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MANNHEIM B-CURVE COUPLES IN MINKOWSKI 3-SPACE

KAZIM İLARSLAN, ALI UÇUM, EMILIJA NEŠOVIĆ AND NIHAL KILIÇ ASLAN

Abstract. In this paper, we define null Cartan Mannheim and pseudo null Mannheim curves in Minkowski 3-space according to their Bishop frames. We obtain the necessary and the sufficient conditions for pseudo null curves to be Mannheim B-curves in terms of their Bishop curvatures. We prove that there are no null Cartan curves in Minkowski 3-space which are Mannheim B-curves, by considering the cases when their Mannheim B-mate curves are spacelike, timelike, null Cartan and pseudo null curves. Finally, we give some examples of pseudo null Mannheim B-curve pairs.

1. Introduction

The Bishop frame or relatively parallel adapted frame $\{T, N_1, N_2\}$ of a regular curve in Euclidean 3-space is introduced by R.L. Bishop in [1]. It contains the tangential vector field T and two relatively normal vector fields N_1 and N_2 whose derivatives N'_1 and N'_2 with respect to the arc-length parameter s of the curve are collinear with the tangential vector field T . The Bishop frame is also known as the frame with *minimal rotation property*, since N'_1 and N'_2 make minimal rotations in the planes $N_1^\perp$ and $N_2^\perp$ respectively. A new version of the Bishop frame, type-2 Bishop frame in $\mathbb{E}^3$, is introduced in [18]. In Minkowski space-time $\mathbb{E}_1^4$ and Euclidean space $\mathbb{E}^4$, the Bishop frame is studied in [6] and [8]. In Minkowski 3-space $\mathbb{E}_1^3$, the Bishop frame (parallel frame) of the timelike curve and the spacelike curve with non-null principal normal is obtained in [16]. Recently, the Bishop frames of pseudo null curves and null Cartan curves in $\mathbb{E}_1^3$ are derived in [11] and the Bishop frame of a null Cartan curve in $\mathbb{E}_1^4$ is introduced in [12].

It is well known that in the Euclidean space $\mathbb{E}^3$, there are many associated curves (Bertrand mates, Mannheim mates, spherical images, evolutes, the principal-direction curves, etc.) whose Frenet's frame vectors satisfy some extra conditions. *Mannheim curves* in $\mathbb{E}^3$ are defined by the property that their principal normal lines coincide with the binormal lines of their mate

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curves at the corresponding points [5, 13]. Mannheim curves and their partner curves in 3-dimensional space forms are studied in [3]. In the Euclidean 4-space and Minkowski space-time $\mathbb{E}_1^4$, the notion of Mannheim curves is generalized in [7, 10, 14]. It is proved in [9] that the only pseudo null Mannheim curves according to Frenet frame in Minkowski 3-space are the pseudo null circles whose mate curves are pseudo null straight lines. It is also proved in [9] that there are no null Cartan curves in Minkowski 3-space which are Mannheim curves according to their Cartan frame.

In this paper, we define null Cartan Mannheim and pseudo null Mannheim curves according to their Bishop frames in Minkowski 3-space. We call them *null Cartan Mannheim B-curves* and *pseudo null Mannheim B-curves*. We obtain the necessary and the sufficient conditions for pseudo null curves to be Mannheim B-curves in terms of their Bishop curvatures. In the related examples, we show that there are infinity many pairs of pseudo null Mannheim B-curves. We prove that mate curves of pseudo null Mannheim B-curves can not be spacelike, timelike, or null Cartan curves. We also prove that there are no null Cartan curves which are Mannheim B-curves, by considering the cases when their mate curves are the spacelike, the timelike, the null Cartan and the pseudo null curves.

