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A-STATISTICAL CORE OF A SEQUENCE

Abstract. In this paper we extend the concepts of statistical limit superior and inferior (as introduced by Fridy and Orhan) to A-statistical limit superior and inferior and give some A-statistical analogue of properties of statistical limit superior and inferior for a sequence of real numbers. Also we extend the concept of statistical core to A-statistical core and get necessary and sufficient conditions on a matrix T so that the Knopp core of Tx is contained in the A-statistical core of a bounded complex number sequence x.

1. Introduction

If K is a set of positive integers, K_n will denote the set $\{k \in K : k \leq n\}$ and $|K_n|$ will denote the cardinality of K_n. The natural density of K [16] is given by $\delta(K) := \lim_n \frac{1}{n}|K_n|$, if it exists. The number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\epsilon > 0$ the set $K := K(\epsilon) := \{k \in N : |x_k - L| \geq \epsilon\}$ has natural density zero [6], [8]. In this case, we write $st - \lim x = L$. Hence x is statistically convergent to L iff $(C_1\chi_{K(\epsilon)})_n \to 0$, (as $n \to \infty$, for every $\epsilon > 0$), where C_1 is the Cesáro mean of order one and χ_K is the characteristic function of the set K. Properties of statistical convergent sequences have been studied in [2], [3], [8], [17].

Statistical convergence can be generalized by using a regular nonnegative summability matrix A in place of C_1.

Following Freedman and Sember [7], we say that a set $K \subseteq N$ has. A-density if $\delta_A(K) := \lim_n(A\chi_K)_n = \lim_n \sum_{k\in K} a_{nk}$ exists where A is a nonnegative regular matrix.

The number sequence $x = (x_k)$ is A-statistically convergent to L provided that for every $\epsilon > 0$ the set $K(\epsilon)$ has A-density zero [3]. In this case we write $st_A - \lim x = L$. This idea was studied by Freedman and Sember [7], Connor [4], [5] and Kolk [14].

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In [5] a A-statistical cluster point of a sequence x is defined as a number γ such that for every $\epsilon > 0$ the set $\{k \in N : |x_k - \gamma| < \epsilon\}$ does not have A-density zero.

In this paper we give definitions of the conceps of A-statistical limit superior and inferior and develop some A-statistical analogues of properties of statistical limit superior and inferior for a sequence of real numbers. Also we introduce the concept of A-statistical core a sequence and get necessary and sufficient conditions on a matrix T so that the Knopp core of Tx is contained in the A-statistical core of a bounded complex number sequence x.

2. A-statistical limit superior and limit inferior

In this section we study the concepts of A-statistical limit superior and inferior. The result are analogues to those given by Fridy and Orhan [10].

Throughout the paper $A = (a_{nk})$ will denote a summability matrix which transforms a number sequence $x = (x_k)$ into the sequence Ax whose n-th term is given by $(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k$. As usual, N will denote the positive integers, R the real numbers and C the complex numbers.In this section x will denote real number sequence.

Throughout the paper $A = (a_{nk})$ will be a nonnegative regular matrix summability method.

For a number sequence $x = (x_k)$ let B_x denote the set

$$B_x := \{b \in R : \delta_A\{k : x_k > b\} \neq 0\}.$$

Similarly,

$$A_x := \{a \in R : \delta_A\{k : x_k < a\} \neq 0\}.$$

Note that throughout the paper the statement $\delta_A(K) \neq 0$ means that either $\delta_A(K) > 0$ or K fails to have A-density.

Definition 1. Let $A = (a_{nk})$ be a nonnegative regular matrix summability method and x be a number sequence. Then the A-statistical limit superior of x is given by

$$st_A - \limsup x := \begin{cases} \sup B_x, & \text{if } B_x \neq \emptyset, \\ -\infty, & \text{if } B_x = \emptyset. \end{cases}$$

Also, the A-statistical limit inferior of x is given by

$$st_A - \liminf x := \begin{cases} \inf A_x, & \text{if } A_x \neq \emptyset, \\ +\infty, & \text{if } A_x = \emptyset. \end{cases}$$

Definition 2. The number sequence x is said to be A-statistically bounded if there is a number B such that $\delta_A\{k : |x_k| > B\} = 0$.

