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Two fixed point results on F-metric spaces

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Abstract: This paper presents two fixed point results for both single valued and multivalued mappings on F-metric spaces, which is introduced by Jleli and Samet [1]. First, we consider the concept of P-contraction for single valued mappings. Then using the Feng-Liu's technique, we present a fixed point result for F-closed set valued mappings on F-metric spaces.

Keywords: Fixed point, F-metric space, P-contraction, multivalued mapping

MSC: 54H25, 47H10

1 Introduction and Preliminaries

The field of fixed point theory is an important branch of nonlinear analysis, because it offers powerful tools for solving various problems including integral and differential equations. The main two research directions are related, on the one hand, to obtain more general results by generalizing contraction conditions for single valued or multivalued mappings, and on the other to obtain new results by relaxing the structure of space.

In recent years, mathematicians sought to find new generalization of metric space. For example, Czerwik [2] introduced the notion of b-metric by considering the following condition instead of the triangular inequality: for every $x, y \in X$ and $s \geq 1$,

$$d(x, y) \leq s \left[d(x, z) + d(z, y)\right].$$

In [3] Fagin et al introduced the notion of s-relaxed$_p$ metric by using the following condition instead of the triangular inequality: for every $x, z, x_1, x_2, \cdots, x_n$ and $s \geq 1$,

$$d(x, z) \leq s \left[d(x, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, z)\right].$$

Mustafa and Sims [4] introduced the notion of G-metric, which is a mapping defined on the product set $X \times X \times X$, and satisfying certain conditions.

Mathews [5] introduced partial metric space. While the distance of a point to itself is zero on metric space, the distance of a point to itself is not always zero on partial metric space. Therefore, it is interesting conclusion.

Jleli and Samet [6] replaced triangle inequality of metric to following condition: there exists a coefficient $s > 0$ such that

$$\text{if } \lim_{n\to\infty} d(x_n, x) = 0 \text{ and } y \in X \text{ then } d(x, y) \leq s[\limsup_{n\to\infty} d(x_n, y)]$$

and hence they introduced JS-metric space.

Very recently, Jleli and Samet [1] introduced F-metric structure on a nonempty set X. Then, they show that every metric and every s-relaxed$_p$ metric is an F-metric but the converse is not true. To show this, they provide an example of F-metric that it is not an s-relaxed$_p$ metric (see Example 2.4 in [1]).

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Now we recall the F-metric structure.

Let $\mathcal{F}$ be the set of functions $f : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

$(\mathcal{F}_1)$ f is non decreasing, i.e, $0 < s < t \Rightarrow f(s) \leq f(t)$,

$(\mathcal{F}_2)$ For every sequence $\{t_n\} \subset (0, \infty)$, we have

$$\lim_{n\to\infty} t_n = 0 \Rightarrow \lim_{n\to\infty} f(t_n) = -\infty.$$

Definition 1 ([1]). *Let X be a nonempty set, and let $D : X \times X \to [0, \infty)$ be a given mapping. Suppose that there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that*

$(D_1)(x, y) \in X \times X$, $D(x, y) = 0 \Leftrightarrow x = y$.

$(D_2)D(x, y) = D(y, x)$, for all $(x, y) \in X \times X$.

(D_3)For every $(x, y) \in X \times X$, for every $N \in \mathbb{N}$, $N \geq 2$, and for every $(u_i)_{i=1}^N \subset X$ with $(u_1, u_N) = (x, y)$, we have

$$D(x, y) > 0 \Rightarrow f\left(D(x, y)\right) \leq f\left(\sum_{i=1}^{N-1} D(u_i, u_{i+1})\right) + \alpha.$$

Then D is said to be an F-metric on X, and the pair (X, D) is said to be an F-metric space.

We can easily observe that every metric on X is an F-metric. Following example shows that the converse is not true.

Example 1 ([1]). *Let $X = \mathbb{N}$ and $D : X \times X \to [0, \infty)$ be the mapping defined by*

$$D(x, y) = \begin{cases} (x - y)^2 & \text{if } (x, y) \in [0, 3] \times [0, 3] \\ |x - y| & \text{if } (x, y) \notin [0, 3] \times [0, 3] \end{cases}$$

for all $(x, y) \in X \times X$. Then D is an F-metric but not a metric on X.