2. Preliminaries

Minkowski space $\mathbb{E}_1^3$ is the real vector space $\mathbb{E}^3$ equipped with the standard indefinite flat metric $\langle \cdot, \cdot \rangle$ given by

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3,$$

for any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{E}_1^3$. Since $\langle \cdot, \cdot \rangle$ is an indefinite metric, an arbitrary vector $x \in \mathbb{E}_1^3$ can have one of three causal characters: it can be a *spacelike*, a *timelike*, or a *null (lightlike)*, if $\langle x, x \rangle > 0$, $\langle x, x \rangle < 0$, or $\langle x, x \rangle = 0$ and $x \neq 0$ respectively. In particular, the vector $x = 0$ is said to be spacelike. The *norm* (length) of vector $x \in \mathbb{E}_1^3$ is given by $\|x\| = \sqrt{|\langle x, x \rangle|}$. If $\|x\| = 1$, the vector x is called a *unit*. An arbitrary curve $\alpha : I \rightarrow \mathbb{E}_1^3$ can be the *spacelike*, the *timelike* or the *null (lightlike)*, if all of its velocity vectors α' are the spacelike, the timelike or the null ([15]).

A spacelike curve $\alpha : I \rightarrow \mathbb{E}_1^3$ is called a *pseudo null curve*, if its principal normal vector field N and binormal vector field B are null vector fields satisfying the condition $\langle N, B \rangle = 1$. The Frenet formulae of a non-geodesic pseudo null curve α have the form ([17])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & \tau & 0 \\ -\kappa & 0 & -\tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad (2.1)$$

where the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$ is an arbitrary function in arc-length parameter s of α .

A curve $\beta : I \rightarrow \mathbb{E}_1^3$ is called a *null curve*, if its tangent vector $\beta' = T$ is a null vector. A null curve β is called a *null Cartan curve*, if it is parameterized by the pseudo-arc function s defined by ([2])

$$s(t) = \int_0^t \sqrt{\|\beta''(u)\|} du. \quad (2.2)$$

There exists a unique Cartan frame $\{T, N, B\}$ along a non-geodesic null Cartan curve β satisfying the Cartan equations ([4])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\tau & 0 & \kappa \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad (2.3)$$

where the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$ is an arbitrary function in pseudo-arc parameter s .

The Frenet equations of a timelike curve, or a spacelike curve with non-null principal normal in $\mathbb{E}_1^3$ according to Bishop frame (parallel transport frame) $\{T_1, N_1, N_2\}$ have the form ([16])

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & -\epsilon_1 k_1 & -\epsilon_2 k_2 \\ \epsilon_0 k_1 & 0 & 0 \\ \epsilon_0 k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \quad (2.4)$$

where T_1, N_1, N_2 are mutually orthogonal vectors satisfying the conditions $\langle T_1, T_1 \rangle = \epsilon_0$, $\langle N_1, N_1 \rangle = \epsilon_1$, $\langle N_2, N_2 \rangle = \epsilon_2$ and $\epsilon_0, \epsilon_1, \epsilon_2 \in \{-1, 1\}$. In particular, it holds $\epsilon_0 \epsilon_1 \epsilon_2 = -1$. The functions $k_1(s)$ and $k_2(s)$ are called the *first* and the *second Bishop curvature* of the curve, respectively.

The Bishop frames of pseudo null and null Cartan curves are obtained in [11]. If $\{T_1, N_1, N_2\}$ is the Bishop frame of a pseudo null curve, then it satisfies the conditions

$$\begin{aligned} \langle T_1, T_1 \rangle &= 1, & \langle N_1, N_1 \rangle &= \langle N_2, N_2 \rangle = 0, \\ \langle T_1, N_1 \rangle &= \langle T_1, N_2 \rangle = 0, & \langle N_1, N_2 \rangle &= 1. \end{aligned} \quad (2.5)$$

Analogously, if $\{T_1, N_1, N_2\}$ is the Bishop frame of null Cartan curve, then it satisfies the conditions ([11])

$$\begin{aligned} \langle T_1, T_1 \rangle &= \langle N_2, N_2 \rangle = 0, & \langle N_1, N_1 \rangle &= 1, \\ \langle T_1, N_2 \rangle &= -1, & \langle T_1, N_1 \rangle &= \langle N_1, N_2 \rangle = 0. \end{aligned} \quad (2.6)$$

Also, the next two theorems are proved in [11].