Note that if we take $A = C_1$ (the Cesáro matrix of order 1) in Definitions 1 and 2, then we get Definitions 1 and 2 of [10]. When $A = C_1$

we shall simply write δ instead of δ_{C_1} and $st - \limsup / \liminf$ instead of $st_{C_1} - \limsup / \liminf$.

We shall provide an example to illustrate the concepts just defined. Consider the matrix $A = (a_{nk})$ defined by

$$a_{nk} = \begin{cases} \frac{n}{2(n+1)}, & \text{if } k = n^2 + 1, \\ 1 - \frac{n}{2(n+1)}, & \text{if } k = n^2 + 2, \\ 0, & \text{otherwise.} \end{cases}$$

Let us define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} \frac{1}{k}, & \text{if } k = n^2 + 1, \\ 2, & \text{if } k = n^2 + 2, \\ k, & \text{if } k = n^2, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $A = (a_{nk})$ is a nonnegative regular matrix summability method. Although x is unbounded above, it is A-statistically bounded for $\delta_A\{k : |x_k| > 2\} = 0$.

If we write $D := \{k = n^2 : n = 1, 2, \ldots\}$, $E := \{k = n^2 + 1 : n = 1, 2, \ldots\}$ and $F := \{k = n^2 + 2 : n = 1, 2, \ldots\}$, then we have $\delta(D) = 0$, $\delta(E) = 0$, $\delta(F) = 0$, $\delta(E \cup F) = 0$, $\delta(D \cup E \cup F) = 0$. Also we have $\delta_A(D) = 0$, $\delta_A(E) = \frac{1}{2}$, $\delta_A(F) = \frac{1}{2}$, $\delta_A(E \cup F) = 1$, $\delta_A(D \cup E \cup F) = 1$. Thus $B_x = (-\infty, 2)$, $A_x = (0, +\infty)$, $st_A - \limsup x = 2$ and $st_A - \liminf x = 0$. Also, x is not A-statistically convergent since it has two (disjoint) subsequences of positive A-density that converge to 0 and 2 respectively, (see Theorem 3.2 of [14]). Observe that, x is statistically bounded since $\delta\{k \in N : x_k \neq 0\} = 0$. Moreover it is statistically convergent to 0 since $st - \limsup x = st - \liminf x = 0$ [10]. We note that the set of A-statistical cluster points of x is $\Gamma_x^A := \{0, 2\}$ and $st_A - \limsup x$ equals the greatest element while $st_A - \liminf x$ is the least element of this set. This observation suggests following result which can be proved by straightforward least upper bound argument.

THEOREM 1. *If $\beta = st_A - \limsup x$ is finite, then for every positive number ϵ*

$$\delta_A\{k : x_k > \beta - \epsilon\} \neq 0 \quad \textit{and} \quad \delta_A\{k : x_k > \beta + \epsilon\} = 0. \tag{1}$$

Conversely, if (1) *holds for every positive ϵ then $\beta = st_A - \limsup x$.*

The dual statement for $st_A - \liminf x$ is as follows.

THEOREM 2. *If $\alpha = st_A - \liminf x$ is finite, then for every positive number ϵ*

$$\delta_A\{k : x_k < \alpha + \epsilon\} \neq 0 \quad \textit{and} \quad \delta_A\{k : x_k < \alpha - \epsilon\} = 0. \tag{2}$$

Conversely, if (2) *holds for every positive ϵ then $\alpha = st_A - \liminf x$.*

Now we have the following

THEOREM 3. *For any real number sequence x,*

$$st_A - \liminf x \leq st_A - \limsup x.$$

Proof. First consider the case in which $st_A - \limsup x = -\infty$. Hence we have $B_x = \emptyset$, so for every b in R, $\delta_A\{k : x_k > b\} = 0$ which implies that $\delta_A\{k : x_k \leq b\} = 1$, so for every a in R, $\delta_A\{k : x_k < a\} \neq 0$. Hence, $st_A - \liminf x = -\infty$.

The case in which $st_A - \limsup x = +\infty$ needs no proof, so we next assume that $\beta = st_A - \limsup x$ is finite, and let $\alpha := st_A - \liminf x$. Given $\epsilon > 0$ we show that $\beta + \epsilon \in A_x$, so that $\alpha \leq \beta + \epsilon$. By Theorem 1, $\delta_A\{k : x_k > \beta + \frac{\epsilon}{2}\} = 0$ because $\beta = lubB_x$. This implies $\delta_A\{k : x_k \leq \beta + \frac{\epsilon}{2}\} = 1$ which, in turn, yields that $\delta_A\{k : x_k < \beta + \epsilon\} = 1$. Hence $\beta + \epsilon \in A_x$. By definition $\alpha = \inf A_x$, so we conclude that $\alpha \leq \beta + \epsilon$; and since ϵ is arbitrary this proves that $\alpha \leq \beta$.