Let (X, D) be an F-metric space. A subset Y of X is said to be F-open if for every $x \in Y$, there is some $r > 0$ such that $B(x, r) \subset Y$, where

$$B(x, r) = \{y \in X : D(x, y) < r\}.$$

A subset Z of X is F-closed if $X \setminus Z$ is F-open. It can be easily seen that the collection of all F-open subsets, say τ_F, of X is a topology, which is first countable, on X.

Proposition 1 ([1]). *Let (X, D) be an F-metric space. Then, for any nonempty subset A of X, the following statements are equivalent:*

(i) A is F-closed

(ii) For any sequence $\{x_n\} \subset A$, we have

$$\lim_{n\to\infty} D(x_n, x) = 0, x \in X \Rightarrow x \in A.$$

Let (X, D) be an F-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If $\{x_n\}$ converges to x with respect to τ_F, then we say that $\{x_n\}$ is F-convergent to x. In this case x is said to be limit of $\{x_n\}$. Therefore we have $\{x_n\}$ is F-convergent to x if and only if $\lim_{n\to\infty} D(x_n, x) = 0$. By the routine calculation we can see that the limit of F-convergent sequence in an F-metric space is unique.

Definition 2 ([1]). *Let (X, D) be an F-metric space and $\{x_n\}$ be a sequence in X.*

(i) We say that $\{x_n\}$ is F-Cauchy, if

$$\lim_{n\to\infty} D(x_n, x_m) = 0.$$

(ii) We say that (X, D) is F-complete, if every F-Cauchy sequence in X is F-convergent to certain element in X.

After the introduction of F-metric space, Jleli and Samet [1] proved the following fixed point result, which is actually F-metric version of Banach fixed point theorem.

Theorem 1. *Let (X, D) be an F-complete F-metric space and $T : X \to X$ be a mapping. If there exists $k \in (0, 1)$ such that*

$$D(Tx, Ty) \le kD(x, y) \tag{1.1}$$

for all $x, y \in X$, then T has a unique fixed point in X. Moreover, every Picard sequence F-converges to the fixed point of T.

Proposition 2 ([1])**.** *Let (X, D) be an F-metric space. Then for any nonempty subset A of X, we have*

$$x \in \overline{A},\ r > 0 \Rightarrow B(x, r) \cap A \neq \emptyset,$$

where $\overline{A}$ is the closure of A with respect to τ_F.

Proposition 3. *Let (X, D) be an F-metric space, $A \subset X$ and $x \in X$. Then $x \in \overline{A} \Leftrightarrow D(x, A) = 0$, where $D(x, A) = \inf_{y \in A} D(x, y)$.*

Proof.

$$\begin{aligned} x \in \overline{A} &\Leftrightarrow B(x, r) \cap A \neq \emptyset \text{ for all } r > 0 \\ &\Leftrightarrow D(x, x_r) < r \text{ for all } r > 0 \text{ and some } x_r \in A \\ &\Leftrightarrow \inf\{D(x, y) : y \in A\} = 0 \\ &\Leftrightarrow D(x, A) = 0. \end{aligned}$$

□

If a subset A of an F-metric space (X, D) is compact with respect to τ_F, then it is said to F-compact set. Jleli and Samet [1] proved that a sunset A is F-compact if and only if it is sequentially F-compact, that is, every sequence in A has an F-convergent subsequence in A. For further information and recent results about F-metric space we refer to [7–11].

In this paper, we present two fixed point results for both single valued and multivalued mappings on F-metric spaces.

2 Single valued maps

In this section we present a fixed point theorem for single-valued P-contraction mappings on F-metric space. For this, we will use the concept of E-contraction in metric space, first introduced by Popescu (see the references of [12, 13]) and later developed by Fulga and Proca [12, 13]. We prefer to use this contraction as P-contraction to cite Popescu. See [14, 15] for more studies about this new idea. As the parallel manner on metric space, a self mapping T of an F-metric space (X, d) is called P-contraction, if for all $x, y \in X$ the following inequality holds:

$$D(Tx, Ty) \le k\left[D(x, y) + \left|D(x, Tx) - D(y, Ty)\right|\right] \tag{2.1}$$

for some $k \in (0, 1)$.