Theorem 1. *Let α be a pseudo null curve in $\mathbb{E}_1^3$ parameterized by arc-length parameter s with the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$. Then the Bishop frame $\{T_1, N_1, N_2\}$ and the Frenet frame $\{T, N, B\}$ of α are related by:*

(i)

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 \\ 0 & 0 & k_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \quad (2.7)$$

and the Frenet equations of α according to the Bishop frame read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_2 & k_1 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} \quad (2.8)$$

where $k_1(s) = 0$ and $k_2(s) = c_0 e^{\int \tau(s) ds}$, $c_0 \in \mathbb{R}_0^+$ are the first and the second Bishop curvature;

(ii)

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -k_1 \\ 0 & -\frac{1}{k_1} & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \quad (2.9)$$

and the Frenet equations of α according to the Bishop frame read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_2 & k_1 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} \quad (2.10)$$

where $k_1(s) = c_0 e^{\int \tau(s) ds}$, $c_0 \in \mathbb{R}_0^-$ and $k_2(s) = 0$ are the first and the second Bishop curvature.

Theorem 2. Let α be a null Cartan curve in $\mathbb{E}_1^3$ parametrized by pseudo-arc s with the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$. Then the Bishop frame $\{T_1, N_1, N_2\}$ and the Cartan frame $\{T, N, B\}$ of α are related by:

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -k_2 & 1 & 0 \\ \frac{k_2^2}{2} & -k_2 & 1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \quad (2.11)$$

and the Cartan equations of α according to the Bishop frame read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} k_2 & k_1 & 0 \\ 0 & 0 & k_1 \\ 0 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} \quad (2.12)$$

where the first Bishop curvature $k_1(s) = 1$ and the second Bishop curvature satisfies Riccati differential equation $k_2'(s) = -\frac{1}{2}k_2^2(s) - \tau(s)$.

3. Null Cartan and pseudo null Mannheim B-curves

In this section we define null Cartan Mannheim and pseudo null Mannheim curves according to their Bishop frames in Minkowski 3-space and call them *Mannheim B-curves*. We obtain the necessary and sufficient conditions for pseudo null curves to be Mannheim B-curves and provide the related examples. We prove that mate curves of pseudo null Mannheim B-curves can not be spacelike, timelike or null Cartan curves. We also prove that there are no null Cartan curves which are Mannheim B-curves, by considering the cases when their Mannheim B-mate curves are the spacelike, timelike, null Cartan and pseudo null curves.

Definition 1. Let β be a pseudo null curve or a null Cartan curve in $\mathbb{E}_1^3$ with the Bishop frame $\{T_1, N_1, N_2\}$ and β^* an arbitrary curve with the Bishop frame $\{T_1^*, N_1^*, N_2^*\}$. If the Bishop vector N_1 is collinear with the Bishop vector N_2^* at the corresponding points of the curves β and β^* , then β is called the *Mannheim B-curve*, β^* is called *Mannheim B-mate curve* of β and (β, β^*) curve couple is called *Mannheim B-pair*.

In the first theorem, we give the necessary and the sufficient conditions for pseudo null curve couple (β, β^*) to be Mannheim B-pair of curves with non-zero Bishop curvatures κ_2 and κ_1^* .

Theorem 3. Let β and β^* be two pseudo null curves in $\mathbb{E}_1^3$ parameterized by arc-length parameters s and s^* respectively with the Bishop curvatures $\kappa_1 = \kappa_2^* = 0$, κ_2 and κ_1^* . Then (β, β^*) curve couple is a Mannheim B-pair if and only if

$$\kappa_2 + \lambda'' = c\kappa_1^*,$$

where $\lambda \neq 0$ is an arbitrary differentiable function and $c \in \mathbb{R}_0$.

