Mediterr. J. Math. (2023) 20:263
https://doi.org/10.1007/s00009-023-02459-2
1660-5446/23/050001-21
published online July 10, 2023

Mediterranean Journal of Mathematics

Approximation Results for Hadamard-Type Exponential Sampling Kantorovich Series

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Abstract. The present paper deals with construction of a new family of exponential sampling Kantorovich operators based on a suitable fractional-type integral operators. We study convergence properties of newly constructed operators and give a quantitative form of the rate of convergence thanks to logarithmic modulus of continuity. To obtain an asymptotic formula in the sense of Voronovskaja, we consider locally regular functions. The rest of the paper devoted to approximations of newly constructed operators in logarithmic weighted space of functions. By utilizing a suitable weighted logarithmic modulus of continuity, we obtain a rate of convergence and give a quantitative form of Voronovskaja-type theorem via remainder of Mellin–Taylor's formula. Furthermore, some examples of kernels which satisfy certain assumptions are presented and the results are examined by illustrative numerical tables and graphical representations.

Mathematics Subject Classification. 41A35, 30D10, 94A20, 41A25, 26A33, 44A15.

Keywords. Exponential sampling Kantorovich series, Hadamard-type fractional integral operators, rate of convergence, modulus of continuity, logarithmic weighted space of functions, Voronovskaja-type formulae.

1. Introduction

The generalized sampling-type operators were developed by E. T. Whittaker [58], V. A. Kotel'nikov [49] and C. E. Shannon [56]. The theory and applications of sampling-type operators are among the most challenging areas in the field of approximation theory, especially in the domains of image and signal processing (see [13,27]). The family of operators is given by

$$\left(S_w^{\mathrm{sinc}} g\right)(\tilde{x}) := \sum_{k\in\mathbb{Z}} g\left(\frac{k}{w}\right) \operatorname{sinc}\left(w\tilde{x}-k\right), \quad \tilde{x}\in\mathbb{R}, w>0, \tag{1.1}$$

where sinc function is defined by $\operatorname{sinc}(\tilde{x}) = \frac{\sin(\pi\tilde{x})}{\pi\tilde{x}}$ for $\tilde{x} \in \mathbb{R}\backslash(0)$, and $\operatorname{sinc}(0) = 1$.

Later, P. L. Butzer and his school (see [29,35–37]) substituted the widely used sinc function by suitable kernels, denoted by ϑ, satisfying the typical assumptions of approximate identities and introduced the family of generalized sampling series given by

$$\left(S_w^{\vartheta} g\right)(\tilde{x}) := \sum_{k\in\mathbb{Z}} g\left(\frac{k}{w}\right)\vartheta\left(w\tilde{x}-k\right) \tag{1.2}$$

for any function $g : \mathbb{R} \to \mathbb{R}$ for which the series is absolutely convergent. For the rest of the sampling theory, see [45,59]. In the field of classical sampling theory, numerous results have been published in the papers [1–4,10–12,20,40–42,57].

In the 1980s, a group of optical physicists and engineers introduced a mathematical method so-called "exponential sampling series" to solve certain problems in optical phenomena (see, e.g., [28,39,43,55]). Actually, it was aimed to find a solution of integral equations of the type

$$h(x) = \int_0^{\infty} \mathcal{N}(xt)\, f(t)\, dt,$$

where h is the data function, $\mathcal{N}$ is a kernel and f is the unknown function. An alternative way of expressing this is the Mellin transform theory which was the most useful tool for solving these problems. To do this, Butzer and Jansche [31] established a Mellin transform analysis that was exactly independent of Fourier transform analysis. They defined the family of mathematical exponential sampling operators given by

$$\left(E_w^{\mathrm{lin}_c} f\right)(x) := \sum_{k\in\mathbb{Z}} f\left(e^{\frac{k}{w}}\right)\mathrm{lin}_{c/w}\left(e^{-k}x^{w}\right),\ w>0, x\in\mathbb{R}^{+}, c\in\mathbb{R}, \tag{1.3}$$

where lin_c function is defined by $\mathrm{lin}_c(x) := \frac{x^{-c}}{2\pi i}\frac{x^{\pi i}-x^{-\pi i}}{\log x}, x \neq 1$ and $\mathrm{lin}_c(1) := 1$.

In 2017, Bardaro et al. [21] reconstructed the series (1.3) by utilizing an arbitrary continuous function ϕ satisfies the certain properties instead of lin_c function. The reconstructed form of (1.3) is given by

$$\left(E_w^{\phi} f\right)(x) := \sum_{k\in\mathbb{Z}} f\left(e^{\frac{k}{w}}\right)\phi\left(e^{-k}x^{w}\right),\ w>0, x\in\mathbb{R}^{+}, \tag{1.4}$$

where $f : \mathbb{R}^{+} \to \mathbb{R}$ is any function for which the series is absolutely convergent on the positive semi-real axis.

It is well known that Kantorovich's idea was the first which allows to approximate not necessarily continuous function. This idea is based on using integral mean values of the corresponding function instead of its natural sample values. Using this idea, Kumar and Bajpeyi [50] have introduced a Kantorovich form of (1.4). For locally integrable function defined on $\mathbb{R}^{+}$, Kantorovich form of (1.4) was given by

$$\left(K_w^{\phi} f\right)(x) := \sum_{k \in \mathbb{Z}} \phi\left(e^{-k} x^w\right)\left[w \int_{\frac{k}{w}}^{\frac{k+1}{w}} f\left(e^u\right)\right] du. \tag{1.5}$$

Recently, numerous results have been published on the exponential sampling series and its different forms (see, e.g., [5–9,14–18,25,26,51]).

In the last century, there has been increasing interest in fractional-type calculus and its applications. The fractional-type calculus aims to study fractional degree derivatives and integral operators on complex or real fields, as well as their applications. There are several different definitions of fractional-type integrals for example, Riemann–Liouville fractional integrals [38], Hadamard-type fractional integrals [44] and Weyl fractional integrals [48]. For $\alpha, x > 0$, Riemann–Liouville fractional-type integral operators are given by

$$\left(I_{0_+}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-v)^{\alpha-1} f(v)\, dv, \tag{1.6}$$

where Γ is the gamma function. In [52], the authors have recently constructed a new Kantorovich form of exponential sampling series using the (1.6) given by

$$\left(I_{w,\alpha}^{\phi} f\right)(x) := \sum_{k \in \mathbb{Z}} \phi\left(e^{-k} x^w\right) \alpha \int_0^1 (1-v)^{\alpha-1} f\left(e^{\frac{k+v}{w}}\right) dv. \tag{1.7}$$

Also for $\alpha, x > 0$, Hadamard-type fractional integral operators are defined by

$$\left(J_{0_+}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \left(\log \frac{x}{v}\right)^{\alpha-1} f(v) \frac{dv}{v}. \tag{1.8}$$

In the recent paper [53], the authors constructed the Riemann–Liouville fractional integral-type Szász–Mirakyan–Kantorovich operators by utilizing the operators (1.6). For more details about fractional-type operators and its applications, we refer the readers to [19,33,34,46,47].

In the present paper, we aim to construct Hadamard-type fractional exponential sampling Kantorovich operators via (1.8). First, we obtain pointwise and uniform convergence for newly constructed operators and give a quantitative form of the rate of convergence using logarithmic modulus of continuity. Moreover, we present a Voronovskaja-type theorem using the Mellin–Taylor formula based on Mellin derivatives. Second, convergence of the series is also investigated for functions belonging to logarithmic weighted space of functions. Finally, we give some examples of kernels which satisfy suitable properties and examine the results of this paper by numerical tables and graphical representations.

2. Basic Notations and Preliminaries

By $\mathbb{N}, \mathbb{Z}$, we will denote the sets of all positive integers and integers, respectively. Furthermore, by $\mathbb{R}$ and $\mathbb{R}^+$, we denote the sets of all real and positive real numbers, respectively.

By $C(\mathbb{R}^+)$, we denote the space of all continuous functions defined on $\mathbb{R}^+$ and by $C_B(\mathbb{R}^+)$ the space of all bounded functions $f \in C(\mathbb{R}^+)$ with the usual sup-norm $\|f\|_\infty$. We will say that a function $f \in C_B(\mathbb{R}^+)$ is log-uniformly continuous on $\mathbb{R}^+$, if for any positive ε there exists positive δ such that $|f(x) - f(y)| < \varepsilon$, whenever $|\log x - \log y| \leq \delta$ (see [21,54]). By $\mathcal{C}(\mathbb{R}^+)$, we denote the subspaces of $C_B(\mathbb{R}^+)$ containing of all functions which are log-uniformly continuous. We note that, in general, a log-uniformly continuous function is not necessarily uniformly continuous and conversely. But, these two notations are equivalent on compact intervals of $\mathbb{R}^+$ (see [21]).