Theorem 2. *(X, D) be an F-complete F-metric space, and $T : X \to X$ be a continuous P-contraction mapping. Then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ be an arbitrary point and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ then x_{n_0} is a fixed point of T. Suppose $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$ and hence $D(x_n, x_{n+1}) > 0$. Now we have

$$\begin{aligned} D(x_{n+1}, x_{n+2}) &= D(Tx_n, Tx_{n+1}) \\ &\le k\left[D(x_n, x_{n+1}) + \left|D(x_n, Tx_n) - D(x_{n+1}, Tx_{n+1})\right|\right] \\ &= k\left[D(x_n, x_{n+1}) + \left|D(x_n, x_{n+1}) - D(x_{n+1}, x_{n+2})\right|\right] \end{aligned} \tag{2.2}$$

for all $n \in \mathbb{N}$. If $D(x_n, x_{n+1}) \leq D(x_{n+1}, x_{n+2})$ for some n, then from (2.2)

$$\begin{aligned} D(x_{n+1}, x_{n+2}) &\leq k\left[D(x_n, x_{n+1}) + \left|D(x_n, x_{n+1}) - D(x_{n+1}, x_{n+2})\right|\right] \\ &\leq k\left[D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2}) - D(x_n, x_{n+1})\right] \\ &= kD(x_{n+1}, x_{n+2}), \end{aligned}$$

which is a contradiction. Hence $D(x_n, x_{n+1}) > D(x_{n+1}, x_{n+2})$ for all $n \in \mathbb{N}$ and so from (2.2) we get

$$\begin{aligned} D(x_{n+1}, x_{n+2}) &\leq k\left[D(x_n, x_{n+1}) + \left|D(x_n, x_{n+1}) - D(x_{n+1}, x_{n+2})\right|\right] \\ &\leq k\left[D(x_n, x_{n+1}) + D(x_n, x_{n+1}) - D(x_{n+1}, x_{n+2})\right] \\ &= 2kD(x_n, x_{n+1}) - kD(x_{n+1}, x_{n+2}). \end{aligned}$$

Thus we have

$$D(x_{n+1}, x_{n+2}) \leq \frac{2k}{1+k} D(x_n, x_{n+1})$$

for all $n \in \mathbb{N}$. Let $\frac{2k}{1+k} = \lambda$, then $0 < \lambda < 1$ and so we get

$$\begin{aligned} D(x_{n+1}, x_{n+2}) &\leq \lambda D(x_n, x_{n+1}) \\ &\leq \lambda^2 D(x_{n-1}, x_n) \\ &\vdots \\ &\leq \lambda^{n+1} D(x_0, x_1) \end{aligned}$$

for all $n \in \mathbb{N}$. Hence we have for $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} \sum_{i=n}^{m-1} D(x_i, x_{i+1}) &\leq \sum_{i=n}^{m-1} \lambda^i D(x_0, x_1) \\ &\leq \frac{\lambda^n}{1-\lambda} D(x_0, x_1). \end{aligned}$$

Since $0 < \lambda < 1$, for all $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$0 < \frac{\lambda^n}{1-\lambda} D(x_0, x_1) < \delta.$$

Now let $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ be such that (D_3) is satisfied. Let $\varepsilon > 0$ be fixed, then by $(\mathcal{F}_2)$ there exists $\eta > 0$ such that

$$0 < t < \eta \Rightarrow f(t) < f(\varepsilon) - \alpha.$$

Considering δ as η we get

$$f\left(\frac{\lambda^n}{1-\lambda} D(x_0, x_1)\right) < f(\varepsilon) - \alpha.$$

