ResearchGate

See discussions, stats, and author profiles for this publication at: https://www.researchgate.net/publication/27189917

Some Fixed Point Theorems for Multivalued Maps Satisfying an Implicit Relation on Metrically Convex Spaces

Article *in* Kyungpook mathematical journal · September 2008

DOI: 10.5666/KMJ.2008.48.3.367 · Source: OAI

CITATIONS
24

READS
47

3 authors, including:

Ishak Altun
Kırıkkale University
174 PUBLICATIONS **4,734** CITATIONS

SEE PROFILE

Duran Turkoglu
Gazi University
116 PUBLICATIONS **2,170** CITATIONS

SEE PROFILE

All content following this page was uploaded by Ishak Altun on 03 October 2015.

The user has requested enhancement of the downloaded file.

A fixed point theorem for multi-maps satisfying an implicit relation on metrically convex metric spaces

I. ALTUN*, H. A. HANCER† AND D. TURKOGLU‡

Abstract. *In this paper, we give a fixed point theorem for multi-valued mapping satisfying an implicit relation on metrically convex metric spaces. This result extends and generalizes some fixed point theorem in the literature.*

Key words: *fixed point, multi-maps, implicit relation, metrically convex metric space*

AMS subject classifications: Primary 54H25; Secondary 47H10

Received February 8, 2006 Accepted March 8, 2006

1. Introduction

Let (X, d) be a metric space. Then X is said to be metrically convex if for every pair $x, y \in X, x \neq y$, there is a point $z \in X$ such that $d(x, y) = d(x, z) + d(z, y)$. We need the following lemma in the sequel.

Lemma 1 [[1]]. *Let K be a non-empty and closed subset of a metrically convex metric space X. Then for any $x \in K$ and $y \notin K$, there exists a point $z \in \partial K$ such that $d(x, y) = d(x, z) + d(z, y)$, where ∂K denotes the boundary of K.*

Let $CB(X)$ denote the family of all non-empty closed and bounded subsets of X. Denote for $A, B \in CB(X)$

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}, \tag{1}$$

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\} \tag{2}$$

and

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}. \tag{3}$$

*Department of Mathematics, Faculty of Science and Arts, Gazi University, 06500 Teknikokullar, Ankara, Turkey, e-mail: ialtun@gazi.edu.tr

†Department of Mathematics, Faculty of Science and Arts, Gazi University, 06500 Teknikokullar, Ankara, Turkey, e-mail: haslan@gazi.edu.tr

‡Department of Mathematics, Faculty of Science and Arts, Gazi University, 06500 Teknikokullar, Ankara, Turkey, e-mail: dturkoglu@gazi.edu.tr

Note that $D(A,B) \leq H(A,B) \leq \delta(A,B)$. Function H is a metric on $CB(X)$ and is called a Hausdorff metric. It is well known that if X is a complete metric space, then so is the metric space $(CB(X), H)$.

Itoh [4] proved a fixed point theorem for non-self maps $F : K \to CB(X)$ satisfying certain contraction condition in terms of Hausdorff metric H on $CB(X)$ under the boundary condition $F(\partial K) \subset K$. Rhoades [7] generalized this result to a wider class of non-self multi-maps on K. Recently Dhage [2] has proved a fixed point theorem for non-self multi-maps on K satisfying a slightly stronger contraction condition than that in Rhoades [7] and under a weaker boundary condition. In *Section 2* of this paper we give an implicit relation and some examples for this relation. In *Section 3*, we prove a fixed point theorem for non-self multi-maps on K satisfying an implicit relation.

2. Implicit relation

Implicit relations on metric space have been used in many articles (see [3], [5], [6], [8]).

Let R_+ be the set of all non-negative real numbers and let $\mathcal{T}$ be the set of all continuous functions $T : R_+^5 \to R$ satisfying the following conditions:

T_1 : $T(t_1, ..., t_5)$ is non-decreasing in t_1 and non-increasing in $t_2, ..., t_5$.

T_2 : there exist two constants $a, b \geq 0$, $2a + 3b < 1$ such that the inequality

$$T(u, v, v, w, v + w) \leq 0 \tag{4}$$

implies $u \leq \max\{(a + b)v + bw, (a + b)w + bv\}$.

T_3 : $T(u, 0, 0, u, u) > 0, T(u, 0, u, 0, u) > 0$ and $T(u, u, 0, 0, 2u) > 0$, $\forall u > 0$.