We denote by $C^{(r)}(\mathbb{R}^+), r \in \mathbb{N}$, the class comprising all the $r-$differentiable functions over $\mathbb{R}^+$. Moreover, we will say that a function f is locally of class $C^{(r)}$ at a point $x \in \mathbb{R}^+$, if f is $(r-1)$-times differentiable in a neighborhood of x and the derivative $f^{(r)}$ exists.

By $M(\mathbb{R}^+)$ and $B(\mathbb{R}^+)$ will be denoted the class of all Lebesgue measurable functions and the space of all bounded functions on $\mathbb{R}^+$, respectively. Furthermore, by $L^p(\mathbb{R}^+), 1 \leq p < \infty$, we will denote the space of all the Lebesgue measurable and p-integrable functions on $\mathbb{R}^+$ endowed with the usual norm $\|f\|_p$.

For $c \in \mathbb{R}$, let us consider the space

$$X_c = \left\{ f : \mathbb{R}^+ \to \mathbb{C} : \ f(\cdot)(\cdot)^{c-1} \in L^1\left(\mathbb{R}^+\right) \right\}$$

endowed with the norm

$$\|f\|_{X_c} = \left\| f(\cdot)(\cdot)^{c-1} \right\|_1 = \int_0^\infty |f(u)| \, u^c \frac{du}{u}.$$

Then, the Mellin transform of $f \in X_c$ is given by

$$[f]_{\tilde{M}}(c + it) := \int_0^\infty f(u) \, u^{c+it} \frac{du}{u}, \ \ c, t, \in \mathbb{R},$$

where i is the complex unit (see [30]).

For $f : \mathbb{R}^+ \to \mathbb{R}$, the first Mellin derivative of f is defined by

$$(\Theta f)(x) := x f^{'}(x)$$

provided the usual derivative $f^{'}$ exists at the point $x \in \mathbb{R}^+$. Furthermore, by iterating, the Mellin differential of order $r \in \mathbb{N}$ can be written by

$$\Theta^r := \Theta\left(\Theta^{r-1}\right).$$

For convenience, put $\Theta^0 := I, I$ denotes the identity. For more information, we refer the readers to [30].

Throughout the rest of paper, $\phi : \mathbb{R}^+ \to \mathbb{R}$ is called a kernel, if it satisfies the following properties:

$\phi_1)$ for every $u \in \mathbb{R}^+$,

$$\sum_{k \in \mathbb{Z}} \phi\left(e^{-k} u\right) = 1;$$

$\phi_2)$

$$M_0(\phi) := \sup_{u\in\mathbb{R}^+}\sum_{k\in\mathbb{Z}}\left|\phi\left(e^{-k}u\right)\right| < \infty;$$

$\phi_3)$ for every $\kappa > 0$,

$$\lim_{\kappa\to\infty}\sum_{|k-\log u|>\kappa}\left|\phi\left(e^{-k}u\right)\right| = 0$$

holds uniformly with respect to $u \in \mathbb{R}^+$.

By Ψ, we denote the class of all functions satisfying the properties $\phi_1), \phi_2)$ and $\phi_3)$.

Let $r \in \mathbb{N}$. Then, the algebraic moment of order r of $\varphi \in \Psi$ is given by

$$m_r(\phi, u) := \sum_{k\in\mathbb{Z}}\phi\left(e^{-k}u\right)(k-\log u)^r$$

and for $\gamma > 0$, the absolute moment of arbitrary order γ is defined by

$$M_\gamma(\phi, u) := \sum_{k\in\mathbb{Z}}\left|\phi\left(e^{-k}u\right)\right||k-\log u|^\gamma.$$

We put

$$M_\gamma(\phi) := \sup_{u\in\mathbb{R}^+} M_\gamma(\phi, u).$$

Remark 1. Note that if $M_\gamma(\phi) < \infty$, then $M_{\tilde{\gamma}}(\phi) < \infty$ for $0 < \tilde{\gamma} < \gamma$ (see [26]).

Now for $\phi \in \Psi$, considering the operators (1.8), we define the Hadamard-type fractional exponential sampling Kantorovich series by

$$\left(I^{\phi}_{w,\alpha,\mu}f\right)(x) := \sum_{k\in\mathbb{Z}}\phi\left(e^{-k}x^w\right)\frac{\mu^\alpha}{\Gamma(\alpha)}\int_0^1 v^\mu\left(\log\frac{1}{v}\right)^{\alpha-1} f\left(e^{\frac{k+v}{w}}\right)\frac{dv}{v},\ \ w,\alpha,\mu > 0, x\in\mathbb{R}^+, \tag{2.1}$$

where $f : \mathbb{R}^+ \to \mathbb{R}$ is integrable function such that the above series is absolutely convergent for every $x \in \mathbb{R}^+$ and Γ is the gamma function.

Remark 2. If we choose $\alpha = 1$ and then $\mu = \alpha$, we obtain the Riemann–Liouville fractional-type Kantorovich exponential sampling series given by (1.7).

Remark 3. Considering the definition of operators (2.1) and the equality

$$\frac{1}{\Gamma(\alpha)}\int_0^1 v^\mu\left(\log\frac{1}{v}\right)^{\alpha-1}\frac{dv}{v} = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-\mu v}t^{\alpha-1}dv = \mu^{-\alpha},$$

we immediately have that $\left(I^{\phi}_{w,\alpha,\mu}\mathbf{1}\right)(x) = 1$, where $\mathbf{1}(x) = 1$ for every $x \in \mathbb{R}^+$.

Remark 4. We easily see that the above series (2.1) is well defined for $f \in B(\mathbb{R}^+)$. Indeed, considering the assumption $\phi_2)$, we have

$$\left|\left(I^{\phi}_{w,\alpha,\mu}f\right)(x)\right| \leq M_0(\phi)\|f\|_\infty.$$

3. Main Results for $I^{\phi}_{w,\alpha,\mu}$

This section deals with the convergence properties of the operators (2.1).

Theorem 1. *Let $f \in M(\mathbb{R}^+) \cap B(\mathbb{R}^+)$ and $\phi \in \Psi$ be a kernel. Suppose that $x \in \mathbb{R}^+$ is a continuity point of f. Then, we have*

$$\lim_{w\to\infty} \left(I^{\phi}_{w,\alpha,\mu} f\right)(x) = f(x). \tag{3.1}$$

Furthermore, if $f \in \mathcal{C}(\mathbb{R}^+)$, there holds

$$\lim_{w\to\infty} \left\| I^{\phi}_{w,\alpha,\mu} f - f \right\|_{\infty} = 0. \tag{3.2}$$

Proof. From (2.1) and the Remark 3, we have that

$$\left|\left(I^{\phi}_{w,\alpha,\mu} f\right)(x) - f(x)\right| \leq \sum_{k\in\mathbb{Z}} \left|\phi\left(e^{-k}x^{w}\right)\right| \frac{\mu^{\alpha}}{\Gamma(\alpha)}$$
$$\int_0^1 v^{\mu}\left(\log\frac{1}{v}\right)^{\alpha-1} \left|f\left(e^{\frac{k+v}{w}}\right) - f(x)\right| \frac{dv}{v}.$$

Let $x \in \mathbb{R}^+$ be a continuity point of f and $\varepsilon > 0$ be fixed. Thus, there exists $\delta > 0$ such that $\left|f\left(e^{\frac{k+v}{w}}\right) - f(x)\right| < \varepsilon$, whenever $\left|\frac{k+v}{w} - \log x\right| \leq \delta$. We can fix $\tilde{w}$ in such a way that $\frac{1}{w} \leq \frac{\delta}{2}$ for every $w > \tilde{w}$. Now for $w > \tilde{w}$, we have

$$\left|\frac{k+v}{w} - \log x\right| \leq \left|\frac{k}{w} - \log x\right| + \frac{v}{w} \leq \delta,$$

whenever $\left|\frac{k}{w} - \log x\right| \leq \frac{\delta}{2}$. Then, we rewrite

$$\begin{aligned}
&\left|\left(I^{\phi}_{w,\alpha,\mu} f\right)(x) - f(x)\right| \\
&\leq \sum_{|k-w\log x|\leq \frac{w\delta}{2}} \left|\phi\left(e^{-k}x^{w}\right)\right| \frac{\mu^{\alpha}}{\Gamma(\alpha)} \int_0^1 v^{\mu}\left(\log\frac{1}{v}\right)^{\alpha-1} \left|f\left(e^{\frac{k+v}{w}}\right) - f(x)\right| \frac{dv}{v} \\
&\quad + \sum_{|k-w\log x|> \frac{w\delta}{2}} \left|\phi\left(e^{-k}x^{w}\right)\right| \frac{\mu^{\alpha}}{\Gamma(\alpha)} \int_0^1 v^{\mu}\left(\log\frac{1}{v}\right)^{\alpha-1} \left|f\left(e^{\frac{k+v}{w}}\right) - f(x)\right| \frac{dv}{v} \\
&:= I_1 + I_2.
\end{aligned}$$

We first estimate I_1. We have

$$I_1 \leq \varepsilon M_0(\phi).$$

As to I_2, by ϕ_3), we get

$$\begin{aligned}
I_2 &\leq 2\|f\|_{\infty} \sum_{|k-w\log x|> \frac{w\delta}{2}} \left|\phi\left(e^{-k}x^{w}\right)\right| \\
&\leq \varepsilon 2\|f\|_{\infty}
\end{aligned}$$

for sufficiently large $w > 0$. Hence, we get (3.1).