From Theorem 3 and the above definition, it is clear that

$$\liminf x \leq st_A - \liminf x \leq st_A - \limsup x \leq \limsup x$$

for any number sequence x.

A-statistical limit point of a sequence x is defined in [5] as the limit of a subsequence of x whose indices do not have A-density zero. One cannot say that $st_A - \limsup x$ is equal to the greatest A-statistical limit points of x. (See, for $A = C_1$, Example 4 in [9]).

Note that A–statistical boundedness implies that $st_A - \limsup$ and $st_A - \liminf$ are finite, so Properties (1) and (2) of Theorem 1 and 2 hold.

THEOREM 4. *The A-statistically bounded sequence x is A-statistically convergent if and only if*

$$st_A - \liminf x = st_A - \limsup x.$$

Proof. Let $\alpha := st_A - \liminf x$ and $\beta := st_A - \limsup x$. First suppose that $st_A - \lim x = L$ and $\epsilon > 0$. Then $\delta_A\{k : |x_k - L| \geq \epsilon\} = 0$, so $\delta_A\{k : x_k > L + \epsilon\} = 0$, which implies that $\beta \leq L$. We also have $\delta_A\{k : x_k < L - \epsilon\} = 0$ which yields that $L \leq \alpha$. Therefore $\beta \leq \alpha$. Combining with Theorem 3 we conclude that $\alpha = \beta$.

Now assume $\alpha = \beta$ and define $L := \alpha$. If $\epsilon > 0$ then (1) and (2) of Theorem 1 and 2 imply $\delta_A\{k : x_k > L + \frac{\epsilon}{2}\} = 0$ and $\delta_A\{k : x_k < L - \frac{\epsilon}{2}\} = 0$. Hence $st_A - \lim x = L$.

In [1] Buck proved the following: a sequence which is C_1-summable to its limit superior is statistically convergent. Also, in [10] Fridy and Orhan have presented a statistical analogue of this result. Now we have the following

THEOREM 5. *If the number sequence x is bounded and A-summable to the number $L := st_A - \limsup x$, then x is A-statistically convergent to L.*

Proof. Suppose that x is not A-statistically convergent to L. Then by Theorem 4, $st_A - \liminf x < L$, so there is a number $M < L$ such that $\delta_A\{k : x_k < M\} \neq 0$. Let $K' := \{k : x_k < M\}$. By the definition of L, $\delta_A\{k : x_k > L + \epsilon\} = 0$ for every $\epsilon > 0$. Define

$$K'' := \{k : M \leq x_k \leq L + \epsilon\} \quad \text{and} \quad K''' := \{k : x_k > L + \epsilon\},$$

and let $B := \sup_k x_k < \infty$. Since $\delta_A(K') \neq 0$, there are infinitely many n such that $\limsup_n \sum_{k \in K'} a_{nk} \geq d > 0$. Furthermore, for each n, $\sum_{k=1}^{\infty} |a_{nk} x_k| < \infty$. Hence we have

$$\begin{aligned}
(Ax)_n &= \sum_{k=1}^{\infty} a_{nk} x_k = \sum_{k \in K'} a_{nk} x_k + \sum_{k \in K''} a_{nk} x_k + \sum_{k \in K'''} a_{nk} x_k \\
&\leq M \sum_{k \in K'} a_{nk} + (L + \epsilon) \sum_{k \in K''} a_{nk} + B \sum_{k \in K'''} a_{nk} \\
&= M \sum_{k \in K'} a_{nk} + (L + \epsilon) \sum_{k=1}^{\infty} a_{nk} - (L + \epsilon) \sum_{k \in K'} a_{nk} + 0(1) \\
&= - \sum_{k \in K'} a_{nk} [-M + (L + \epsilon)] + (L + \epsilon) \sum_{k=1}^{\infty} a_{nk} + 0(1) \\
&\leq L \sum_{k=1}^{\infty} a_{nk} - d(L - M) + \epsilon \Big(\sum_{k=1}^{\infty} a_{nk} - d \Big) + 0(1).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary it fallows that

$$\liminf Ax \leq L - d(L - M) < L.$$

Hence, x is not A-summable to L whence the result.