Remark 1. *Note that, if $u = w$ in T_2, then the inequality $T(u, v, v, w, v+w) \leq 0$ implies $u \leq \dfrac{a+b}{1-b} v$.*

Now we give some examples.

Example 1. *Let $T(t_1, ...t_5) = t_1 - \alpha \max\{t_2, t_3, t_4\} - \beta t_5$, where $\alpha, \beta \geq 0$ and $2\alpha + 3\beta < 1$.*

T_1 : Obvious. T_2 : Let $T(u, v, v, w, w + v) = u - \alpha \max\{w, v\} - \beta(w + v) \leq 0$. Thus $u \leq \max\{(\alpha + \beta)v + \beta w, (\alpha + \beta)w + \beta v\}$. T_3 : $T(u, 0, 0, u, u) = T(u, 0, u, 0, u) = u(1-\alpha-\beta) > 0$ and $T(u, u, 0, 0, 2u) = u(1-\alpha-2\beta) > 0, \forall u > 0$. Therefore $T \in \mathcal{T}$.

Example 2. *Let $T(t_1, ..., t_5) = t_1 - m \max\{t_2, t_3, t_4, \frac{1}{2}t_5\}$, where $0 \leq m < \frac{1}{2}$.*

T_1 : Obvious. T_2 : Let $T(u, v, v, w, w + v) = u - m \max\{w, v\} \leq 0$. Thus $u \leq \max\{mw, mv\}$ and so T_2 is satisfying with $a = m$, $b = 0$. T_3 : $T(u, 0, 0, u, u) = T(u, 0, u, 0, u) = T(u, u, 0, 0, 2u) = u(1 - m) > 0, \forall u > 0$. Therefore $T \in \mathcal{T}$.

Example 3. *Let $T(t_1, ..., t_5) = t_1 - (\alpha t_2 + \beta t_3 + \gamma t_4)$, where $\alpha, \beta, \gamma \geq 0, 2\alpha + 2\beta + \gamma < 1$ and $\alpha + \beta - \gamma \geq 0$.*

T_1 : Obvious. T_2 : Let $T(u, v, v, w, w + v) = u - (\alpha v + \beta v + \gamma w) \leq 0$. Thus $u \leq (\alpha+\beta)v+\gamma w \leq \max\{(\alpha+\beta)v+\gamma w, (\alpha+\beta)w+\gamma v\}$ and so T_2 is satisfying with

$a = \alpha+\beta-\gamma, b = \gamma$. $T_3 : T(u, 0, 0, u, u) = u(1-\gamma) > 0$, $T(u, 0, u, 0, u) = u(1-\beta) > 0$ *and* $T(u, u, 0, 0, 2u) = u(1-\alpha) > 0, \forall u > 0$. *Therefore* $T \in \mathcal{T}$.

Example 4. *Let* $T(t_1, ...t_5) = t_1 - \alpha t_2 - \beta \max\{t_3, t_4\} - \gamma t_5$, *where* $\alpha, \beta, \gamma \geq 0$ *and* $2\alpha + 2\beta + 3\gamma < 1$.

T_1 : *Obvious.* T_2 : *Let* $T(u, v, v, w, w+v) = u-\alpha v-\beta \max\{w, v\}-\gamma(w+v) \leq 0$. *Thus* $u \leq \max\{(\alpha+\beta+\gamma)v+\beta w, (\alpha+\beta+\gamma)w+\beta v\}$ *and so* T_2 *is satisfying with* $a = \alpha+\beta+\gamma, b = \gamma$. $T_3 : T(u, 0, 0, u, u) = T(u, 0, u, 0, u) = u(1-\beta-\gamma) > 0$ *and* $T(u, u, 0, 0, 2u) = u(1-\alpha-2\gamma) > 0, \forall u > 0$. *Therefore* $T \in \mathcal{T}$.

3. Main result

Now we give our main theorem.

Theorem 1. *Let* (X, d) *be a metrically convex complete metric space and* K *a non-empty closed subset of* X. *Let* $F : K \to CB(X)$ *be a multi-map satisfying*

$$T(\delta(Fx, Fy), d(x, y), D(x, Fx), D(y, Fy), D(x, Fy) + D(y, Fx)) \leq 0, \tag{5}$$

for all $x, y \in K$, *where* $T \in \mathcal{T}$. *Further, if* $Fx \cap K \neq \phi$ *for each* $x \in \partial K$, *then* F *has a fixed point* $p \in K$ *such that* $Fp = \{p\}$ *and* F *is continuous at* p *in the Hausdorff metric on* X.