As for uniform continuity (3.2), we can prove in a similar way as the above proof if we choose δ is independent of x. □

4. Rate of Convergence of $\left(I^{\phi}_{w,\alpha,\mu}\right)$

In this section, we investigate the rate of convergence of the family of operators $\left(I^{\phi}_{w,\alpha,\mu}\right)$. First, we recall the definition of the logarithmic modulus of continuity.

For $f \in \mathcal{C}\left(\mathbb{R}^{+}\right)$, the logarithmic modulus of continuity is defined by

$$\omega_L\left(f,\delta\right) := \sup\left\{\left|f\left(x\right)-f\left(y\right)\right| : \ \left|\log x - \log y\right| \leq \delta, \ x, y \in \mathbb{R}^{+}, \delta > 0\right\}. \tag{4.1}$$

The modulus of continuity given in (4.1) has all the properties by the classical modulus of continuity (see [24]).

Theorem 2. *Let ϕ be a kernel satisfying the assumption $M_1\left(\varphi\right) < \infty$. If we suppose that $f \in \mathcal{C}\left(\mathbb{R}^{+}\right)$, then we have*

$$\left\|I^{\phi}_{w,\alpha,\mu}f - f\right\|_{\infty} \leq \omega_L\left(f, \frac{1}{w}\right)\left[M_0\left(\phi\right)\left(1 + \frac{\mu^{\alpha}}{\left(\mu+1\right)^{\alpha}}\right) + M_1\left(\phi\right)\right].$$

Proof. From (2.1), we immediately have

$$\begin{aligned}&\left|\left(I^{\phi}_{w,\alpha,\mu}f\right)\left(x\right) - f\left(x\right)\right| \\ &\quad\leq \sum_{k\in\mathbb{Z}}\left|\phi\left(e^{-k}x^{w}\right)\right|\frac{\mu^{\alpha}}{\Gamma\left(\alpha\right)}\int_0^1 v^{\mu}\left(\log\frac{1}{v}\right)^{\alpha-1}\left|f\left(e^{\frac{k+v}{w}}\right) - f\left(x\right)\right|\frac{dv}{v} := I.\end{aligned}$$

Thanks to inequality $\omega_L\left(f;\lambda\delta\right) \leq \left(\lambda+1\right)\omega_L\left(f,\delta\right), \lambda > 0$, we obtain

$$\begin{aligned}I &\leq \sum_{k\in\mathbb{Z}}\left|\phi\left(e^{-k}x^{w}\right)\right|\frac{\mu^{\alpha}}{\Gamma\left(\alpha\right)}\int_0^1 v^{\mu}\left(\log\frac{1}{v}\right)^{\alpha-1}\omega_L\left(f, \left|\frac{k+v}{w} - \log x\right|\right)\frac{dv}{v} \\ &\leq \omega_L\left(f,\delta\right)\sum_{k\in\mathbb{Z}}\left|\phi\left(e^{-k}x^{w}\right)\right|\frac{\mu^{\alpha}}{\Gamma\left(\alpha\right)}\int_0^1 v^{\mu}\left(\log\frac{1}{v}\right)^{\alpha-1}\left(1 + \frac{\left|k+v-w\log x\right|}{w\delta}\right)\frac{dv}{v} \\ &\leq \omega_L\left(f,\delta\right)\left\{M_0\left(\phi\right) + \frac{M_1\left(\phi\right)}{w\delta} + \frac{\mu^{\alpha}}{\left(\mu+1\right)^{\alpha}w\delta}M_0\left(\phi\right)\right\}.\end{aligned}$$

Choosing $\delta = \frac{1}{w}$, we get the desired result. □

The result given in Theorem 2 cannot be applied to the Mellin–Fejer kernel since its absolute moment of order 1 is not finite. For this reason, to obtain a similar estimate, we consider the space of log-holderian functions of order $0 < \beta < 1$ defined by

$$Log_H\left(\beta\right) = \left\{f \in \mathcal{C}\left(\mathbb{R}^{+}\right) : \ \omega_L\left(f,\delta\right) = \mathcal{O}\left(\delta^{\beta}\right), \ \text{as } \delta \to 0^{+}\right\}.$$

Corollary 1. *For $f \in Log_H\left(\beta\right)$, as a consequence of Theorem 2, we immediately have that*

$$\left\|I^{\phi}_{w,\alpha,\mu}f - f\right\|_{\infty} = \mathcal{O}\left(\frac{1}{w^{\beta}}\right), \text{as } w \to \infty.$$

5. Voronovskaja-Type Theorem

This section is devoted to a Voronovskaja-type theorem using the Mellin–Taylor formula based on Mellin derivatives. To do this, we recall the Mellin–Taylor formula for any $f \in C_B(\mathbb{R}^+)$ belonging to $C^{(r)}, r \in \mathbb{N}$, locally at the point $x \in \mathbb{R}^+$. The Mellin–Taylor formula is defined by (see [22,54])

$$f(t) := f(x) + (\Theta f)(x)(\log t - \log x) + \frac{(\Theta^2 f)(x)}{2!}(\log t - \log x)^2 + \ldots + \frac{(\Theta^r f)(x)}{r!}(\log t - \log x)^r + h\left(\frac{t}{x}\right)(\log t - \log x)^r, \tag{5.1}$$

where $h : \mathbb{R}^+ \to \mathbb{R}$ is a bounded function such that $\lim_{t\to x} h\left(\frac{t}{x}\right) = 0$.

For $f \in C^{(r)}(\mathbb{R}^+), r \in \mathbb{N}$, not necessarily locally at the point $x \in \mathbb{R}^+$, we consider the following expression:

$$f(t) := \sum_{j=0}^{r} \frac{1}{j!}\left(\Theta^j f\right)(x)(\log t - \log x)^j + R_r(t), \tag{5.2}$$

where

$$R_r(t) = \frac{(\Theta^r f)(\xi) - (\Theta^r f)(x)}{r!}(\log t - \log x)^r$$

is the Lagrange remainder in Mellin–Taylor formula at the point $x \in \mathbb{R}^+$ and ξ is a suitable number lying between t and x.

Now, we consider

φ_4)

$$\lim_{\kappa\to\infty} \sum_{|k-\log u|>\kappa} \left|\phi\left(e^{-k}u\right)\right| |k - \log u| = 0,$$

uniformly with respect to $u \in \mathbb{R}^+$.

From now on, by $\tilde{\Psi}$, we denote the class of functions satisfying the properties $\phi_1), \phi_2)$ and $\phi_4)$.

Theorem 3. *Let $f \in C^{(1)}(\mathbb{R}^+)$ and $\phi \in \tilde{\Psi}$ be a kernel such that $m_1(\phi, x^w) := m_1(\phi) < \infty$ is independent of x. Let $m_1(\phi) \neq -\frac{\mu^\alpha}{(\mu+1)^\alpha}$. If we suppose that $M_1(\phi) < \infty$, then we have*

$$\lim_{w\to\infty} w\left[\left(I^{\phi}_{w,\alpha,\mu} f\right)(x) - f(x)\right] = (\Theta f)(x)\left[m_1(\phi) + \frac{\mu^\alpha}{(\mu+1)^\alpha}\right] + o\left(\frac{1}{w}\right)$$

for $\alpha, \mu > 0$.

