1)^2 \alpha^{2n+r+s}}{(1-q)^2} (1-q^r) q^{n+1} (\beta^* \alpha^* - \alpha^* \beta^* q^s) \\ &\quad + k \left(\left(\mathbb{H}\mathcal{L}_{k,n}^q - \mathbb{H}\mathcal{L}_{k,n+r}^q \right) I + I \left(\mathbb{H}\mathcal{L}_{k,n+r+s}^q - \mathbb{H}\mathcal{L}_{k,n+s}^q \right) \right). \end{aligned}$$

Proof. From the Binet formula of q -generalized Leonardo hybrid numbers, we have

$$\begin{aligned} &\mathbb{H}\mathcal{L}_{k,n+r}^q \mathbb{H}\mathcal{L}_{k,n+s}^q - \mathbb{H}\mathcal{L}_{k,n}^q \mathbb{H}\mathcal{L}_{k,n+r+s}^q \\ &= \frac{(k+1)^2}{\alpha^2(1-q)^2} \left(\left(\left(\alpha^{n+r+1} \alpha^* - (\alpha q)^{n+r+1} \beta^* \right) \left(\alpha^{n+s+1} \alpha^* - (\alpha q)^{n+s+1} \beta^* \right) \right) \right. \\ &\quad \left. - \left(\alpha^{n+1} \alpha^* - (\alpha q)^{n+1} \beta^* \right) \left(\alpha^{n+r+s+1} \alpha^* - (\alpha q)^{n+r+s+1} \beta^* \right) \right) \\ &\quad - \frac{(k+1)}{\alpha(1-q)} \left(\alpha^{n+r+1} \alpha^* - (\alpha q)^{n+r+1} \beta^* - \alpha^{n+1} \alpha^* + (\alpha q)^{n+1} \beta^* \right) kI \\ &\quad - \frac{(k+1)}{\alpha(1-q)} kI \left(\alpha^{n+s+1} \alpha^* - (\alpha q)^{n+s+1} \beta^* - \alpha^{n+r+s+1} \alpha^* + (\alpha q)^{n+r+s+1} \beta^* \right) \\ &= \frac{(k+1)^2 \alpha^{2n+r+s}}{(1-q)^2} (1-q^r) q^{n+1} (\beta^* \alpha^* - \alpha^* \beta^* q^s) \\ &\quad + \frac{k(k+1)}{(1-q)} \left(\left(\alpha^{n+1} \alpha^* - (\alpha q)^{n+1} \beta^* \right) - \left(\alpha^{n+r+1} \alpha^* - (\alpha q)^{n+r+1} \beta^* \right) \right) I \\ &\quad + \frac{k(k+1)}{(1-q)} I \left(\left(\alpha^{n+r+s+1} \alpha^* - (\alpha q)^{n+r+s+1} \beta^* \right) - \left(\alpha^{n+s+1} \alpha^* - (\alpha q)^{n+s+1} \beta^* \right) \right) \\ &= \frac{(k+1)^2 \alpha^{2n+r+s}}{(1-q)^2} (1-q^r) q^{n+1} (\beta^* \alpha^* - \alpha^* \beta^* q^s) \\ &\quad + k \left(\left(\mathbb{H}\mathcal{L}_{k,n}^q - \mathbb{H}\mathcal{L}_{k,n+r}^q \right) I + I \left(\mathbb{H}\mathcal{L}_{k,n+r+s}^q - \mathbb{H}\mathcal{L}_{k,n+s}^q \right) \right). \end{aligned}$$

□

The following identity corresponds to Catalan's identity for the q -generalized Leonardo hybrid numbers.