Proof. Let be arbitrary and consider a sequence $\{x_n\}$ in K as follows: Let $x_0 = x$ and take a point $x_1 \in Fx_0 \cap K$ if $Fx_0 \cap K \neq \phi$. Otherwise choose a point $x_1 \in \partial K$ such that

$$d(x_0, x_1') = d(x_0, x_1) + d(x_1, x_1') \tag{6}$$

for some $x_1' \in Fx_0 \subset X\backslash K$. Similarly, pick $x_2 \in Fx_1 \cap K$ if $Fx_1 \cap K \neq \phi$, otherwise choose a point $x_2 \in \partial K$ such that

$$d(x_1, x_2') = d(x_1, x_2) + d(x_2, x_2') \tag{7}$$

for some $x_2' \in Fx_1 \subset X\backslash K$. Continuing this way we have

$$x_{n+1} \in Fx_1 \cap K \text{ if } Fx_1 \cap K \neq \phi, \tag{8}$$

or $x_{n+1} \in \partial K$ satisfying

$$d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+1}') = d(x_n, x_{n+1}') \tag{9}$$

for some $x_{n+1}' \in Fx_n \subset X\backslash K$.

By the construction of $\{x_n\}$ we can write

$$\{x_n\} = P \cup Q \subset K, \tag{10}$$

where

$$P = \{x_n \in \{x_n\} : x_n \in Fx_{n-1}\} \tag{11}$$

and

$$Q = \{x_n \in \{x_n\} : x_n \in \partial K, x_n \notin Fx_{n-1}\}. \tag{12}$$

Then for any two consecutive terms x_n, x_{n+1} of the sequence $\{x_n\}$, we observe that there are only the following three possibilities:

(i) $x_n, x_{n+1} \in P$,

(ii) $x_n \in P, x_{n+1} \in Q$, and

(iii) $x_n \in Q$, $x_{n+1} \in P$.

First we show that $\{x_n\}$ is a Cauchy sequence in K. Now for any $x_n, x_{n+1} \in \{x_n\}$, we have the following estimates:

Case 1: Suppose that $x_n, x_{n+1} \in P$. Now since $x_{n-1}, x_n \in K$, we can use the inequality (5), then we have

$$\begin{aligned} &T\left(\delta(Fx_{n-1}, Fx_n), d(x_{n-1}, x_n), D(x_{n-1}, Fx_{n-1}),\right. \\ &\left. D(x_n, Fx_n), D(x_{n-1}, Fx_n) + D(x_n, Fx_{n-1})\right) \leq 0 \end{aligned} \tag{13}$$

and so

$$T(d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n)+d(x_n, x_{n+1})) \leq 0. \tag{14}$$

From *Remark 1* there exist two constants $a, b \geq 0$, $2a+3b < 1$ such that $d(x_n, x_{n+1}) \leq \dfrac{a+b}{1-b} d(x_{n-1}, x_n)$, where $\dfrac{a+b}{1-b} < \frac{1}{2}$ since $2a + 3b < 1$.

Case 2: Let $x_n \in P$ and $x_{n+1} \in Q$. Then $d(x_n, x_{n+1}) + d(x_{n+1}, x'_{n+1}) = d(x_n, x'_{n+1})$ for some $x'_{n+1} \in Fx_n$. Clearly,

$$\begin{cases} d(x_n, x_{n+1}) \leq d(x_n, x'_{n+1}) \\ d(x_n, x'_{n+1}) \leq \delta(Fx_{n-1}, Fx_n). \end{cases} \tag{15}$$

Now following arguments similar to those in Case 1, we obtain

$$d(x_n, x'_{n+1}) \leq \frac{a+b}{1-b} d(x_{n-1}, x_n). \tag{16}$$

From (15) and (16) it follows that

$$d(x_n, x_{n+1}) \leq \frac{a+b}{1-b} d(x_{n-1}, x_n). \tag{17}$$

Case 3: Suppose that $x_n \in Q$ and $x_{n+1} \in P$. Note that then $x_{n-1} \in P$ and there is a point $x'_n \in Fx_{n-1}$ such that

$$d(x_{n-1}, x_n) + d(x_n, x'_n) = d(x_{n-1}, x'_n). \tag{18}$$

Now,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, x'_n) + d(x'_n, x_{n+1}) \\ &\leq d(x_{n-1}, x'_n) + \delta(Fx_{n-1}, Fx_n). \end{aligned} \tag{19}$$