Proof. Let $f \in C^{(1)}(\mathbb{R}^+)$. Thanks to the Mellin–Taylor formula (5.1) up to the first order, we have

$$f\left(e^{\frac{k+v}{w}}\right) = f(x) + (\Theta f)(x)\left(\frac{k+v}{w} - \log x\right) + h\left(\frac{e^{\frac{k+v}{w}}}{x}\right)\left(\frac{k+v}{w} - \log x\right).$$

Using the definition of the operators (2.1), we get

$$\begin{aligned}
&\left(I^{\phi}_{w,\alpha,\mu} f\right)(x) - f(x) \\
&\quad = \sum_{k\in\mathbb{Z}} \phi\left(e^{-k}x^{w}\right) \frac{\mu^{\alpha}}{\Gamma(\alpha)} \int_0^1 v^{\mu} \left(\log\frac{1}{v}\right)^{\alpha-1} (\Theta f)(x) \left(\frac{k - w\log x + v}{w}\right) \frac{dv}{v} \\
&\qquad + \sum_{k\in\mathbb{Z}} \phi\left(e^{-k}x^{w}\right) \frac{\mu^{\alpha}}{\Gamma(\alpha)} \int_0^1 v^{\mu} \left(\log\frac{1}{v}\right)^{\alpha-1} h\left(\frac{e^{\frac{k+v}{w}}}{x}\right) \left(\frac{k - w\log x + v}{w}\right) \frac{dv}{v} \\
&\quad := J_1 + J_2.
\end{aligned}$$

We first consider J_1. We immediately have

$$\begin{aligned}
J_1 &= \sum_{k\in\mathbb{Z}} \phi\left(e^{-k}x^{w}\right) \frac{\mu^{\alpha}}{\Gamma(\alpha)} \int_0^1 v^{\mu} \left(\log\frac{1}{v}\right)^{\alpha-1} (\Theta f)(x) \left(\frac{k - w\log x + v}{w}\right) \frac{dv}{v} \\
&= \frac{(\Theta f)(x)}{w} \left\{ m_1(\phi) + \frac{\mu^{\alpha}}{1+\mu^{\alpha}} \right\}.
\end{aligned}$$

Now, let us consider J_2. Here, we note that the pointwise notions of continuity and log −continuity are equivalent. We fix $\varepsilon > 0$. Then, there exists $\delta > 0$ such that $\left| h\left(\frac{e^{\frac{k+v}{w}}}{x}\right) \right| < \varepsilon$, whenever $\left|\frac{k+v}{w} - \log x\right| < \delta$. Let us fix $\tilde{w}$ in such a way that $\frac{1}{w} \leq \frac{\delta}{2}$ for every $w > \tilde{w}$. Then, there holds

$$\begin{aligned}
|J_2| \leq & \sum_{|k - w\log x| \leq \frac{w\delta}{2}} \left|\phi\left(e^{-k}x^{w}\right)\right| \frac{\mu^{\alpha}}{\Gamma(\alpha)} \int_0^1 v^{\mu} \left(\log\frac{1}{v}\right)^{\alpha-1} \\
& \qquad \times \left| h\left(\frac{e^{\frac{k+v}{w}}}{x}\right) \right| \frac{|k - w\log x + v|}{w} \frac{dv}{v} \\
& + \sum_{|k - w\log x| > \frac{w\delta}{2}} \left|\phi\left(e^{-k}x^{w}\right)\right| \frac{\mu^{\alpha}}{\Gamma(\alpha)} \int_0^1 v^{\mu} \left(\log\frac{1}{v}\right)^{\alpha-1} \\
& \qquad \times \left| h\left(\frac{e^{\frac{k+v}{w}}}{x}\right) \right| \frac{|k - w\log x + v|}{w} \frac{dv}{v} \\
& := J_{2,1} + J_{2,2}.
\end{aligned}$$

Let us consider $J_{2,1}$. We obtain

$$\begin{aligned}
J_{2,1} \leq & \sum_{|k - w\log x| \leq \frac{w\delta}{2}} \left|\phi\left(e^{-k}x^{w}\right)\right| \frac{\mu^{\alpha}}{\Gamma(\alpha)} \int_0^1 v^{\mu} \left(\log\frac{1}{v}\right)^{\alpha-1} \left| h\left(\frac{e^{\frac{k+v}{w}}}{x}\right) \right| \\
& \qquad \times \left(\frac{|k - w\log x|}{w} + \frac{v}{w}\right) \frac{dv}{v} < \frac{\varepsilon}{w} \left(M_1(\phi) + \frac{\mu^{\alpha}}{(\mu+1)^{\alpha}} M_0(\phi) \right).
\end{aligned}$$

Now, we consider $J_{2,2}$. Considering the fact that h is a bounded function, we have

$$J_{2,2} \leq \frac{\varepsilon \|h\|_{\infty}}{w} \left[1 + \frac{\mu^{\alpha}}{(\mu+1)^{\alpha}} \right]$$

for sufficiently large $w > 0$. This completes the proof. □

As a consequence of Theorem 3, we have the following Corollary.

Corollary 2. *Let $\phi \in \Psi$ be a kernel satisfying the assumption $m_j(\phi, x) := m_j(\phi) = 0, j = 1, 2, \ldots, r-1$ and $m_j(\phi, x) \neq 0, j = r$, is independent of $x \in \mathbb{R}^+$. Let $m_r(\phi) \neq -\frac{\mu^\alpha}{(\mu+r)^\alpha}$. If $f \in C^{(r)}(\mathbb{R}^+), r \in \mathbb{N}$, and $M_r(\phi) < \infty$, we get*

$$\lim_{w\to\infty} w^r \left[\left(I^{\phi}_{w,\alpha,\mu} f\right)(x) - f(x)\right] = (\Theta^r f)(x) \left[m_r(\phi) + \frac{\mu^\alpha}{(\mu+r)^\alpha}\right] + o\left(\frac{1}{w^r}\right)$$

for $\alpha, \mu > 0$.

Proof. For proof of Corollary, if we use the equality

$$\frac{\mu^\alpha}{\Gamma(\alpha)} \int_0^1 v^{\mu+r} \left(\log \frac{1}{v}\right)^{\alpha-1} \frac{dv}{v} = \frac{\mu^\alpha}{(\mu+r)^\alpha}, \ \alpha, \mu > 0$$

and use the hypothesis of the Corollary, we immediately get the assertion. □

6. Logarithmic Weighted Approximation

This section is devoted to main results in logarithmic weighted space of functions and its subspaces. Note that the logarithmic weighted space of functions is more general space of functions than $\mathcal{C}(\mathbb{R}^+)$.

Now, we recall the logarithmic weighted space of functions and its subspaces. Thanks to weight function $1+\log^2(\cdot)$ on $\mathbb{R}^+$, the logarithmic weighted space of functions and its subspaces are given by (see [14])

$$\begin{aligned}
\tilde{B}_2\left(\mathbb{R}^+\right) &:= \left\{ f : \mathbb{R}^+ \to \mathbb{R} : \frac{|f(x)|}{1+\log^2 x} \leq M \text{ for every } x \in \mathbb{R}^+ \right\}, \\
\tilde{C}_2\left(\mathbb{R}^+\right) &:= C\left(\mathbb{R}^+\right) \cap \tilde{B}_2\left(\mathbb{R}^+\right), \\
\tilde{C}_2^*\left(\mathbb{R}^+\right) &:= \left\{ f \in \tilde{C}_2\left(\mathbb{R}^+\right) : \lim_{x\to\infty} \frac{|f(x)|}{1+\log^2 x} \in \mathbb{R} \right\}.
\end{aligned}$$

The linear space $\tilde{B}_2(\mathbb{R}^+)$ is a normed linear space with the norm

$$\|f\|_L := \sup_{x>0} \frac{|f(x)|}{1+\log^2 x}.$$

Theorem 4. *Let $\phi \in \Psi$ be a kernel satisfying the assumption $M_2(\phi) < \infty$. Then, for $w > 0$, $I^{\phi}_{w,\alpha,\mu}$ is a linear operator from $B_2(\mathbb{R}^+)$ to $B_2(\mathbb{R}^+)$ and it is bounded.*

Proof. Let us fix $w > 0$ and $f \in \tilde{B}_2(\mathbb{R}^+)$. From (2.1), we obtain

$$\begin{aligned}
\left|\left(I^{\phi}_{w,\alpha,\mu} f\right)(x)\right| &\leq \sum_{k\in\mathbb{Z}} \left|\phi\left(e^{-k} x^w\right)\right| \frac{\mu^\alpha}{\Gamma(\alpha)} \int_0^1 v^\mu \left(\log\frac{1}{v}\right)^{\alpha-1} \left|f\left(e^{\frac{k+v}{w}}\right)\right| \frac{dv}{v} \\
&\leq \|f\|_L \sum_{k\in\mathbb{Z}} \left|\phi\left(e^{-k} x^w\right)\right| \frac{\mu^\alpha}{\Gamma(\alpha)} \int_0^1 v^\mu \left(\log\frac{1}{v}\right)^{\alpha-1} \left(1 + \left(\frac{k+v}{w}\right)^2\right) \frac{dv}{v} \\
&= \|f\|_L \sum_{k\in\mathbb{Z}} \left|\phi\left(e^{-k} x^w\right)\right| \frac{\mu^\alpha}{\Gamma(\alpha)} \int_0^1 v^\mu \left(\log\frac{1}{v}\right)^{\alpha-1} \left(1 + \frac{k^2+2kv+v^2}{w^2}\right) \frac{dv}{v}.
\end{aligned}$$