Corollary 1. For nonnegative integers n and m with $n \geq m$, we have

$$\begin{aligned} \mathbb{H}\mathcal{L}_{k,n-m}^q \mathbb{H}\mathcal{L}_{k,n+m}^q - \left(\mathbb{H}\mathcal{L}_{k,n}^q \right)^2 &= -\frac{(k+1)^2 \alpha^{2n}}{(1-q)^2} (1-q^m) q^{n+1} (\beta^* \alpha^* q^{-m} - \alpha^* \beta^*) \\ &\quad + k \left(\left(\mathbb{H}\mathcal{L}_{k,n}^q - \mathbb{H}\mathcal{L}_{k,n-m}^q \right) I + I \left(\mathbb{H}\mathcal{L}_{k,n}^q - \mathbb{H}\mathcal{L}_{k,n+m}^q \right) \right). \end{aligned}$$

Proof. If we take $r, s \rightarrow m$ and $n \rightarrow n - m$ in Theorem 6, we obtain

$$\begin{aligned} \mathbb{H}\mathcal{L}_{k,n}^q \mathbb{H}\mathcal{L}_{k,n}^q - \mathbb{H}\mathcal{L}_{k,n-m}^q \mathbb{H}\mathcal{L}_{k,n+m}^q &= \frac{(k+1)^2 \alpha^{2n}}{(1-q)^2} (1-q^m) q^{n-m+1} (\beta^* \alpha^* - \alpha^* \beta^* q^m) \\ &\quad + k \left(\left(\mathbb{H}\mathcal{L}_{k,n-m}^q - \mathbb{H}\mathcal{L}_{k,n}^q \right) I + I \left(\mathbb{H}\mathcal{L}_{k,n+m}^q - \mathbb{H}\mathcal{L}_{k,n}^q \right) \right). \end{aligned}$$

Thus, we obtain the desired result. $\square$

The following identity corresponds to Cassini's identity for the q -generalized Leonardo hybrid numbers.

Corollary 2. For positive integer n , we have

$$\begin{aligned} \mathbb{H}\mathcal{L}_{k,n-1}^q \mathbb{H}\mathcal{L}_{k,n+1}^q - \left(\mathbb{H}\mathcal{L}_{k,n}^q\right)^2 &= -\frac{(k+1)^2 \alpha^{2n}}{1-q} q^n (\beta^* \alpha^* - \alpha^* \beta^* q) \\ &\quad + k \left(\left(\mathbb{H}\mathcal{L}_{k,n}^q - \mathbb{H}\mathcal{L}_{k,n-1}^q\right) I + I \left(\mathbb{H}\mathcal{L}_{k,n}^q - \mathbb{H}\mathcal{L}_{k,n+1}^q\right) \right). \end{aligned}$$

Proof. If we take $r = s = 1$ and $n \rightarrow n - 1$ in Theorem 6, we obtain

$$\begin{aligned} \mathbb{H}\mathcal{L}_{k,n}^q \mathbb{H}\mathcal{L}_{k,n}^q - \mathbb{H}\mathcal{L}_{k,n-1}^q \mathbb{H}\mathcal{L}_{k,n+1}^q &= \frac{(k+1)^2 \alpha^{2n}}{1-q} q^n (\beta^* \alpha^* - \alpha^* \beta^* q) \\ &\quad + k \left(\left(\mathbb{H}\mathcal{L}_{k,n-1}^q - \mathbb{H}\mathcal{L}_{k,n}^q\right) I + I \left(\mathbb{H}\mathcal{L}_{k,n+1}^q - \mathbb{H}\mathcal{L}_{k,n}^q\right) \right). \end{aligned}$$

Thus, we obtain the desired result. $\square$

The following identity corresponds to d'Ocagne's identity for the q -generalized Leonardo hybrid numbers.

Corollary 3. For nonnegative integers n and m with $m \geq n$, we have

$$\begin{aligned} \mathbb{H}\mathcal{L}_{k,m}^q \mathbb{H}\mathcal{L}_{k,n+1}^q - \mathbb{H}\mathcal{L}_{k,n}^q \mathbb{H}\mathcal{L}_{k,m+1}^q &= \frac{(k+1)^2 \alpha^{n+m+1}}{(1-q)^2} (1-q^{m-n}) q^{n+1} (\beta^* \alpha^* - \alpha^* \beta^* q) \\ &\quad + k \left(\left(\mathbb{H}\mathcal{L}_{k,n}^q - \mathbb{H}\mathcal{L}_{k,m}^q\right) I + I \left(\mathbb{H}\mathcal{L}_{k,m+1}^q - \mathbb{H}\mathcal{L}_{k,n+1}^q\right) \right). \end{aligned}$$

Proof. If we take $r = m - n$ and $s = 1$ in Theorem 6, we obtain the desired result. $\square$

Remark 3. It should be noted that Theorem 6 is more general than the results given in [30]. If we take $k = 1$ in the above corollaries, we obtain the identities for the q -Leonardo hybrid numbers in ([30] Theorem 3.6, Theorem 3.7, Corollary 3.2).