By symmetry we have the dual result.

COROLLARY 1. *If the number sequence x is bounded and A-summable to the number $L := st_A - \liminf x$, then x is A-statistically convergent to L.*

If we take $A = C_1$ in the Theorem 5, then Example 2 of [10] shows that the boundedness cannot be omitted or even replaced by the weaker assumption of a A-statistical boundedness.

3. A-statistical core

In [10] Fridy and Orhan introduced the concept of the statistical core a real number sequence, and proved the statistical core theorem. Those results have also been extended [11] to the complex case by them too. Using the same technique as in [11], we introduce the concept of A-statistical core of a sequence and get necessary and sufficient conditions on a matrix T so that the Knopp core of Tx is contained in the A-statistical core of x for every bounded complex number sequence.In this section x, y and z will denote complex number sequences and $A = (a_{nk})$ will denote nonnegative regular matrix summability method.

In [12] and [13] the Knopp core the sequence x is defined by

$$\mathcal{K} - core\{x\} := \bigcap_{n \in N} C_n(x),$$

where $C_n(x)$ is the closed convex hull of $\{x_k\}_{k \geq n}$. In [18] it is shown that for every bounded x

$$\mathcal{K} - core\{x\} = \bigcap_{z \in C} B_x^*(z),$$

where $B_x^*(z) := \{w \in C : |w - z| \leq \limsup_k |x_k - z|\}$.

The next definition is an A-statistical analogue of statistical core [11] of a sequence.

Note that, if x and y are sequences such that

$$\delta_A\{k \in N : x_k = y_k\} = 1,$$

then we write "$x_k = y_k \ A - a.a.k.$"

DEFINITION 3. For any complex sequence x let $H_A(x)$ be the collection of all closed half-planes that contain $x_k \ A - a.a.k$; i.e.,

$$H_A(x) := \{H : H \text{ is closed half-plane and } \delta_A\{k \in N : x_k \notin H\} = 0\},$$

then the A-statistical core of x is given by

$$st_A - core\{x\} := \bigcap_{H \in H_A(x)} H.$$

It should be note that in the definition of the $\mathcal{K} - core\{x\}$, the closed convex hull $C_n(x)$ is the intersection of all closed half-planes that contain $\{x_k\}_{k \geq n}$; in defining the $st_A - core\{x\}$ we have simply replaced $\{x_k\}_{k \geq n}$ by an arbitrary subsequence of A-density one. Therefore it follows that for all x, $st_A - core\{x\} \subseteq \mathcal{K} - core\{x\}$. It is clear that for any A-statistically bounded real sequence x

$$st_A - core\{x\} = [st_A - \liminf x, st_A - \limsup x].$$

The next theorem is A-statistical analogue of the Lemma of [11].

THEOREM 6. *Let x be A-statistically bounded sequence; for each $z \in C$ let*

$$B_x(z) := \{w \in C : |w - z| \leq st_A - \limsup_k |x_k - z|\};$$

then

$$st_A - core\{x\} = \bigcap_{z \in C} B_x(z).$$

P r o o f. From the definition of $st_A - \limsup x$ and Theorem 1, we observe that the disk $B_x(z)$ is equal to the intersection of all closed disks centered at ze that contain x_k $A-a.a.k$. First assume $w \notin \bigcap_{z\in C} B_x(z)$, say $w \notin B_x(z^*)$ for some z^*. Let H be the half-plane containing $B_x(z^*)$ whose boundary line is perpendicular to the line containing w and z^* and tangent to the circular boundary of $B_x(z^*)$. Since $B_x(z^*) \subset H$ and $B_x(z^*)$ contains x_k $A-a.a.k$, it follows that $H \in H_A(x)$. Since $w \notin H$ this implies $w \notin \bigcap_{H\in H_A(x)} H$. Hence, $st_A - core\{x\} \subseteq \bigcap_{z\in C} B_x(z)$.