Since $k^2 = (k - w\log x)^2 + 2w\log x\,(k - w\log x) + w^2\log^2 x$ and $k = (k - w\log x + w\log x)$, we get

$$\begin{aligned}
&\left|\left(I^{\phi}_{w,\alpha,\mu}f\right)(x)\right| \\
&\quad\le \|f\|_L \Bigg\{M_0(\phi) \\
&\qquad + \sum_{k\in\mathbb{Z}}\left|\phi\left(e^{-k}x^w\right)\right| \frac{\mu^\alpha}{\Gamma(\alpha)} \int_0^1 v^\mu \left(\log\frac{1}{v}\right)^{\alpha-1} \\
&\qquad \left(\frac{(k-w\log x)^2 + 2w\left|\log x\right|\left|k-w\log x\right| + w^2\log^2 x}{w^2}\right)\frac{dv}{v} \\
&\qquad + \sum_{k\in\mathbb{Z}}\left|\phi\left(e^{-k}x^w\right)\right| \frac{\mu^\alpha}{\Gamma(\alpha)} \int_0^1 v^\mu \left(\log\frac{1}{v}\right)^{\alpha-1} \left(\frac{2\left(\left|k-w\log x\right| + w\left|\log x\right|\right)v}{w^2}\right)\frac{dv}{v} \\
&\qquad + \sum_{k\in\mathbb{Z}}\left|\phi\left(e^{-k}x^w\right)\right| \frac{\mu^\alpha}{\Gamma(\alpha)} \int_0^1 v^\mu \left(\log\frac{1}{v}\right)^{\alpha-1} \frac{v^2}{w^2}\frac{dv}{v}\Bigg\}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\left|\left(I^{\phi}_{w,\alpha,\mu}f\right)(x)\right| \\
&\le \|f\|_L \left\{M_0(\phi) + \frac{M_2(\phi) + 2w\left|\log x\right| M_1(\phi) + w^2\log^2 x M_0(\phi)}{w^2}\right. \\
&\left. + \frac{\mu^\alpha}{(\mu+1)^\alpha}\frac{2M_1(\phi) + 2w\left|\log x\right| M_0(\phi)}{w^2} + \frac{\mu^\alpha}{(\mu+2)^\alpha}\frac{M_0(\phi)}{w^2}\right\}.
\end{aligned}$$

Dividing the both sides by $1+\log^2 x$, we have

$$\begin{aligned}
&\frac{\left|\left(I^{\phi}_{w,\alpha,\mu}f\right)(x)\right|}{1+\log^2 x} \\
&\qquad \le \|f\|_L \left[M_0(\phi)\left(1 + \frac{\mu^\alpha}{w(\mu+1)^\alpha} + \frac{\mu^\alpha}{w^2(\mu+2)^\alpha}\right)\right. \\
&\qquad\quad \left. + M_1(\phi)\left(\frac{1}{w} + \frac{2\mu^\alpha}{(\mu+1)^\alpha}\right) + \frac{M_2(\phi)}{w^2}\right].
\end{aligned}$$

Since $M_2(\phi) < \infty$, this completes the proof. □

Now, we determine the rate of convergence. To do this, we recall the weighted logarithmic modulus of continuity for $f \in \tilde{C}_2(\mathbb{R}^+)$ given in [14]. It is defined by

$$\tilde{\Omega}(f;\delta) := \sup_{\substack{|\log t|\le\delta \\ x>0}} \frac{\left|f(tx) - f(x)\right|}{\left(1+\log^2 t\right)\left(1+\log^2 x\right)}. \tag{6.1}$$

The modulus of continuity defined in (6.1) has the following properties:

Lemma 1. *([14]) Let $\delta > 0$. Then*

a) for $f \in \tilde{C}_2(\mathbb{R}^+)$, $\tilde{\Omega}(f,\delta) < \infty$,
b) for $f \in \tilde{C}^*_2(\mathbb{R}^+)$, $\lim_{\delta\to 0}\tilde{\Omega}(f,\delta) = 0$,

c) for any $\lambda > 0$,

$$\tilde{\Omega}\left(f, \lambda\delta\right) \leq 2\left(1+\lambda\right)^{3}\left(1+\delta^{2}\right)\tilde{\Omega}\left(f,\delta\right),$$

d) for all $f \in \tilde{C}_2\left(\mathbb{R}^+\right)$,

$$\begin{aligned}&\left|f\left(t\right)-f\left(x\right)\right|\\&\quad\leq 16\left(1+\log^{2}x\right)\left(1+\delta^{2}\right)^{2}\left(1+\frac{\left|\log t-\log x\right|^{5}}{\delta^{5}}\right)\tilde{\Omega}\left(f,\delta\right),\ t,x>0\end{aligned}$$

hold.

Theorem 5. *Let* $f \in \tilde{C}_2^*\left(\mathbb{R}^+\right)$ *and* $\phi \in \Psi$ *be a kernel satisfying the assumption* $M_5\left(\phi\right) < \infty$. *Then, the inequality*

$$\left\|I_{w,\alpha,\mu}^{\phi}f-f\right\|_{L}\leq 2^{6}\Omega\left(f,\frac{1}{w}\right)\left[M_{0}\left(\phi\right)\left\{1+\frac{\mu^{\alpha}}{\left(\mu+5\right)^{\alpha}}\right\}+16M_{5}\left(\phi\right)\right]$$

holds.

Proof. Thanks to definition of the operators (2.1), we can write

$$\begin{aligned}\left|\left(I_{w,\alpha,\mu}^{\phi}f\right)\left(x\right)-f\left(x\right)\right|&\leq\sum_{k\in\mathbb{Z}}\left|\phi\left(e^{-k}x^{w}\right)\right|\frac{\mu^{\alpha}}{\Gamma\left(\alpha\right)}\int_{0}^{1}v^{\mu}\left(\log\frac{1}{v}\right)^{\alpha-1}\left|f\left(e^{\frac{k+v}{w}}\right)\right.\\&\quad\left.-f\left(x\right)\right|\frac{dv}{v}:=I.\end{aligned}$$

From Lemma 1 d), we get

$$\begin{aligned}I&\leq 16\left(1+\log^{2}x\right)\left(1+\delta^{2}\right)^{2}\tilde{\Omega}\left(f,\delta\right)\\&\quad\times\left[M_{0}\left(\phi\right)+\sum_{k\in\mathbb{Z}}\left|\phi\left(e^{-k}x^{w}\right)\right|\frac{\mu^{\alpha}}{\Gamma\left(\alpha\right)}\int_{0}^{1}v^{\mu}\left(\log\frac{1}{v}\right)^{\alpha-1}\frac{\left|k+v-w\log x\right|^{5}}{w^{5}\delta^{5}}\frac{dv}{v}\right]\\&\leq 16\left(1+\log^{2}x\right)\left(1+\delta^{2}\right)^{2}\tilde{\Omega}\left(f,\delta\right)\left[M_{0}\left(\phi\right)+16\left(\frac{M_{5}\left(\phi\right)}{w^{5}\delta^{5}}+\frac{\mu^{\alpha}M_{0}\left(\phi\right)}{\left(\mu+5\right)^{\alpha}}\right)\right].\end{aligned}$$

If we choose $\delta = w^{-1}$ and consider $\delta < 1$, we obtain which is the desired result. □

From Theorem 5 and Lemma 1 b), we have the following Corollary.

Corollary 3. *For all* $f \in \tilde{C}_2^*\left(\mathbb{R}^+\right)$, *there*

$$\lim_{w\to\infty}\left\|I_{w,\alpha,\mu}^{\phi}f-f\right\|_{L}=0$$

holds.

Finally, we present a quantitative form of Voronovskaja-type formula in the logarithmic weighted space of functions. Let us first recall the following Proposition.