Theorem 7. Let $\Delta := (\alpha - \alpha q)^2$. For nonnegative integers n and r , the following hold:

$$\begin{aligned} (i) \quad \sum_{i=0}^n \binom{n}{i} (-\alpha^2 q)^{n-i} \mathbb{H}\mathcal{L}_{k,2i+r}^q &= \begin{cases} (k+1) \sqrt{\Delta}^n \mathbb{H}F_{n+r+1}^q - k(1-\alpha^2 q)^n I, & \text{if } n \text{ is even,} \\ (k+1) \sqrt{\Delta}^{n-1} \mathbb{H}L_{n+r+1}^q - k(1-\alpha^2 q)^n I, & \text{if } n \text{ is odd.} \end{cases} \\ (ii) \quad \sum_{i=0}^n \binom{n}{i} (-1)^i (-\alpha^2 q)^{n-i} \mathbb{H}\mathcal{L}_{k,2i+r}^q &= (k+1) (-\alpha[2]_q)^n \mathbb{H}F_{n+r+1}^q + (-1)^{n+1} k (1+\alpha^2 q)^n I. \end{aligned}$$

Proof. (i) From the Binet formulas of q -generalized Leonardo hybrid numbers, we have

$$\begin{aligned}
 & \sum_{i=0}^n \binom{n}{i} (-\alpha^2 q)^{n-i} \mathbb{H}\mathcal{L}_{k,2i+r}^q \\
 &= \sum_{i=0}^n \binom{n}{i} (-\alpha^2 q)^{n-i} \left((k+1) \left(\frac{\alpha^{2i+r+1} \alpha^* - (\alpha q)^{2i+r+1} \beta^*}{\sqrt{\Delta}} \right) - kI \right) \\
 &= \frac{k+1}{\sqrt{\Delta}} \left(\alpha^{r+1} \alpha^* \sum_{i=0}^n \binom{n}{i} (-\alpha^2 q)^{n-i} \alpha^{2i} - (\alpha q)^{r+1} \beta^* \sum_{i=0}^n \binom{n}{i} (-\alpha^2 q)^{n-i} (\alpha q)^{2i} \right) \\
 &\quad - kI \sum_{i=0}^n \binom{n}{i} (-\alpha^2 q)^{n-i} \\
 &= \frac{k+1}{\sqrt{\Delta}} \left(\alpha^{r+1} \alpha^* (\alpha^2 - \alpha^2 q)^n - (\alpha q)^{r+1} \beta^* (\alpha^2 q^2 - \alpha^2 q)^n \right) - kI (1 - \alpha^2 q)^n \\
 &= \frac{k+1}{\sqrt{\Delta}} \left(\alpha^{r+1} \alpha^* (\alpha \sqrt{\Delta})^n - (\alpha q)^{r+1} \beta^* (-\alpha q \sqrt{\Delta})^n \right) - kI (1 - \alpha^2 q)^n.
 \end{aligned}$$

For even n , we have

$$\begin{aligned}
 & \sum_{i=0}^n \binom{n}{i} (-\alpha^2 q)^{n-i} \mathbb{H}\mathcal{L}_{k,2i+r}^q \\
 &= (k+1) \sqrt{\Delta}^n \frac{\alpha^{n+r+1} \alpha^* - (\alpha q)^{n+r+1} \beta^*}{\sqrt{\Delta}} - k(1 - \alpha^2 q)^n I \\
 &= (k+1) \sqrt{\Delta}^n \mathbb{H}F_{n+r+1}^q - k(1 - \alpha^2 q)^n I.
 \end{aligned}$$