Conversely, if $w \notin \bigcap_{H\in H_A(x)} H$, let H be a plane in $H_A(x)$ such that $w \notin H$. Let L be the line through w that is perpendicular to the boundary of H, and let p be the midpoint of the segment to L between w and H. Let z be a point of L such that $z \in H$, and consider the disk

$$B(z) = \{\xi \in C : |\xi - z| \leq |p - z|\}.$$

Since x is A-statistically bounded and $x_k \in H$ $A - a.a.k$, we can choose z sufficiently far from p so that $|p - z| = st_A - \limsup_k |x_k - z|$. Thus $B(z)$ is one of the $B_x(z)$ disks, and since $w \notin B(z)$ we get that $w \notin \bigcap_{z\in C} B_x(z)$. This establishes the proof.

We note that Theorem 6 is not necessarily valid if x is not A-statistically bounded. For example, consider the matrix $A = (a_{nk})$ defined by

$$a_{nk} = \begin{cases} 1, & \text{if } k = n^2, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $A = (a_{nk})$ is a nonnegative regular matrix summability method. If $x_k = k$ for all k, then x has no A-statistical cluster point and $st_A - core\{x\} = \phi$. But for any $z \in C$ no disk of finite radius can contain x_k A-a.a.k, so $st_A - \limsup_k |x_k - z| = \infty$ and $B_x(z)$ is the whole plane C, whence

$$\phi = st_A - core\{x\} \neq \bigcap_{z\in C} B_x(z) = C.$$

Now we give necessary and sufficient conditions on a matrix T so that the Knopp core of Tx is contained in the A-statistical core of x for every bounded complex number sequence x. Throughout the remainder of this paper the set of bounded complex number sequence will be denoted by ℓ_∞.

THEOREM 7. *If the matrix T satisfies*

$$\sup_n \sum_{k=1}^{\infty} |t_{nk}| < \infty,$$

then $\mathcal{K} - core\{Tx\} \subseteq st_A - core\{x\}$ for every $x \in \ell_\infty$ if and only if the following conditions hold:

(i) $T \in \tau_A^*$, *i.e., T is regular and* $\lim_n \sum_{k\in E} |t_{nk}| = 0$ *whenever* $\delta_A(E) = 0$;
(ii) $\lim_n \sum_{k=1}^{\infty} |t_{nk}| = 1$.

Proof. (Necessity). If x A-statistically convergent to L, then

$$st_A - core\{x\} \supseteq \mathcal{K} - core\{Tx\}.$$

Since Tx is bounded for all $x \in \ell_\infty$, we must have that

$$\mathcal{K} - core\{Tx\} = \{L\},$$

so Tx is convergent to L. Now the Corollary 5.1 of Kolk [14] imply that T is regular and $\lim_n \sum_{k\in E} |t_{nk}| = 0$ whenever $\delta_A(E) = 0$, i.e., condition (i) holds. Also we have

$$\mathcal{K} - core\{Tx\} \subseteq st_A - core\{x\} \subseteq \mathcal{K} - core\{x\}.$$

Now Theorem 2.1 of [15] for the case $\alpha = 1$ and $K = C$ implies that $\limsup_n \sum_{k=1}^{\infty} |t_{nk}| \le 1$, and this is equivalent to $\lim_n \sum_{k=1}^{\infty} |t_{nk}| = 1$ because T is regular. This proves the necessity of (i) and (ii).

(Sufficiency). Assume (i) and (ii) and let $w \in \mathcal{K} - core\{Tx\}$. For any $z \in C$ we have

$$
\begin{aligned}
(3) \qquad |w - z| &\le \limsup_n |z - (Tx)_n| \\
&= \limsup_n \Big| z - \sum_{k=1}^{\infty} t_{nk} x_k \Big| \\
&= \limsup_n \Big| \sum_{k=1}^{\infty} t_{nk}(z - x_k) + z\Big(1 - \sum_{k=1}^{\infty} t_{nk}\Big) \Big| \\
&\le \limsup_n \Big| \sum_{k=1}^{\infty} t_{nk}(z - x_k) \Big| + \limsup_n |z| \Big| 1 - \sum_{k=1}^{\infty} t_{nk} \Big| \\
&= \limsup_n \Big| \sum_{k=1}^{\infty} t_{nk}(z - x_k) \Big|.
\end{aligned}
$$