Proposition 1. ([6]) *Let* $\left(\Theta f\right) \in \tilde{C}_2\left(\mathbb{R}^+\right)$. *For* $\delta < 1$ *and* $t, x > 0$, *we have*

$$\left|R_{n}\left(t\right)\right|\leq 64\left(1+\log^{2}x\right)\tilde{\Omega}\left(\left(\Theta f\right),\delta\right)\left(\left|\log t-\log x\right|+\frac{\left|\log t-\log x\right|^{6}}{\delta^{5}}\right),\tag{6.2}$$

where R *is the Lagrange remainder of Mellin–Taylor formula given in (5.2).*

Theorem 6. *Let* $(\Theta f) \in \tilde{C}_2(\mathbb{R}^+)$ *and* $\phi \in \Psi$ *be a kernel such that* $m_1(\phi, x) := m_1(\phi) < \infty$. *Let* $m_1(\phi) \neq -\frac{\mu^\alpha}{(\mu+1)^\alpha}$. *If* $M_6(\phi) < \infty$, *then we have*

$$\left| w\left[\left(I^{\phi}_{w,\alpha,\mu} f\right)(x) - f(x)\right] - (\Theta f)(x)\left(m_1(\phi) + \frac{\mu^\alpha}{(\mu+1)^\alpha}\right)\right|$$
$$\leq 64\left(1+\log^2 x\right)\tilde{\Omega}\left((\Theta f), \frac{1}{w}\right)$$
$$\left\{\mu^\alpha M_0(\phi)\left(\frac{1}{(\mu+1)^\alpha} + \frac{32}{(\mu+6)^\alpha}\right) + M_1(\phi) + \frac{32\mu^\alpha M_6(\phi)}{(\mu+1)^\alpha}\right\}$$

for $\alpha, \mu > 0$.

Proof. Thanks to (5.2), we obtain

$$\left(I^{\phi}_{w,\alpha,\mu} f\right)(x) - f(x)$$
$$= \sum_{k\in\mathbb{Z}} \phi\left(e^{-k}x^w\right)\frac{\mu^\alpha}{\Gamma(\alpha)}\int_0^1 v^\mu\left(\log\frac{1}{v}\right)^{\alpha-1}(\Theta f)(x)\left(\frac{k - w\log x + v}{w}\right)\frac{dv}{v}$$
$$+ \sum_{k\in\mathbb{Z}} \phi\left(e^{-k}x^w\right)\frac{\mu^\alpha}{\Gamma(\alpha)}\int_0^1 v^\mu\left(\log\frac{1}{v}\right)^{\alpha-1} R\left(e^{\frac{k+v}{w}}\right)\frac{dv}{v}$$
$$= \frac{(\Theta f)(x)}{w}\left(m_1(\phi) + \frac{\mu^\alpha}{(\mu+1)^\alpha}\right)$$
$$+ \underbrace{\sum_{k\in\mathbb{Z}} \phi\left(e^{-k}x^w\right)\frac{\mu^\alpha}{\Gamma(\alpha)}\int_0^1 v^\mu\left(\log\frac{1}{v}\right)^{\alpha-1} R\left(e^{\frac{k+v}{w}}\right)\frac{dv}{v}}_{J}.$$

Now, we estimate J. From the inequality (6.2), we have

$$|J| \leq \sum_{k\in\mathbb{Z}} \left|\phi\left(e^{-k}x^w\right)\right|\frac{\mu^\alpha}{\Gamma(\alpha)}\int_0^1 v^\mu\left(\log\frac{1}{v}\right)^{\alpha-1}\left|R\left(e^{\frac{k+v}{w}}\right)\right|\frac{dv}{v}$$
$$\leq 64\left(1+\log^2 x\right)\tilde{\Omega}((\Theta f), \delta)\sum_{k\in\mathbb{Z}}\left|\phi\left(e^{-k}x^w\right)\right|\frac{\mu^\alpha}{\Gamma(\alpha)}\int_0^1 v^\mu\left(\log\frac{1}{v}\right)^{\alpha-1}$$
$$\times\left\{\frac{|k - w\log x| + v}{w} + \frac{32}{w^6\delta^5}\left[|k - w\log x|^6 + v^6\right]\right\}\frac{dv}{v}$$
$$\leq 64\left(1+\log^2 x\right)\tilde{\Omega}((\Theta f), \delta)$$
$$\left\{\frac{M_1(\phi)}{w} + \frac{\mu^\alpha M_0(\phi)}{(\mu+1)^\alpha w} + \frac{32}{w^6\delta^5}\left[\frac{\mu^\alpha M_6(\phi)}{(\mu+1)^\alpha} + \frac{\mu^\alpha M_0(\phi)}{(\mu+6)^\alpha}\right]\right\}.$$

Setting $\delta = \frac{1}{w}$, the proof is completed. □

7. Some Examples of Kernels with Applications

This section is devoted to some examples of kernel satisfying the assumptions employed in this paper. Furthermore, we present numerical examples and graphical representations.

7.1. Mellin-Spline Kernel

For $n \in \mathbb{N}$ and $t > 0$, Mellin-Spline kernel of order n is defined by

$$S_n(t) := \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(\frac{n}{2} + \log t - j\right)_+^{n-1},$$

where $(\cdot)_+$ denotes the positive part of the numbers $(\cdot)$. B_n is a kernel with compact support on $[e^{-a}, e^a], a > 0$. We have that this kernel is continuous and $B_n \in X_c$ for every $c \in \mathbb{R}$. Thus, we can compute the Mellin transform of B_n. For $c \in \mathbb{R}$, the Mellin transform of B_n is given by

$$[B_n]_{\tilde{M}}(c + i\tilde{v}) = \left(\frac{\sin(\tilde{v}/2)}{\tilde{v}/2}\right)^r, \ \tilde{v} \in \mathbb{R}\backslash\{0\}$$

(see [21]).

Using the Mellin–Poisson summation formula (see [32]) which is the form

$$(i)^s \sum_{k\in\mathbb{Z}} B_n\left(e^k t\right)(k - \log t)^s = \sum_{k\in\mathbb{Z}} [B_n]_{\tilde{M}}(2k\pi i)\, t^{-2k\pi i}, \tag{7.1}$$

we can show that

$$\sum_{k\in\mathbb{Z}} B_n\left(e^k t\right) = 1$$

for every $t \in \mathbb{R}^+$. Since B_n has a compact support, we have all the absolute moments are finite. From (7.1), one has $m_1(B_n) = 0$ and other algebraic moments of B_n can be computed in a similar way. Moreover, $m_r(B_n, x)$ are independent of $x \in \mathbb{R}^+$ for $r = 1, 2 \ldots n$ (see [23]). Since the all absolute moments of B_b are finite, the property ϕ_4) is satisfies.

As an example, we consider the Mellin-Spline kernel of order 3 given by

$$B_3(t) = \begin{cases} \frac{1}{2}\left(\frac{3}{2} + \log t\right)^2, & e^{-3/2} < t \le e^{-1/2} \\ \frac{3}{4} - \log^2 t, & e^{-1/2} < t \le e^{1/2} \\ \frac{1}{2}\left(\frac{3}{2} - \log t\right)^2, & e^{1/2} < t < e^{3/2} \\ 0, & \text{otherwise} \end{cases}.$$

Let us consider the function $f : \mathbb{R}^+ \to \mathbb{R}, f(x) = \frac{1}{1+x^2}$. In the Table 1, we compare with the operators (1.7) and our operators (2.1) at random values for $\alpha = 3, \mu = \frac{1}{2}$.

Now, we consider the function $g : \mathbb{R}^+ \to \mathbb{R}, g(x) = \log^2 x$. In the Table 2, we compare with the operators (1.7) and our operators (2.1) at random values for $\alpha = 2, \mu = \frac{1}{3}$.

Finally, we consider the function $h : \mathbb{R}^+ \to \mathbb{R}, h(x) = 1 + \log^2 x$. Figure 1 shows approximation of the function h by $\left(I^{B_3}_{5,2,\frac{1}{3}} h\right)$ and $\left(I^{B_3}_{5,2} h\right)$, respectively.

Figure 2 shows approximation to the function h of the family of operators $\left(I^{B_3}_{w,\alpha,\mu}\right)$ with respect to μ.