For odd n , we have

$$\begin{aligned}
 & \sum_{i=0}^n \binom{n}{i} (-\alpha^2 q)^{n-i} \mathbb{H}\mathcal{L}_{k,2i+r}^q \\
 &= (k+1) \frac{(\alpha^{r+1} \alpha^* (\alpha \sqrt{\Delta})^n + (\alpha q)^{r+1} \beta^* (\alpha q \sqrt{\Delta})^n)}{\sqrt{\Delta}} - k(1 - \alpha^2 q)^n I \\
 &= (k+1) \sqrt{\Delta}^n \frac{\alpha^{n+r+1} \alpha^* + (\alpha q)^{n+r+1} \beta^*}{\sqrt{\Delta}} - k(1 - \alpha^2 q)^n I \\
 &= (k+1) \sqrt{\Delta}^{n-1} \mathbb{H}L_{n+r+1}^q - k(1 - \alpha^2 q)^n I.
 \end{aligned}$$

The identity (ii) can be proven similarly. $\square$

Remark 4. If we take $k = 1$ in Theorem 7, we obtain the identities for the q -Leonardo hybrid numbers in ([30] Theorem 3.3).

3. Leonardo Quaternions over Finite Fields

In this section, we consider the quaternion algebra $\mathbb{Q}_{\mathbb{Z}_p}(-1, -1)$, for simplicity $\mathbb{Q}_{\mathbb{Z}_p}$. Since $\mathbb{Q}_{\mathbb{Z}_p}$ is a split algebra, it is natural to ask about the zero divisors within this quaternion algebra. Now we determine the Leonardo quaternions, which are zero divisors in the quaternion algebra $\mathbb{Q}_{\mathbb{Z}_p}$ for $p = 3$ and $p = 5$. Additionally, we identify certain Leonardo quaternions that are invertible in the quaternion algebra $\mathbb{Q}_{\mathbb{Z}_p}$ for prime integer p with $p \geq 7$.

It should be noted that determining the Leonardo quaternions that are zero divisors and invertible elements is a more challenging task compared to the Fibonacci quaternions, which was studied by Savin in [22], due to the increased complexity of the norm associated with Leonardo quaternions. Therefore, we restrict our focus to the conventional Leonardo quaternions case.

Let QL_n be the n th Leonardo quaternion [24] defined as

$$QL_n = \mathcal{L}_n + \mathcal{L}_{n+1}\mathbf{i} + \mathcal{L}_{n+2}\mathbf{j} + \mathcal{L}_{n+3}\mathbf{k},$$

where the basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfies the multiplication rules $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. By using the definition of Leonardo quaternion and the relations $F_n + F_{n+2} = L_{n+1}$, $F_n^2 + F_{n+1}^2 = F_{2n+1}$ and $F_n + F_{n+4} = 3F_{n+2}$, the norm of Leonardo quaternion can be obtained as follows:

$$\begin{aligned} N(QL_n) &= \mathcal{L}_n^2 + \mathcal{L}_{n+1}^2 + \mathcal{L}_{n+2}^2 + \mathcal{L}_{n+3}^2 \\ &= (2F_{n+1} - 1)^2 + (2F_{n+2} - 1)^2 + (2F_{n+3} - 1)^2 + (2F_{n+4} - 1)^2 \\ &= 4(F_{n+1}^2 + F_{n+2}^2 + F_{n+3}^2 + F_{n+4}^2) - 4(F_{n+1} + F_{n+2} + F_{n+3} + F_{n+4}) + 4 \\ &= 4(F_{2n+3} + F_{2n+7}) - 4(F_{n+3} + F_{n+5}) + 4 \\ &= 4(3F_{2n+5} - L_{n+4} + 1). \end{aligned} \quad (15)$$

Proposition 1. A Leonardo quaternion QL_n is a zero divisor in quaternion algebra $Q_{\mathbb{Z}_3}$ if and only if $n \equiv 0, 5, 7 \pmod{8}$. Moreover, in $Q_{\mathbb{Z}_3}$, there are 3 Leonardo quaternions that are zero divisors.