On the other hand, since $x_{n-1}, x_n \in K$, we can use inequality (5), then we have

$$\begin{aligned} T\left(\delta(Fx_{n-1}, Fx_n), d(x_{n-1}, x_n), D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n),\right. \\ \left. D(x_{n-1}, Fx_n) + D(x_n, Fx_{n-1})\right) \leq 0. \end{aligned} \tag{20}$$

Thus we have

$$T\left(d(x'_n, x_{n+1}), d(x_{n-1}, x_n), d(x_{n-1}, x'_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}) + d(x_n, x'_n)\right) \leq 0. \tag{21}$$

Using (18) we have

$$T\left(d(x'_n, x_{n+1}), d(x_{n-1}, x'_n), d(x_{n-1}, x'_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, x'_n)\right) \leq 0 \tag{22}$$

and so

$$T\left(d(x'_n, x_{n+1}), d(x_{n-1}, x'_n), d(x_{n-1}, x'_n), d(x_n, x_{n+1}), d(x_{n-1}, x'_n) + d(x_n, x_{n+1})\right) \leq 0. \tag{23}$$

From T_2 there exist two constants $a, b \geq 0$, $2a + 3b < 1$ such that

$$d(x'_n, x_{n+1}) \leq \max \left\{ \begin{array}{l} (a+b)d(x_{n-1}, x'_n) + bd(x_n, x_{n+1}), \\ (a+b)d(x_n, x_{n+1}) + bd(x_{n-1}, x'_n) \end{array} \right\}. \tag{24}$$

Therefore using (19) we have

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x'_n) + \max \left\{ \begin{array}{l} (a+b)d(x_{n-1}, x'_n) + bd(x_n, x_{n+1}), \\ (a+b)d(x_n, x_{n+1}) + bd(x_{n-1}, x'_n) \end{array} \right\} \tag{25}$$

Now from (16) in Case 2 applied to $n-1$, we have

$$d(x_{n-1}, x'_n) \leq \frac{a+b}{1-b} d(x_{n-2}, x_{n-1}) \tag{26}$$

and hence from (25)

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{a+b}{1-b} d(x_{n-2}, x_{n-1}) \\ &\quad + \max \left\{ \begin{array}{l} \dfrac{(a+b)^2}{1-b} d(x_{n-2}, x_{n-1}) + bd(x_n, x_{n+1}), \\ (a+b)d(x_n, x_{n+1}) + \dfrac{b(a+b)}{1-b} d(x_{n-2}, x_{n-1}) \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} \dfrac{(a+b)(1+a+b)}{1-b} d(x_{n-2}, x_{n-1}) + bd(x_n, x_{n+1}), \\ (a+b)d(x_n, x_{n+1}) + \dfrac{(1+b)(a+b)}{1-b} d(x_{n-2}, x_{n-1}) \end{array} \right\}. \end{aligned}$$

This implies

$$d(x_n, x_{n+1}) \leq \max\{\frac{(a+b)(1+a+b)}{(1-b)^2}, \frac{(1+b)(a+b)}{(1-b)(1-a-b)}\} d(x_{n-2}, x_{n-1}). \tag{27}$$

Note that $q = \max\{\frac{(a+b)(1+a+b)}{(1-b)^2}, \frac{(1+b)(a+b)}{(1-b)(1-a-b)}\} < 1$. To see this, $2a + 3b < 1$ yields

$$\begin{aligned} & a + b < 1 - 2b - a \\ & \Rightarrow a + b + ab + b^2 < 1 - 2b - a + ab + b^2 \\ & \Rightarrow \frac{(a + b + ab + b^2)}{1 - 2b - a + ab + b^2} < 1 \\ & \Rightarrow \frac{(a+b)(1+b)}{(1-b)(1-a-b)} < 1. \end{aligned} \tag{28}$$

Similarly, again from $2a + 3b < 1$ we have

$$\begin{aligned} & 1 > 3b \\ & \Rightarrow \frac{3}{2} > \frac{1}{1-b} \\ & \Rightarrow 1 > \frac{1}{2(1-b)} + \frac{1}{4} \\ & \Rightarrow 1 > (\frac{1}{1-b} + \frac{1}{2})\frac{1}{2} \\ & \Rightarrow 1 > (\frac{1}{1-b} + \frac{a+b}{1-b})\frac{a+b}{1-b} \\ & \Rightarrow 1 > \frac{(1+a+b)(a+b)}{(1-b)^2}. \end{aligned} \tag{29}$$