Table 1. Errors of estimates

w	$\left(I^{B_3}_{w,3,\frac{1}{2}}f\right)(1.1) - f(1.1)$	$\left(I^{B_3}_{w,3}f\right)(1.1) - f(1.1)$	$\left(I^{B_3}_{w,3,\frac{1}{2}}f\right)(4) - f(4)$	$\left(I^{B_3}_{w,3}f\right)(4) - f(4)$
10	0.00171143	0.01218759	0.00015773	0.00243594
50	0.00036219	0.00247059	0.00007197	0.00054007

Table 2. Errors of estimates

w	$\left(I^{B_3}_{w,2,\frac{1}{3}}g\right)(2) - g(2)$	$\left(I^{B_3}_{w,2}g\right)(2) - g(2)$	$\left(I^{B_3}_{w,2,\frac{1}{3}}g\right)(5) - g(5)$	$\left(I^{B_3}_{w,2}g\right)(5) - g(5)$
5	0.02814501	0.10908629	0.05105227	0.23125839XXX
200	0.00043998	0.00232091	0.00101266	0.00537521XXX

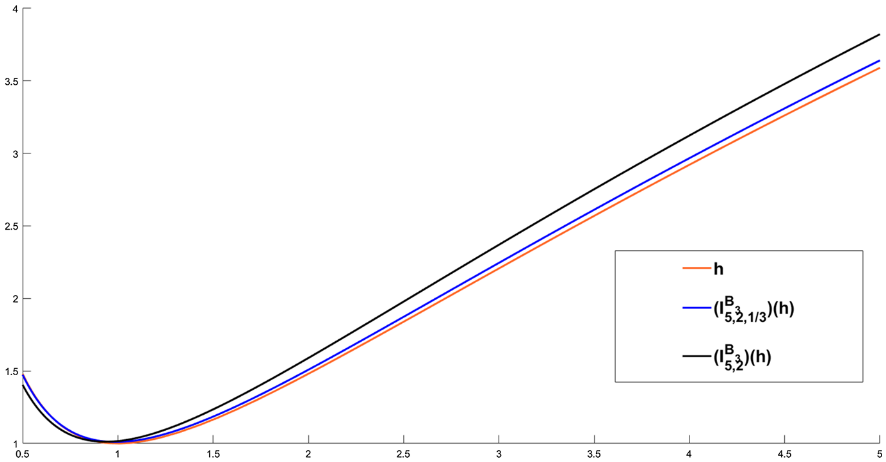

Figure 1. Approximation of the function h by $\left(I^{B_3}_{5,2,\frac{1}{3}}h\right)$

7.2. Mellin–Fejer Kernel

As another example, we present family of Mellin–Fejer kernel defined by

$$F^c_\rho(t) := \frac{t^{-c}}{2\pi}\rho\,\mathrm{sinc}^2\left(\frac{\rho}{\pi}\log\sqrt{t}\right),\ c \in \mathbb{R}, \rho, t \in \mathbb{R}^+,$$

with the continuous extension $F^c_\rho(1) = \frac{\rho}{2\pi}$. The Mellin–Fejer kernel is not with compact support but $F^c_\rho \in X_c$ and, thus, Mellin transform of F^c_ρ is given by

$$\left[F^c_\rho\right]_{\tilde{M}}(c + i\tilde{v}) = \begin{cases} 1 - \frac{|\tilde{v}|}{\rho} & , |\tilde{v}| \leq \rho \\ 0 & , |\tilde{v}| > \rho \end{cases}$$

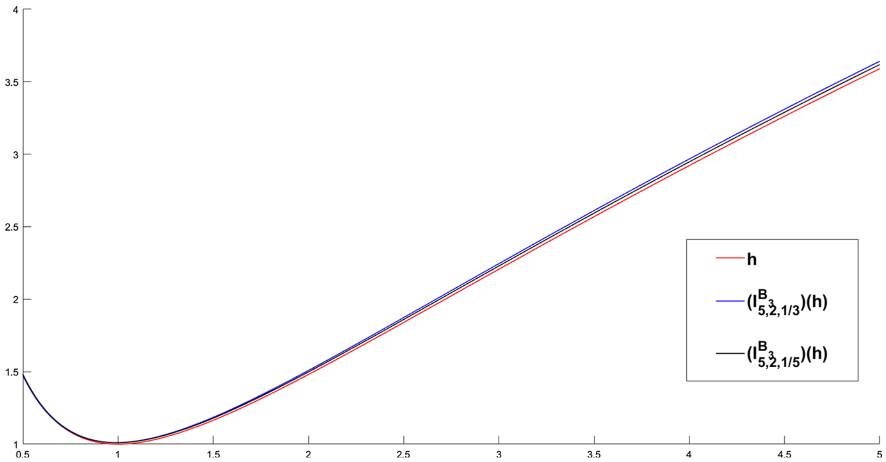

Figure 2. Approximation to the function h of the family of operators $\left(I^{B_3}_{w,\alpha,\mu}\right)$ with respect to μ

(see [21]). Let $c = 0$ and $\rho = 1$. Thanks to (7.1), we get

$$\sum_{k\in\mathbb{Z}} F_1^0\left(e^k t\right) = 1$$

for every $t \in \mathbb{R}^+$. Note that $M_1\left(F_1^0\right)$ is not finite but for $0 < \beta < 1$ we have $M_\beta\left(F_1^0\right) < \infty$.

To have an asymptotic expression, we present the following example modifying the Mellin–Fejer kernel in a suitable way.

7.3. Mellin–Jackson Kernel

We consider the generalized Mellin–Jackson kernel which is the form (see [23])

$$J_{\gamma,\theta}\left(t\right) := d_{\gamma,\theta} t^{-c} \operatorname{sinc}^{2\theta}\left(\frac{\log t}{2\gamma\theta\pi}\right),\ c \in \mathbb{R}, t \in \mathbb{R}^+,$$

where $\theta \in \mathbb{N}, \gamma \geq 1$ and $d_{\gamma,\theta}$ is a normalization constant, i.e.,

$$d_{\gamma,\theta}^{-1} := \int_0^\infty \operatorname{sinc}^{2\theta}\left(\frac{\log t}{2\gamma\theta\pi}\right)\frac{du}{u}.$$

Since $\left\|J_{\gamma,\theta}\right\|_{X_c} = 1$, hence $J_{\gamma,\theta} \in X_c$. Therefore, Mellin transform of $J_{\gamma,\theta}$ is well-defined and $\left[J_{\gamma,\theta}\right]_{\tilde{M}}\left(c + i\tilde{v}\right) = 0$ for $\left|\tilde{v}\right| \geq \gamma$. Thus, we have that $J_{\gamma,\theta}$ is a Mellin band-limited kernel. Moreover, since $\left[J_{\gamma,\theta}\right]_{\tilde{M}}\left(0\right) = 0$ and for $k \neq 0$, $\left[J_{\gamma,\theta}\right]_{\tilde{M}}\left(2k\pi i\right) = 0$, we get

$$\sum_{k\in\mathbb{Z}} J_{\gamma,\theta}\left(e^k t\right) = 1.$$

Also, thanks to (7.1), one has $m_1(J_{\gamma,\theta}) = 0$ and for $\theta > \frac{3}{2}$, $m_2(J_{\gamma,\theta}) := A_{\gamma,\theta} < \infty$. In addition, we have that $M_1(J_{\gamma,\theta}) < \infty, M_2(J_{\gamma,\theta}) < \infty$ and thus ϕ_4) is satisfies (see [23]).

Author contributions SK wrote the main manuscript text and prepared the figures/tables. AA and TA analyzed the theorems and proofs in the paper. All authors reviewed the manuscript.

Data Availability All data generated or analyzed during this study are included in this published article.

Declarations

Conflict of interest The authors declare no competing interests.

References

[1] Acar, T., Alagöz, O., Aral, A., Costarelli, D., Turgay, M., Vinti, G.: Convergence of generalized sampling series in weighted spaces. Demonstr. Math. **55**, 153–162 (2022)

[2] Acar, T., Alagöz, O., Aral, A., Costarelli, D., Turgay, M., Vinti, G.: Approximation by sampling Kantorovich series in weighted spaces of functions. Turk. J. Math. **46**(7), 2663–2676 (2022)

[3] Acar, T., Costarelli, D., Vinti, G.: Linear prediction and simultaneous approximation by m-th order Kantorovich type sampling series. Banach J. Math. Anal. **14**, 1481–1508 (2020)

[4] Acar, T., Draganov, B.R.: A stronge converse inequality for generalized sampling operators. Ann. Funct. Anal. **13**(36), 1–16 (2022)

[5] Acar, T., Eke, A., Kursun, S.: Bivariate generalized Kantorovich-type exponential sampling series. (Submitted) (2023)

[6] Acar, T., Kursun, S.: Pointwise convergence of generalized Kantorovich exponential sampling series. Dolomites Res. Notes Approx. **16**, 1–10 (2023)

[7] Acar, T., Kursun, S.: Approximation properties of exponential sampling series in logarithmic weighted spaces, (Submitted) (2023)

[8] Acar, T., Kursun, S.: Convergence of bivariate exponential sampling series in logarithmic weighted space of functions, (Submitted) (2023)

[9] Acar, T., Kursun, S., Turgay, M.: Multidimensional Kantorovich modifications of exponential sampling series. Quaest. Math. **46**(1), 57–72 (2023)

[10] Acar, T., Turgay, M.: Approximation by bivariate generalized sampling series in weighted spaces of functions. Dolomites Res. Notes Approx. **16**, 11–22 (2023)

[11] Alagöz, O., Turgay, M., Acar, T., Parlak, M.: Approximation by sampling Durrmeyer operators in weighted space of functions. Numer. Funct. Anal. Optim. **43**(10), 1223–1239 (2022)

[12] Angeloni, L., Cetin, N., Costarelli, D., Sambucini, A.R., Vinti, G.: Multivariate sampling Kantorovich operators: quantitative estimates in Orlicz spaces. Constr. Math. Anal. **4**(2), 229–241 (2021)