Proof. Assume that (β, β^*) is Mannheim B-pair of curves. Then we can write the curve β^* as

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda(s)N_1(s), \quad (3.1)$$

where $\lambda(s) \neq 0$ is some differentiable function. Differentiating (3.1) with respect to s , we find

$$T^* f' = T + \lambda' N_1 + \lambda N_1'.$$

By using (2.7), (2.8) and (2.9), the previous relation becomes

$$-T_1^* f' = T_1 + \lambda' N_1. \quad (3.2)$$

By taking the scalar product of (3.2) with $-T_1^* f'$ and using (2.5), we obtain

$$\langle T_1^* f', T_1^* f' \rangle = f'^2 = 1. \quad (3.3)$$

Let us take $f' = 1$. Then we have

$$T_1^\star = -T_1 - \lambda' N_1. \quad (3.4)$$

Differentiating (3.4) with respect to s and using (2.8) and (2.10), we get

$$\kappa_1^\star N_2^\star = -(\kappa_2 + \lambda'') N_1 \quad (3.5)$$

which implies

$$N_2^\star = -\frac{\kappa_2 + \lambda''}{\kappa_1^\star} N_1.$$

Differentiating the last equation with respect to s and using (2.8) and (2.10), we find

$$\kappa_2 + \lambda'' = c\kappa_1^\star,$$

where $c \in \mathbb{R}_0$.

Conversely, assume that $\kappa_2 + \lambda'' = c\kappa_1^\star$, where $\lambda \neq 0$ is an arbitrary differentiable function and $c \in \mathbb{R}_0$. Define the curve $\beta^\star$ by

$$\beta^\star(s) = \beta(s) + \lambda(s)N_1(s). \quad (3.6)$$

Differentiating (3.6) with respect to s and using (2.7) and (2.8), we find

$$\frac{d\beta^\star}{ds} = T_1 + \lambda' N_1, \quad (3.7)$$

which together with relation (2.5) leads to

$$\left\langle \frac{d\beta^\star}{ds}, \frac{d\beta^\star}{ds} \right\rangle = 1.$$

Therefore, the curve $\beta^\star$ is parameterized by arc-length parameter s . Then from (2.9) and (3.7) we have

$$T^\star = -T_1^\star = T_1 + \lambda' N_1. \quad (3.8)$$

Differentiating (3.8) with respect to s and using (2.8) and (2.10), we obtain

$$-\frac{dT_1^\star}{ds} = -\kappa_1^\star N_2^\star = (\kappa_2 + \lambda'') N_1. \quad (3.9)$$

By using the assumption, we get

$$N_2^\star = -\frac{\kappa_2 + \lambda''}{\kappa_1^\star} = -cN_1.$$

Hence $(\beta, \beta^\star)$ is Mannheim B-pair of curves. $\square$

Example 1. Let us consider a pseudo null curve β in $\mathbb{E}_1^3$ with parameter equation

$$\beta(s) = \left(\frac{s^3}{3} + \frac{s^2}{2}, \frac{s^3}{3} + \frac{s^2}{2}, s \right)$$

and the Frenet frame

$$\begin{aligned} T(s) &= (s^2 + s, s^2 + s, 1), \\ N(s) &= (2s + 1)(1, 1, 0), \\ B(s) &= \left(-\frac{(s^2 + s)^2 + 1}{2(2s + 1)}, \frac{1 - (s^2 + s)^2}{2(2s + 1)}, -\frac{s^2 + s}{2s + 1} \right). \end{aligned}$$

A straightforward calculation shows that Frenet curvatures of β read $\kappa(s) = 1$, $\tau(s) = \frac{2}{2s+1}$. According to statement (i) of Theorem 1, the Bishop curvatures of β are given by $\kappa_1(s) = 0$ and $\kappa_2(s) = c_0(2s + 1)$, $c_0 \in \mathbb{R}_0^+$. In particular, the Bishop frame of β has the form

$$\begin{aligned} T_1(s) &= (s^2 + s, s^2 + s, 1), \\ N_1(s) &= \frac{1}{c_0}(1, 1, 0), \\ N_2(s) &= c_0 \left(-\frac{(s^2 + s)^2 + 1}{2}, \frac{1 - (s^2 + s)^2}{2}, -s^2 - s \right). \end{aligned}$$