Now for any $n \in N$, we have

$$d(x_{2n}, x_{2n+1}) \leq qd(x_{2n-2}, x_{2n}) \leq q^n d(x_0, x_1). \tag{30}$$

Since n is arbitrary, one has

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1). \tag{31}$$

Then from Cases 1-3, it easily follows that $\{x_n\}$ is a Cauchy sequence in K. As K is closed, it is complete and hence $\lim_n x_n = p$ exists. We show that p is a fixed point of F. Without loss of generality, we may assume that $x_{n+1} \in Fx_n$ for some $n \in N$. Then using (5) we have

$$T(\delta(Fx_n, Fp), d(x_n, p), D(x_n, Fx_n), D(p, Fp), D(x_n, Fp) + D(p, Fx_n)) \leq 0, \tag{32}$$

and letting $n \to \infty$ we have

$$T(D(p, Fp), 0, 0, D(p, Fp), D(p, Fp)) \leq 0. \tag{33}$$

From T_3 we have $D(p, Fp) = 0$ and so $p \in Fp$.

Further, we have

$$T(\delta(Fp, Fp), d(p, p), D(p, Fp), D(p, Fp), D(p, Fp) + D(p, Fp)) \leq 0, \tag{34}$$

and so

$$T(\delta(Fp, Fp), 0, 0, 0, 0) \leq 0. \tag{35}$$

Again from T_1 and T_3 we have $\delta(Fp, Fp) = 0$ and so $Fp = \{p\}$.

To show the uniqueness of p, let $q(\neq p)$ be another fixed point of F. Then

$$T(\delta(Fp, Fq), d(p, q), D(p, Fp), D(q, Fq), D(p, Fq) + D(q, Fp)) \leq 0, \tag{36}$$

and so

$$T(d(p, q), d(p, q), 0, 0, 2d(p, q)) \leq 0. \tag{37}$$

Again from T_3 we have $p = q$.

Finally, we prove the continuity of F at p. Let $\{z_n\} \subset X$ be any sequence such that $z_n \to p$ as $n \to \infty$. Now

$$T(\delta(Fz_n, Fp), d(z_n, p), D(z_n, Fz_n), D(p, Fp), D(z_n, Fp) + D(p, Fz_n)) \leq 0 \quad (38)$$

and letting $n \to \infty$ we have

$$T(\lim_n H(Fz_n, Fp), 0, \lim_n H(Fp, Fz_n), 0, \lim_n H(Fp, Fz_n)) \leq 0. \quad (39)$$

From T_3 we have $\lim_n H(Fz_n, Fp) = 0$, showing that F is continuous at p. This completes the proof. □

Remark 2. *Theorem 1 of [2] follows from* Example 1 *and* Theorem 1.

Remark 3. *We can have some new results from other examples and* Theorem 1.

References

[1] L. M. BLUMENTHAL, *Theory and Applications of Distance Geometry*, Clarendon Press, Oxford, 1943.

[2] B. C. DHAGE, *A fixed point theorem for non-self multi-maps in metric spaces*, Comment. Math. Univ. Carolinae **40**(1999), 251-258.

[3] M. IMDAD, S. KUMAR, M. S. KHAN, *Remarks on some fixed point theorems satisfying implicit relations*, Rad. Math. **11**(2002), 135-143.

[4] S. ITOH, *Multi-valued generalized contraction and fixed point theorems*, Comment. Math. Univ. Carolinae **18**(1977), 247-248.

[5] V. POPA, *Some fixed point theorems for compatible mappings satisfying an implicit relation*, Demonstratio Math. **32**(1999), 157-163.

[6] V. POPA, *A general coincidence theorem for compatible multivalued mappings satisfying an implicit relation*, Demonstratio Math. **33**(2000), 159-164.

[7] B. E. RHOADES, *A fixed point theorem for a multi-valued non-self mappings*, Comment. Math. Univ. Carolinae **37**(1996), 401-404.

[8] S. SHARMA, B. DESPHANDE, *On compatible mappings satisfying an implicit relation in common fixed point consideration*, Tamkang J. Math. **33**(2002), 245-252.

View publication stats