By $(\mathcal{F}_1)$ we have

$$f\left(\sum_{i=n}^{m-1} D(x_i, x_{i+1})\right) \leq f\left(\frac{\lambda^n}{1-\lambda} D(x_0, x_1)\right) < f(\varepsilon) - \alpha$$

for all $m, n \in \mathbb{N}$ with $m > n \geq n_0$. Using (D_3) and this last inequality we get for $m > n \geq n_0$,

$$D(x_n, x_m) > 0 \Rightarrow f(D(x_n, x_m) \leq f\left(\sum_{i=n}^{m-1} D(x_i, x_{i+1})\right) + \alpha < f(\varepsilon),$$

which implies by $(\mathcal{F}_1)$ that $D(x_n, x_m) < \varepsilon$ for $m > n \geq n_0$. Therefore $\{x_n\}$ is F-Cauchy. Since (X, D) is F-complete there exists $z \in X$ such that $\{x_n\}$ is F-convergent to z, that is, $\lim_{n\to\infty} D(x_n, z) = 0$. Since T is continuous, which is equivalent to sequential continuous, and then $\lim_{n\to\infty} D(Tx_n, Tz) = \lim_{n\to\infty} D(x_{n+1}, Tz) = 0$. The

uniqueness of the limit we have $z = Tz$. Now suppose w is another fixed point of T, then $Tw = w$ and $D(z, w) > 0$. Hence

$$\begin{aligned} D(z, w) &= D(Tz, Tw) \\ &\leq k\left[D(z, w) + \left|D(z, Tz) - D(w, Tw)\right|\right] \\ &= kD(z, w), \end{aligned}$$

which is a contradiction. Therefore the fixed point of T is unique. □

Now we present an illustrative example.

Example 2. *Let $X = \{1, 2, 3, \cdots\}$ and $D : X \times X \to [0, \infty)$ be the mapping defined by*

$$D(x, y) = \begin{cases} (x - y)^2 & \text{if } x, y \in \{1, 2\} \\ 2\,|x - y| & \text{if otherwise} \end{cases}.$$

Then (X, D) is an F-metric space which is also F-complete. Note that, since $D(x, y) \geq 1$ for all $x, y \in X$ with $x \neq y$, then every F-Cauchy sequence in X should be eventually constant and hence they should be F-convergent. For the same reason the induced topology τ_F is discrete topology, and so every self mapping on X are continuous. Now define a mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \\ \frac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}.$$

Then by the routine calculation we can show that T is P-contraction mapping with $k \geq \frac{1}{2}$. Hence by Theorem 2, T has a unique fixed point.

Remark 1. *Since the inequality (2.1) is more general then (1.1), than we can take notice Theorem 1 is a special case of Theorem 2.*

3 Multivalued maps

In this section, we present Feng-Liu [16] type fixed point result for multivalued mapping on F-metric spaces. Here, we will use the following notations: Let (X, D) be an F-metric space, $T : X \to N(X)$ be a mapping, where $N(X)$ denotes the collection of all nonempty subsets of X, and $x \in X$. For a constant $b \in (0, 1)$, define the set $I_b^x \subset X$ as

$$I_b^x = \{y \in Tx : bD(x, y) \leq D(x, Tx)\}.$$

Let (X, τ) be a topological space, $x_0 \in X$ and $f : X \to \mathbb{R}$ be a function. Then f is called lower semi-continuous at x_0, if for any $\{x_n\} \subset X$ with $x_n \to x_0$ implies $f(x) \leq \lim\limits_{n\to\infty} \inf f(x_n)$. If f is lower semi-continuous at every point of X, then it is said to be lower semi-continuous on X.

Theorem 3. *Let (X, D) be an F-complete F-metric space, $T : X \to C(X)$ ($C(X)$ is the collection of all nonempty F-closed subsets of X) be a multivalued mapping. If there exists a constant $c \in (0, 1)$ such that for any $x \in X$, there is $y \in I_b^x$ satisfying*

$$D(y, Ty) \leq cD(x, y)$$

then T has a fixed point in X provided $c < b$ and $f(x) = D(x, Tx)$ is lower semi continuous.