Let us take $\lambda(s) = -\frac{c_0 s^3}{3}$. Define the curve β^* by

$$\beta^*(s) = \beta(s) + \lambda(s)N_1(s) = \left(\frac{s^2}{2}, \frac{s^2}{2}, s \right).$$

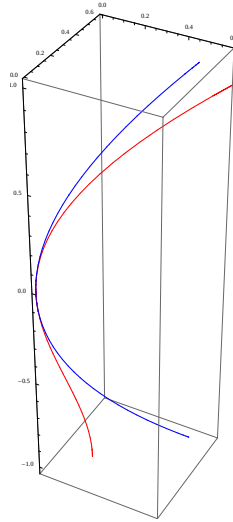
Therefore, β^* is pseudo null circle with Frenet curvatures $\kappa^*(s) = 1$, $\tau^*(s) = 0$. By using statement (ii) of Theorem 1, the Bishop curvatures of β^* are given by

$$\kappa_1^*(s) = c_1, \quad \kappa_2^*(s) = 0,$$

and the Bishop frame of β^* reads

$$\begin{aligned} T_1^*(s) &= -(s, s, 1), \\ N_1^*(s) &= -c_1 \left(\frac{-1 - s^2}{2}, \frac{1 - s^2}{2}, -s \right), \\ N_2^*(s) &= -\frac{1}{c_1}(1, 1, 0), \end{aligned}$$

where $c_1 \in \mathbb{R}_0^-$. Since N_1 and N_2^* are collinear, (β, β^*) is Mannheim B-pair of curves (Figure 1). It can be easily verified that the equation $\kappa_2 + \lambda'' = \frac{c_0}{c_1} \kappa_1^*$ holds.

Figure 1: Mannheim B-pair of curves (β, β^*) .

In the next theorem, we give the necessary and the sufficient conditions for pseudo null curve couple (β, β^*) to be Mannheim B-pair of curves with non-zero Bishop curvatures κ_1 and κ_2^* .

Theorem 4. *Let β and β^* be two pseudo null curves in $\mathbb{E}_1^3$ parameterized by arc-length parameters s and s^* respectively with the Bishop curvatures κ_1 , κ_2^* and $\kappa_2 = \kappa_1^* = 0$. Then (β, β^*) curve couple is a Mannheim B-pair if and only if*

$$\kappa_2^* = -\frac{\kappa_1 (\lambda')^2}{c(1 + \lambda\kappa_1)^3} \quad (3.10)$$

where c is a non-zero real number and $\lambda(s)$ is differentiable function satisfying differential equation

$$\lambda'(s) \int \kappa_1(s) ds + 2(1 + \lambda(s)\kappa_1(s)) = 0. \quad (3.11)$$

Proof. Assume that (β, β^*) is Mannheim B-pair of curves. Then we can write the curve β^* as

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda(s)N_1(s), \quad (3.12)$$

where $\lambda(s)$ is some differentiable function. Differentiating (3.12) with respect to s and using (2.7), (2.8) and (2.9), we get

$$T_1^* f' = (-1 - \lambda\kappa_1) T_1 + \lambda' N_1. \quad (3.13)$$

By taking the scalar product of (3.13) with $T_1^* f'$ and using (2.5), we obtain

$$f'^2 = (1 + \lambda\kappa_1)^2. \quad (3.14)$$

Let us take $f' = 1 + \lambda\kappa_1$. Then $1 + \lambda\kappa_1 \neq 0$. Putting $a = \lambda'/f'$ in (3.13), we find

$$T_1^* = -T_1 + aN_1. \quad (3.15)$$

Differentiating (3.15) with respect to s and using (2.8) and (2.10), we obtain

$$\kappa_2^* f' N_1^* = -a\kappa_1 T_1 + a' N_1 - \kappa_1 N_2. \quad (3.16)$$

Since N_1^* is a null vector, relations (2.5) and (3.16) imply

$$-2a'\kappa_1 + a^2\kappa_1^2 = 0.$$

Integrating the previous equation, we obtain

$$a(s) = -\frac{2}{\int \kappa_1(s) ds}.$$

Substituting $a = \lambda'/f'$ in the last equation, it follows that λ satisfies differential equation (3.11).