Proof. We recall that the cycle of Lucas numbers modulo 3 is

$$2, 1, 0, 1, 1, 2, 0, 2.$$

Therefore, the cycle length of Lucas numbers modulo 3 is 8.

A Leonardo quaternion QL_n is a zero divisor in quaternion algebra $Q_{\mathbb{Z}_3}$ if and only if $N(QL_n) = \bar{0}$ in $\mathbb{Z}_3$. By using Equation (15), we have

$$L_{n+4} \equiv 1 \pmod{3} \Leftrightarrow n + 4 \equiv 1, 3, 4 \pmod{8} \Leftrightarrow n \equiv 0, 5, 7 \pmod{8}.$$

There are 81 elements in the quaternion algebra $Q_{\mathbb{Z}_3}$. From [35], the number of zero divisors in $Q_{\mathbb{Z}_p}$ is $p^3 + p^2 - p$. Thus, from 81 quaternions, 33 quaternions are zero divisors in $Q_{\mathbb{Z}_3}$. From those, only 3 quaternions are zero divisor Leonardo quaternions, namely:

$$\begin{aligned} QL_0 &= \bar{\mathcal{L}}_0 + \bar{\mathcal{L}}_1\mathbf{i} + \bar{\mathcal{L}}_2\mathbf{j} + \bar{\mathcal{L}}_3\mathbf{k} = \bar{1} + \mathbf{i} + \bar{2}\mathbf{k}, \\ QL_5 &= \bar{\mathcal{L}}_5 + \bar{\mathcal{L}}_6\mathbf{i} + \bar{\mathcal{L}}_7\mathbf{j} + \bar{\mathcal{L}}_8\mathbf{k} = \mathbf{i} + \bar{2}\mathbf{j} + \mathbf{k}, \\ QL_7 &= \bar{\mathcal{L}}_7 + \bar{\mathcal{L}}_8\mathbf{i} + \bar{\mathcal{L}}_9\mathbf{j} + \bar{\mathcal{L}}_{10}\mathbf{k} = \bar{2} + \mathbf{i} + \mathbf{j}. \end{aligned}$$

□

Proposition 2. A Leonardo quaternion QL_n is a zero divisor in quaternion algebra $Q_{\mathbb{Z}_5}$ if and only if $n \equiv 2, 5, 7, 16 \pmod{20}$.

Proof. We recall that the cycle of Fibonacci numbers modulo 5 is

$$0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1.$$

Therefore, the cycle length of Fibonacci numbers modulo 5 is 20. See ([17] A082116).

A Leonardo quaternion QL_n is a zero divisor in quaternion algebra $Q_{\mathbb{Z}_5}$ if and only if $N(QL_n) = \bar{0}$ in $\mathbb{Z}_5$. By using Equation (15), we have

$$F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 1 \pmod{5}. \quad (16)$$

To find n such that the congruence (16) is satisfied, we need to consider the following five cases:

Case 1: If $n \equiv 0 \pmod{5}$, then $F_{n+5} \equiv 0 \pmod{5}$, $F_{2n+5} \equiv 0 \pmod{5}$. Therefore, we obtain that the congruence (16) is true if and only if $F_{n+3} \equiv 1 \pmod{5} \Leftrightarrow n+3 \equiv 8 \pmod{20} \Leftrightarrow n \equiv 5 \pmod{20}$.

Case 2: If $n \equiv 1 \pmod{5}$, then we have four subcases:

- If $n \equiv 1 \pmod{20}$, then $F_{n+3} \equiv 3 \pmod{5}$, $F_{n+5} \equiv 3 \pmod{5}$, $F_{2n+5} \equiv 3 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 2 \pmod{5}$, therefore, the congruence (16) is not satisfied.
- If $n \equiv 6 \pmod{20}$, then $F_{n+3} \equiv 4 \pmod{5}$, $F_{n+5} \equiv 4 \pmod{5}$, $F_{2n+5} \equiv 2 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 2 \pmod{5}$, therefore, the congruence (16) is not satisfied.
- If $n \equiv 11 \pmod{20}$, then $F_{n+3} \equiv 2 \pmod{5}$, $F_{n+5} \equiv 2 \pmod{5}$, $F_{2n+5} \equiv 3 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 0 \pmod{5}$, therefore, the congruence (16) is not satisfied.
- If $n \equiv 16 \pmod{20}$, then $F_{n+3} \equiv 1 \pmod{5}$, $F_{n+5} \equiv 1 \pmod{5}$, $F_{2n+5} \equiv 2 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 1 \pmod{5}$, therefore, the congruence (16) is satisfied.