Proof. Since $Tx \in C(X)$ for any $x \in X$, then there exists $y \in Tx$ such that

$$bD(x, y) \leq D(x, Tx).$$

Hence $I_b^x \neq \emptyset$. For any initial point $x_0 \in X$, there exist $x_1 \in I_b^{x_0}$ such that

$$D(x_1, Tx_1) \leq cD(x_0, x_1)$$

Similarly, for $x_1 \in X$ there exist $x_2 \in I_b^{x_1}$ such that

$$D(x_2, Tx_2) \leq cD(x_1, x_2).$$

Continuing this process, we can construct a sequence $\{x_n\}$ in X such that, for all $n \in \mathbb{N}$, $x_{n+1} \in Tx_n$,

$$D(x_{n+1}, Tx_{n+1}) \leq cD(x_n, x_{n+1}) \text{ and } bD(x_n, x_{n+1}) \leq D(x_n, Tx_n). \tag{3.1}$$

From (3.1), we have

$$D(x_{n+1}, Tx_{n+1}) \leq \frac{c}{b} D(x_n, Tx_n) \tag{3.2}$$

for all $n \in \mathbb{N}$. On the other hand, thanks to definition of sequence $\{x_n\}_{n=0}^{\infty}$ we get

$$D(x_n, x_{n+1}) \leq \frac{c}{b} D(x_{n-1}, x_n) \tag{3.3}$$

for all $n \in \mathbb{N}$. By using the inequalities (3.2) and (3.3) we have

$$D(x_n, x_{n+1}) \leq \frac{c^n}{b^n} D(x_0, x_1), \ n = 0, 1, 2, \cdots$$

$$D(x_n, Tx_n) \leq \frac{c^n}{b^n} D(x_0, Tx_0), \ n = 0, 1, 2, \cdots .$$

Then for $m, n \in \mathbb{N}$, $m > n$, we have

$$\begin{aligned}
\sum_{i=n}^{m-1} D(x_i, x_{i+1}) &= D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2}) + \cdots + D(x_{m-1}, x_m) \\
&\leq \frac{c^n}{b^n} D(x_0, x_1) + \frac{c^{n+1}}{b^{n+1}} D(x_0, x_1) + \cdots + \frac{c^{m-1}}{b^{m-1}} D(x_0, x_1) \\
&= a^n (1 + a + \cdots + a^{m-n-1}) D(x_0, x_1) \ (\text{for } a = \frac{c}{b}) \\
&\leq \frac{a^n}{1-a} D(x_0, x_1).
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{a^n}{1-a} D(x_0, x_1) = 0$, then for every $\delta > 0$, there exist $N \in \mathbb{N}$ such that

$$0 < \frac{a^n}{1-a} D(x_0, x_1) < \delta, \ n \geq N. \tag{3.4}$$

Now, let $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ be such that (D_3) is satisfied. Let $\varepsilon > 0$ be fixed. By $(\mathcal{F}_2)$, there exist $\delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f(\varepsilon) - \alpha. \tag{3.5}$$

Hence by (3.5) and $(\mathcal{F}_1)$, we get

$$f\left(\sum_{i=1}^{m-1} D(x_i, x_{i+1})\right) \leq f\left(\frac{a^n}{1-a} D(x_0, x_1)\right) < f(\varepsilon) - \alpha \ (\text{for } m > n \geq N). \tag{3.6}$$

Using (D_3) and (3.6), we obtain for $m > n \geq N$,

$$D(x_n, x_m) > 0 \Rightarrow f(D(x_n, x_m)) \leq f(\varepsilon)$$

which implies by $(\mathcal{F}_1)$ that

$$D(x_n, x_m) < \varepsilon.$$

Therefore $\{x_n\}_{n=0}^{\infty}$ is an F-Cauchy sequence. Since (X, D) is F-complete, there exist $x \in X$ such that $\{x_n\}_{n=0}^{\infty}$ is F-convergent to x. i.e.

$$\lim_{n\to\infty} D(x_n, x) = 0.$$

Now since $f(x) = D(x, Tx)$ is lower semi-continuous and $x_n \to x$, we have, from (3.2)

$$0 \leq D(x, Tx) = f(x) \leq \liminf_{n\to\infty} f(x_n) = \liminf_{n\to\infty} D(x_n, Tx_n) = 0.$$

Therefore $D(x, Tx) = 0$. Since Tx is F-closed, then we get $x \in Tx$. □

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