By taking the scalar product of (3.16) with $N_1 = \mu N_2^*$, where $\mu(s) \neq 0$ is some differentiable function and using (2.5), we find

$$\mu\kappa_2^* f' = -\kappa_1. \quad (3.17)$$

Substituting (3.17) in (3.16), we get

$$N_1^* = a\mu T_1 - \frac{\mu a'}{\kappa_1} N_1 + \mu N_2. \quad (3.18)$$

Differentiating (3.18) with respect to s and using (2.8) and (2.10), we find

$$(a\mu' + 2\mu a') T_1 - \left(\frac{\mu a'}{\kappa_1}\right)' N_1 + (\mu' + a\mu\kappa_1) N_2 = 0,$$

which implies that

$$a\mu' + 2\mu a' = 0, \quad \frac{\mu a'}{\kappa_1} = \text{constant}, \quad \mu' + a\mu\kappa_1 = 0. \quad (3.19)$$

From the first equation of (3.19) we find

$$\mu = \frac{c}{a^2}, \quad (3.20)$$

where $c \in \mathbb{R} \setminus \{0\}$. The other two equations of (3.19) hold automatically. Substituting relations $f' = 1 + \lambda\kappa_1$ and (3.20) in (3.17), we get that Bishop curvature κ_2^* satisfies (3.10).

Conversely, assume relation (3.10) holds and that λ satisfies differential equation (3.11). Define a curve β^* by

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda N_1(s). \quad (3.21)$$

Differentiating (3.21) with respect to s and using (2.7), (2.8) and (2.9), we obtain

$$T_1^* f' = (-1 - \lambda \kappa_1) T_1 + \lambda' N_1. \quad (3.22)$$

By taking the scalar product of (3.22) with $T_1^* f'$ and using (2.5), we find $f'^2 = (1 + \lambda \kappa_1)^2$. Thus we may take $f' = 1 + \lambda \kappa_1$. Then from (3.22) we have

$$T_1^* = -T_1 + a N_1, \quad (3.23)$$

where $a = \lambda' / f'$. Differentiating (3.23) with respect to s and using (2.8) and (2.10), we find

$$\kappa_2^* f' N_1^* = -a \kappa_1 T_1 + a' N_1 - \kappa_1 N_2. \quad (3.24)$$

Therefore,

$$N_1^* = -\frac{c}{a} T_1 + \frac{ca'}{a^2 \kappa_1} N_1 + \frac{c}{a^2} N_2. \quad (3.25)$$

By using the conditions $\langle N_2^*, N_2^* \rangle = \langle N_2^*, T_1^* \rangle = 0$, $\langle N_1^*, N_2^* \rangle = 1$ and relations (3.23) and (3.25), we find

$$N_2^* = \frac{a^2}{c} N_1 = \frac{1}{\mu} N_1.$$

Hence (β, β^*) is Mannheim B-pair of curves. □

Example 2. Consider a unit speed pseudo null curve in $\mathbb{E}_1^3$ with parameter equation

$$\beta(s) = (s^3, s^3, s).$$

The Frenet frame of β reads

$$\begin{aligned} T(s) &= (3s^2, 3s^2, 1), \\ N(s) &= 6s(1, 1, 0), \\ B(s) &= \left(-\frac{9s^4 + 1}{12s}, \frac{1 - 9s^4}{12s}, -\frac{s}{2}\right), \end{aligned}$$

and the Frenet curvatures of β are given by $\kappa(s) = 1$, $\tau(s) = \frac{1}{s}$. According to statement (ii) of Theorem 1, the Bishop curvatures of β read

$$\kappa_1(s) = c_0 s, \quad \kappa_2(s) = 0, \quad c_0 \in \mathbb{R}_0^-.$$

Hence the Bishop frame of β has the form

$$\begin{aligned} T_1(s) &= -(3s^2, 3s^2, 1), \\ N_1(s) &= c_0 \left(\frac{1 + 9s^4}{12}, \frac{9s^4 - 1}{12}, \frac{s^2}{2} \right), \end{aligned}$$