Therefore, in Case 2, we have $N(Q\mathcal{L}_n) = \bar{0} \Leftrightarrow n \equiv 16 \pmod{20}$.

Case 3: If $n \equiv 2 \pmod{5}$, then we have four subcases:

- If $n \equiv 2 \pmod{20}$, then $F_{n+3} \equiv 0 \pmod{5}$, $F_{n+5} \equiv 3 \pmod{5}$, $F_{2n+5} \equiv 4 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 1 \pmod{5}$, therefore, the congruence (16) is satisfied.
- If $n \equiv 7 \pmod{20}$, then $F_{n+3} \equiv 0 \pmod{5}$, $F_{n+5} \equiv 4 \pmod{5}$, $F_{2n+5} \equiv 1 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 1 \pmod{5}$, therefore, the congruence (16) is satisfied.
- If $n \equiv 12 \pmod{20}$, then $F_{n+3} \equiv 0 \pmod{5}$, $F_{n+5} \equiv 2 \pmod{5}$, $F_{2n+5} \equiv 4 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 0 \pmod{5}$, therefore, the congruence (16) is not satisfied.
- If $n \equiv 17 \pmod{20}$, then $F_{n+3} \equiv 0 \pmod{5}$, $F_{n+5} \equiv 1 \pmod{5}$, $F_{2n+5} \equiv 1 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 3 \pmod{5}$, therefore, the congruence (16) is not satisfied.

Therefore, in Case 3, we have $N(Q\mathcal{L}_n) = \bar{0} \Leftrightarrow n \equiv 2, 7 \pmod{20}$.

Case 4: If $n \equiv 3 \pmod{5}$, then we have four subcases:

- If $n \equiv 3 \pmod{20}$, then $F_{n+3} \equiv 3 \pmod{5}$, $F_{n+5} \equiv 1 \pmod{5}$, $F_{2n+5} \equiv 4 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 2 \pmod{5}$, therefore, the congruence (16) is not satisfied.
- If $n \equiv 8 \pmod{20}$, then $F_{n+3} \equiv 4 \pmod{5}$, $F_{n+5} \equiv 3 \pmod{5}$, $F_{2n+5} \equiv 1 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 4 \pmod{5}$, therefore, the congruence (16) is not satisfied.
- If $n \equiv 13 \pmod{20}$, then $F_{n+3} \equiv 2 \pmod{5}$, $F_{n+5} \equiv 4 \pmod{5}$, $F_{2n+5} \equiv 4 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 4 \pmod{5}$, therefore, the congruence (16) is not satisfied.
- If $n \equiv 18 \pmod{20}$, then $F_{n+3} \equiv 1 \pmod{5}$, $F_{n+5} \equiv 2 \pmod{5}$, $F_{2n+5} \equiv 1 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 0 \pmod{5}$, therefore, the congruence (16) is not satisfied.

Therefore, in Case 4, we have $N(Q\mathcal{L}_n) \neq \bar{0}$.

Case 5: If $n \equiv 4 \pmod{5}$, then we have four subcases:

- If $n \equiv 4 \pmod{20}$, then $F_{n+3} \equiv 3 \pmod{5}$, $F_{n+5} \equiv 4 \pmod{5}$, $F_{2n+5} \equiv 3 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 3 \pmod{5}$, therefore, the congruence (16) is not satisfied.
- If $n \equiv 9 \pmod{20}$, then $F_{n+3} \equiv 4 \pmod{5}$, $F_{n+5} \equiv 2 \pmod{5}$, $F_{2n+5} \equiv 2 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 0 \pmod{5}$, therefore, the congruence (16) is not satisfied.
- If $n \equiv 14 \pmod{20}$, then $F_{n+3} \equiv 2 \pmod{5}$, $F_{n+5} \equiv 1 \pmod{5}$, $F_{2n+5} \equiv 3 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 4 \pmod{5}$, therefore, the congruence (16) is not satisfied.
- If $n \equiv 19 \pmod{20}$, then $F_{n+3} \equiv 1 \pmod{5}$, $F_{n+5} \equiv 3 \pmod{5}$, $F_{2n+5} \equiv 2 \pmod{5}$. It results $F_{n+3} + F_{n+5} + 2F_{2n+5} \equiv 3 \pmod{5}$, therefore, the congruence (16) is not satisfied.

Therefore, in Case 5, we have $N(Q\mathcal{L}_n) \neq \bar{0}$.

Thus, we obtain the desired result. $\square$

Proposition 3. The Leonardo quaternion $Q\mathcal{L}_{p-4}$ is invertible in quaternion algebra $Q_{\mathbb{Z}_p}$ for prime integer p with $p \geq 7$.

Proof. Recall that $p \nmid F_{2p-3}$ for prime integer p with $p \geq 7$. (Since the proof involved many calculations, we skip it.) From (15), in $\mathbb{Z}_p$, we have

$$N(Q\mathcal{L}_{p-4}) = \bar{4}(3\bar{F}_{2p-3} - \bar{L}_p + \bar{1}).$$

Since $L_p \equiv 1 \pmod{p}$ and $F_{2p-3} \not\equiv 0 \pmod{p}$, we have $N(Q\mathcal{L}_{p-4}) \neq \bar{0}$ in $\mathbb{Z}_p$. Thus, $Q\mathcal{L}_{p-4}$ is an invertible element in $Q\mathbb{Z}_p$. $\square$

4. Conclusions

We can summarize the results obtained in this paper under two main headings. Firstly, we introduce a new class of Leonardo hybrid numbers and investigate some of their properties. The main advantage of the proposed family of hybrid numbers that reflect the generalized Leonardo numbers is that it allows the derivation of several hybrid number classes as a special case. In particular, different from the work [30], where the Leonardo hybrid numbers in [29] are obtained by taking $q = -1/\alpha^2$ with $\alpha = \frac{1+\sqrt{5}}{2}$, in this paper, by taking $q = -1/\alpha^2$ with $\alpha = \frac{1+\sqrt{5}}{2}$, we obtain the generalized Leonardo hybrid numbers, which are firstly defined here. Secondly, we study the Leonardo quaternions that are zero divisors and invertible elements in the quaternion algebra $Q\mathbb{Z}_p$ for special values of prime integer p . This part can be seen as an application of Leonardo quaternions over finite fields, and it marks the first instance in the literature of determining Leonardo quaternions that are zero divisors or invertible elements. To provide a brief summary of our findings:

- We introduce the generalized Leonardo hybrid numbers, which are reduced to the conventional Leonardo hybrid numbers in [29] when $k = 1$.
- We derive a new class of Leonardo hybrid numbers, referred to as the q -generalized Leonardo hybrid numbers. When $k = 1$, the q -generalized Leonardo hybrid numbers reduce to the conventional q -Leonardo hybrid numbers in [30].
- We obtain Vajda's identity for q -generalized Leonardo numbers, which generalizes Catalan's identity, Cassini's identity, and d'Ocagne's identity automatically. Thus, this result is even more general than the ones in [30].
- We obtain that a Leonardo quaternion $Q\mathcal{L}_n$ is a zero divisor in quaternion algebra $Q\mathbb{Z}_3$ if and only if $n \equiv 0, 5, 7 \pmod{8}$.
- We obtain that a Leonardo quaternion $Q\mathcal{L}_n$ is a zero divisor in quaternion algebra $Q\mathbb{Z}_5$ if and only if $n \equiv 2, 5, 7, 16 \pmod{20}$.
- We show that the Leonardo quaternion $Q\mathcal{L}_{p-4}$ is invertible in quaternion algebra $Q\mathbb{Z}_p$ for prime $p \geq 7$.

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