Let $r = st_A - \limsup_k |x_k - z|$, suppose $\epsilon > 0$, and let $E := \{k : |z - x_k| > r + \epsilon\}$. Then $\delta_A(E) = 0$, and we write

$$
\begin{aligned}
\Big| \sum_{k=1}^{\infty} t_{nk}(z - x_k) \Big| &= \Big| \sum_{k\in E} t_{nk}(z - x_k) + \sum_{k\notin E} t_{nk}(z - x_k) \Big| \\
&\le \sum_{k\in E} |t_{nk}||z - x_k| + \sum_{k\notin E} |t_{nk}||z - x_k| \\
&\le \sup_k |z - x_k| \sum_{k\in E} |t_{nk}| + (r + \epsilon) \sum_{k\notin E} |t_{nk}|.
\end{aligned}
$$

Now (i) and (ii) imply that

$$\limsup_n \left| \sum_{k=1}^{\infty} t_{nk}(z - x_k) \right| \leq r + \epsilon.$$

If follow from (3) that $|w - z| \leq r + \epsilon$; and since ϵ is arbitrary this yields $|w - z| \leq r$. Hence, $w \in B_x(z)$, so by Theorem 6 we get $w \in st_A - core\{x\}$. Hence the proof is completed.

Since $st_A - core\{x\} \subseteq \mathcal{K} - core\{x\}$ we have the following corollary.

COROLLARY 2. *If the matrix T satisfies* $\sup_n \sum_{k=1}^{\infty} |t_{nk}| < \infty$ *and properties* (i) *and* (ii) *of Theorem* 7*, then*

$$st_A - core\{Tx\} \subseteq st_A - core\{x\}$$

for every $x \in \ell_\infty$.

We will now show that the converse of Corollary 2 is not true but we first need the following lemma.

LEMMA 1. *If x and y are sequences such that $x_k = y_k$ for A-a.a.k, then* $\Gamma_x^A = \Gamma_y^A$.

Proof. Let $\gamma \in \Gamma_x^A$. Define $K := \{k \in N : x_k = y_k\}$. For every $\epsilon > 0$ the set $\{k \in K : |y_k - \gamma| < \epsilon\}$ does not have A-density zero. Since

$$\{k \in K : |y_k - \gamma| < \epsilon\} \subseteq \{k \in N : |y_k - \gamma| < \epsilon\},$$

it follows that for every $\epsilon > 0$ the set $\{k \in N : |y_k - \gamma| < \epsilon\}$ does not have A-density zero. Hence $\gamma \in \Gamma_y^A$. By the symmetry we see that $\Gamma_y^A \subseteq \Gamma_x^A$ whence $\Gamma_x^A = \Gamma_y^A$.

EXAMPLE. Consider the matrix $A = (a_{nk})$ defined by

$$a_{nk} = \begin{cases} 1, & \text{if } k = n^2 + 1, \\ 0, & \text{otherwise,} \end{cases}$$

and define $T = (t_{nk})$ so that $(Tx)_n = x_n$ A-a.a.k. By Lemma 1 the A-statistical cluster points of Tx are the same as those of x, so $st_A - core\{Tx\} = st_A - care\{x\}$. Now define

$$t_{nk} = \begin{cases} 1, & \text{if } k = n \text{ and } n \text{ is a nonsquare,} \\ 1, & \text{if } k \leq n \text{ and } n \text{ is a square,} \\ 0, & \text{otherwise;} \end{cases}$$

then

$$(Tx)_n = \begin{cases} x_n, & \text{if } n \text{ is a nonsquare,} \\ \sum_{k=1}^{n} x_k, & \text{if } n \text{ is a square.} \end{cases}$$

For the matrix $T = (t_{nk})$,

$$\sum_{k=1}^{\infty} |t_{nk}| = \begin{cases} 1, & \text{if } n \text{ is a nonsquare,} \\ n, & \text{if } n \text{ is a square.} \end{cases}$$

Observe that $\sup_n \sum_{k=1}^{\infty} |t_{nk}| = \infty$. Also, it is clear that $\lim_n \sum_{k=1}^{\infty} |t_{nk}|$ in (ii) does not exist. If the set $E := \{k = n^2 : n = 1, 2, \ldots\}$, then $\delta_A(E) = 0$. Since

$$\sum_{k \in E} |t_{nk}| = \begin{cases} 0, & \text{if } n \text{ is a nonsquare,} \\ \sqrt{n}, & \text{if } n \text{ is a square,} \end{cases}$$

$\lim_n \sum_{k=1}^{\infty} |t_{nk}|$ in (i) does not exist.

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