[13] Angeloni, L., Costarelli, D., Vinti, G.: Convergence in variation for the multidimensional generalized sampling series and applications to smoothing for digital image processing. Ann. Acad. Sci. Fenn. Math. **45**, 751–770 (2020)

[14] Aral, A., Acar, T., Kursun, S.: Generalized Kantorovich forms of exponential sampling series, Anal. Math. Phys., **12**(2), 1–19 (2022)

[15] Bajpeyi, S., Angamuthu, S.K.: On approximation by Kantorovich exponential sampling operators. Numer. Funct. Anal. Optim. **42**(9), 1096–1113 (2021)

[16] Bajpeyi, S., Angamuthu, S.K., Mantellini, I.: Approximation by Durrmeyer type exponential sampling operators. Numer. Funct. Anal. Optim. **43**(1), 16–34 (2022)

[17] Balsamo, S., Mantellini, I.: On linear combinations of general exponential sampling series, Results Math., **74**(4), (2019)

[18] Bardaro, C., Bevignani, G., Mantellini, I., Seracini, M.: Bivariate generelized exponential sampling series and applications to seismic waves. Const. Math. Anal. **2**(4), 153–167 (2019)

[19] Bardaro, C., Butzer, P.L., Mantellini, I.: The foundations of fractional calculus in the Mellin transform setting with applications. J. Fourier Anal. Appl. **21**, 961–1017 (2015)

[20] Bardaro, C., Faina, L., Mantellini, I.: Quantitative Voronovskaja formulae for generalized Durrmeyer sampling type series, Math. Nach., 289, 1702–1720 (2016)

[21] Bardaro, C., Faina, L., Mantellini, I.: A generalization of the exponential sampling series and its approximation properties. Math. Slovaca **67**(6), 1481–1496 (2017)

[22] Bardaro, C., Mantellini, I.: A note on the Voronovskaja theorem for Mellin-Fejer convolution operators, Appl. Math. Lett., 24, 2064–2067 (2011)

[23] Bardaro, C., Mantellini, I.: Asymptotic formulae for linear combinations of generalized sampling type operators. Zeitschrift für Anal. und ihre Anwendung **32**, 279–298 (2013)

[24] Bardaro, C., Mantellini, I.: On Mellin convolution operators: a direct approach to the asymptotic formulae. Integral Transforms Spec. Funct. **25**, 182–195 (2014)

[25] Bardaro, C., Mantellini, I.: On a Durrmeyer-type modification of the exponential sampling series. Rend. Circ. Mat. Palermo **70**(3), 1289–1304 (2021)

[26] Bardaro, C., Mantellini, I., Schmeisser, G.: Exponential sampling series: convergence in Mellin- Lebesgue spaces. Results Math., **74**, 1–20 (2019)

[27] Bardaro, C., Mantellini, I., Stens, R., Vautz, J., Vinti, G.: Generalized sampling approximation for multivariate discontinuous signals and application to image processing, New Perspectives on Approximation and Sampling Theory-Festschrift in honor of Paul Butzer's 85th birthday, Birkhauser, 87-114 (2014)

[28] Bertero, M., Pike, E.R.: Exponential sampling method for Laplace and other dilationally invariant transforms I. Singularsystem analysis. II. Examples in photon correction spectroscopy and Frauenhofer diffraction, Inverse Probl., 7, 1-20; 21-41 (1991)

[29] Butzer, P.L., Engels, W., Ries, S., Stens, R.L.: The Shannon sampling series and the reconstruction of signals in terms of linear, quadratic and cubic splines. SIAM J. Appl. Math. **46**(2), 299–323 (1986)

[30] Butzer, P.L., Jansche, S.: A direct approach to the Mellin transform. J. Fourier Anal. Appl. **3**(4), 325–375 (1997)

[31] Butzer, P.L., Jansche, S.: The exponential sampling theorem of signal analysis, Atti Sem. Mat. Fis. Univ. Modena Suppl., 46, (special issue dedicated to Professor Calogero Vinti) 99-122 (1998)

[32] Butzer, P.L., Jansche, S.: The finite Mellin Transform, Mellin-Fourier series and the Mellin-Poisson summation formula. Rend. Circ. Mat. Palermo **52**, 55–81 (1998)

[33] Butzer, P.L., Kilbas, A.A., Trujillo, J.J.: Fractional calculus in the Mellin setting and Hadamard-type fractional integral. J. Math. Anal. Appl. **269**, 1–27 (2002)

[34] Butzer, P.L., Kilbas, A.A., Trujillo, J.J.: Stirling functions of the second kind in the setting of difference and fractional calculus. Numer. Funct. Anal. Optimiz. **4**(7–8), 673–711 (2003)

[35] Butzer, P.L., Splettstosser, W.: A sampling theorem for duration-limited functions with error estimates. Inf. Control. **34**(1), 55–65 (1977)

[36] Butzer, P.L., Stens, R.L.: Linear prediction by samples from the past. In: Advanced topics in Shannon sampling and interpolation theory, New York, Springer, 157-183 (1993)

[37] Butzer, P.L., Stens, R.L.: The sampling theorem and linear prediction in signal analysis. Jahresber. Dtsch. Math. Ver. **90**(1), 1–70 (1998)

[38] Butzer, P.L., Westphal, U.: An introduction to fractional calculus. In: Hifler, H. (eds) Applications of Fractional Calculus in Physics, pp. 1–85. Wordl Scientific Publ, Singapore (2000)

[39] Casasent, D. (ed.).: Optical data processing, pp. 241–282. Springer, Berlin (1978)

[40] Costarelli, D., Vinti, G.: An inverse result of approximation by sampling Kantorovich series. Proc. Edinb. Math. Soc. **62**(1), 265–280 (2019)

[41] Costarelli, D., Vinti, G.: Inverse results of approximation and the saturation order for the sampling Kantorovich series. J. Approx. Theory **24**, 64–82 (2019)

[42] Draganov, B.R.: A fast converging sampling operator. Constr. Math. Anal. **5**(4), 190–201 (2022)

[43] Gori, F.: Sampling in optics. In: Marks, R.J., II. (ed.) Advances Topics in Shannon Sampling and Interpolation Theory, pp. 37–83. Springer, New York (1993)

[44] Hadamard, J.: Essai sur l'etude des fonctions donnees par leur developpement de Taylor. J. Math. Pures et Appl. **4**(8), 101–186 (1892)

[45] Higgins, J.R.: Sampling theory in Fourier and signal analysis. Foundations. Oxford University Press, Oxford (1996)

[46] Kadak, U.: Fractional type multivariate sampling operators. RACSAM **115**, 153 (2021)

[47] Kadak, U.: Fractional sampling operators of multivariate fuzzy functions and applications to image processing. Appl. Soft Comput. **132**, 109901 (2023)

[48] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)

[49] Kotel'nikov, V.A.: On the carrying capacity of "ether" and wire in electrocommunications, Material for the First All-Union Conference on the Questions of Communications, Moscow, (1933)

[50] Kumar, S.A., Bajpeyi, S.: Direct and inverse results for Kantorovich type exponential sampling series, Results Math., **75**(3), 1–17 (2020)

[51] Kursun, S., Turgay, M., Alagoz, O., Acar, T.: Approximation properties of multivariate exponential sampling series. Carpathian Math. Publ. **13**(3), 666–675 (2021)

[52] Kursun, S., Aral, A., Acar, T.: Riemann-Liouville fractional integral type exponential sampling Kantorovich series, (Submitted) (2023)

[53] Mahmudov, N., Kara, M.: Approximation properties of the Riemann-Liouville fractional integral type Szász-Mirakyan-Kantorovich operators. J. Math. Inequalities **16**(4), 1285–1308 (2022)

[54] Mamedov, R.G.: The Mellin transform and approximation theory (in Russian). "Elm", Baku (1991)

[55] Ostrowsky, N., Sornette, D., Parker, P., Pike, E.R.: Exponential sampling method for light scattering polydispersity analysis. Opt. Acta **28**, 1059–1070 (1994)

[56] Shannon, C.E.: Communications in the presence of noise. Proc. IRE. **37**, 10–21 (1949)

[57] Turgay, M., Aral, A., Acar, T.: Approximation properties of the Riemann-Liouville fractional integral type generalized sampling Kantorovich series, (Submitted) (2023)

[58] Whittaker, E.T.: On the functions, which are represented by expansions of the interpolation theory. Proc. R. Soc. Edinb. **35**, 181–194 (1915)

[59] Zayed, A.I.: Advances in Shannon's sampling theory. CRC Press, Boca Raton (1993)

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Received: April 1, 2023.
Revised: June 5, 2023.
Accepted: June 17, 2023.