$$N_2(s) = -\frac{6}{c_0}(1, 1, 0).$$

Substituting $\kappa_1(s) = c_0 s$ in (3.11), we get

$$\lambda(s) = -\frac{4}{3c_0 s}.$$

Let us define pseudo null curve β^* by

$$\beta^*(s) = \beta(s) + \lambda(s)N_1(s).$$

Then β^* has parameter equation

$$\beta^*(s) = \left(-\frac{1}{9s}, \frac{1}{9s}, \frac{s}{3}\right).$$

In particular, the arc-length parameter of β^* is given by $s^* = \frac{s}{3}$. Therefore, the Frenet frame of β^* reads

$$\begin{aligned} T^*(s) &= \left(\frac{1}{27s^{*2}}, -\frac{1}{27s^{*2}}, 1\right), \\ N^*(s) &= \left(-\frac{2}{27s^{*3}}, \frac{2}{27s^{*3}}, 0\right), \\ B^*(s) &= \left(\frac{27}{4}s^{*3} + \frac{1}{108s^*}, \frac{27}{4}s^{*3} - \frac{1}{108s^*}, \frac{s^*}{2}\right), \end{aligned}$$

and Frenet curvatures of β^* have the form $\kappa^*(s) = 1$, $\tau^*(s) = -\frac{3}{s^*}$. Hence statement (i) of Theorem 1 implies that the Bishop curvatures of β^* read

$$\kappa_1^*(s) = 0, \quad \kappa_2^*(s) = \frac{c_1}{s^{*3}}, \quad c_1 \in \mathbb{R}_0^+.$$

Also, according to statement (i) of Theorem 1, the Bishop frame of β^* reads

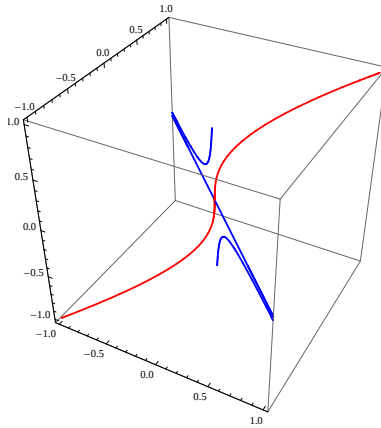
$$\begin{aligned} T_1^*(s) &= T^*, \\ N_1^*(s) &= \frac{s^{*3}}{c_1} \left(-\frac{2}{27s^{*3}}, \frac{2}{27s^{*3}}, 0\right), \\ N_2^*(s) &= \frac{c_1}{s^{*3}} \left(\frac{27}{4}s^{*3} + \frac{1}{108s^*}, \frac{27}{4}s^{*3} - \frac{1}{108s^*}, \frac{s^*}{2}\right). \end{aligned}$$

It can be easily verified that

$$N_1 = \frac{9c_0 s^{*4}}{c_1} N_2^*.$$

Consequently, (β, β^*) is Mannheim B-pair of curves (Figure 2).

Moreover, it can be easily checked that the equation (3.10) is satisfied.

Figure 2: Mannheim B-pair of curves (β, β^*) .

Remark 1. Note that to any pseudo null curve in $\mathbb{E}_1^3$ with Bishop curvatures $k_1 \neq 0$ and $k_2 = 0$, we may assign function λ as the solution of differential equation (3.11). The function λ determines Mannheim mate B-curve $\beta^* = \beta + \lambda N_1$ of β . This means that there are infinity many pseudo null Mannheim B-curve couples.

The next theorem can be proved analogously, so we omit its proof.

Theorem 5. *There are no Mannheim B-pair of curves (β, β^*) in $\mathbb{E}_1^3$, where β and β^* are pseudo null curves with Bishop curvatures $\kappa_1 = \kappa_1^*$ and $\kappa_2 = \kappa_2^*$.*

Theorem 6. *There are no Mannheim B-pair of curves (β, β^*) in $\mathbb{E}_1^3$, where β is a pseudo null curve and β^* is a null Cartan curve.*

Proof. Let β be a pseudo null Mannheim B-curve parametrized by arc-length s with the Bishop curvatures κ_1 and κ_2 and Bishop frame $\{T_1, N_1, N_2\}$. Assume that there exists Mannheim B-pair of curves (β, β^*) , where β^* is a null Cartan curve with Bishop frame $\{T_1^*, N_1^*, N_2^*\}$. Then we can write the curve β^* as

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda(s)N_1(s), \quad (3.26)$$

where $\lambda(s)$ is some differentiable function. Now we may distinguish two possibilities: (i) $\kappa_1 = 0, \kappa_2 \neq 0$ and (ii) $\kappa_1 \neq 0, \kappa_2 = 0$.

(i) If $\kappa_1 = 0$ and $\kappa_2 \neq 0$, differentiating (3.26) with respect to s and using (2.7), (2.8) and (2.11), we get

$$T_1^* f' = T_1 + \lambda' N_1. \quad (3.27)$$

By taking the scalar product of (3.27) with $T_1^* f'$ and using (2.5) and (2.6), we obtain a contradiction.

(ii) If $\kappa_1 \neq 0$ and $\kappa_2 = 0$, differentiating (3.26) with respect to s and using (2.9), (2.10) and (2.11), we obtain

$$T_1^* f' = (-1 - \lambda \kappa_1) T_1 + \lambda' N_1. \quad (3.28)$$

By taking the scalar product of (3.28) with $N_1 = \mu N_2^*$ where $\mu(s) \neq 0$ is some differentiable function and using (2.5) and (2.6), we get $f' \mu = 0$ which is a contradiction. $\square$

The proof of the following theorem follows from the fact that a null Bishop vector N_1 of β can be collinear with non-null Bishop vector N_2^* of β^* .

Theorem 7. *There are no Mannheim B-pair of curves (β, β^*) in $\mathbb{E}_1^3$, where β is a pseudo null curve and β^* is a timelike or spacelike curve with non-null Bishop vector N_2^* .*

Theorem 8. *There are no Mannheim B-pair of curves (β, β^*) in $\mathbb{E}_1^3$, where β is a null Cartan curve and β^* is a timelike curve, or a spacelike curve with a non-null Bishop vector N_2^* .*

Proof. Assume that there exists Mannheim B-pair of curves (β, β^*) . Denote by s and s^* pseudo-arc and arc-length of β and β^* respectively and by $\{T_1, N_1, N_2\}$ and $\{T_1^*, N_1^*, N_2^*\}$ their Bishop frames. It is sufficient to assume that N_2^* is a spacelike vector. Otherwise, we easily get a contradiction. We can write the curve β as

$$\beta(s) = \beta(f(s^*)) = \beta^*(s^*) + \lambda(s^*) N_2^*(s^*), \quad (3.29)$$

where $\lambda(s^*)$ is some differentiable function. Differentiating the relation (3.29) with respect to s^* and using relations (2.4) and (2.11), we obtain

$$T_1 f' = T_1^* + \lambda' N_2^* + \epsilon_0 \lambda \kappa_2^* T_1^*.$$

By taking the scalar product of the previous equation with $N_1 = \mu N_2^*$ where $\mu \neq 0$ is some differentiable function and using (2.6) and the conditions $\langle T_1^*, N_2^* \rangle = 0$, $\langle N_2^*, N_2^* \rangle = 1$, we get $\lambda' = 0$. Hence

$$T_1 f' = (1 + \epsilon_0 \lambda \kappa_2^*) T_1^*.$$

This means that a null vector T_1 is collinear with a non-null vector T_1^* , which is a contradiction. $\square$

The proof of the last theorem follows from the fact that a spacelike Bishop vector N_1 of β can not be collinear with a null Bishop vector N_2^* of β^* .

Theorem 9. *There are no Mannheim B-pair of curves (β, β^*) in $\mathbb{E}_1^3$, where β is null Cartan curve and β^* is Cartan null or pseudo null curve.*

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