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New characterization of weighted inequalities involving superposition of Hardy integral operators

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Abstract

Let $1 \le p < \infty$ and $0 < q, r < \infty$. We characterize the validity of the inequality for the composition of the Hardy operator,

$$\left(\int_a^b \left[\int_a^x \left(\int_a^t f(s)ds\right)^q u(t)dt\right]^{\frac{r}{q}} w(x)dx\right)^{\frac{1}{r}} \le C\left(\int_a^b f(x)^p v(x)dx\right)^{\frac{1}{p}}$$

for all non-negative measurable functions f on (a,b), $-\infty \le a < b \le \infty$. We construct a more straightforward discretization method than those previously presented in the literature, and we provide some new scales of weight characterizations of this inequality in both discrete and continuous forms and we obtain previous characterizations as the special case of the parameter.

KEYWORDS

Copson operator, Hardy operator, inequalities for monotone functions, iterated operators, weighted Hardy inequality

1 | INTRODUCTION AND THE MAIN RESULTS

Let $-\infty \le a < b \le \infty$. Denote by $\mathfrak{M}^+(a,b)$ the set of all non-negative measurable functions on (a,b) and $\mathfrak{M}^{\uparrow}(a,b)$ is the class of non-decreasing elements of $\mathfrak{M}^+(a,b)$.

In the operator theory, weighted inequalities involving operator composition may be found in various topics. Let $0 < q, r < \infty$ and $1 \le p < \infty$. The validity of inequalities

$$\left(\int_0^\infty \left(\int_0^x \left(\int_t^\infty h(s)ds\right)^q u(t)dt\right)^{\frac{r}{q}} w(x)dx\right)^{\frac{1}{r}} \le C\left(\int_0^\infty h(x)^p v(x)dx\right)^{\frac{1}{p}}, \tag{1.1}$$

and

$$\left(\int_0^\infty \left(\int_0^x \left(\int_0^t h(s)ds\right)^q u(t)dt\right)^{\frac{r}{q}} w(x)dx\right)^{\frac{1}{r}} \leq C\left(\int_0^\infty h(x)^p v(x)dx\right)^{\frac{1}{p}}, \tag{1.2}$$

for all $h \in \mathfrak{M}^+(0,\infty)$ are crucial because many classical inequalities can be reduced to them. For example, duality techniques reduce the embeddings between Lorentz, Morrey, and Cesáro-type spaces to the weighted iterated inequalities (e.g., [3, 4, 6, 8, 33]). On the other hand, characterizations of weighted bilinear Hardy and Copson inequalities reduce to the characterizations of iterated Hardy inequalities (see, e.g., [1, 22, 32]).

Various approaches have been used to handle inequalities (1.1) and (1.2), resulting in different conditions. Inequality (1.1) is investigated thoroughly. The most recent results are presented in the paper [19]. Alternative characterizations of inequalities (1.1) and (1.2) are found in [18]. Detailed information on the development and history of this inequality may be found in the recent paper [5].

Note that, when $q = 1$ using Fubini's Theorem, inequality (1.2) reduces to the weighted Hardy-type inequality involving kernel, that is,

$$\left(\int_0^\infty \left(\int_0^x \left(\int_s^x u(t)dt\right) h(s)ds\right)^r w(x)dx\right)^{\frac{1}{r}} \leq C\left(\int_0^\infty h(x)^p v(x)dx\right)^{\frac{1}{p}}, \quad h \in \mathfrak{M}^+(0,\infty). \tag{1.3}$$

Inequality (1.3) was completely characterized in [2, 25, 27, 31] when $1 \leq r, p < \infty$. However, for a long period, there was no adequate characterization in the case when $0 < r < 1 \leq p < \infty$. Several attempts have been made to tackle this case (e.g., [13, 25, 29, 32]); in some works, necessary and sufficient conditions did not match, while characterization had a discrete form or involved auxiliary functions in others. Hence, it was not easily verifiable. Finally, in [21], the missing integral conditions were provided.

In the general cases, Equation (1.2) is characterized in [7], but the conditions are non-standard. It was also considered in [29], but the conditions are not applicable because they involve auxiliary functions. Recently, in [23], a more complicated discretization method has been used to characterize the same inequality that involves iteration of the Copson operators and is restricted to non-degenerate weights, and the case $p = 1$ is presented without a proof. In our approach, the case $p > 1$ is not separated from $p = 1$.

Recently, in [5], with a new and simpler discretization technique that requires neither parameter restrictions nor non-degeneracy conditions, the characterization of Equation (1.1) is given. We adapt this approach to the specific demands of the inequality considered in this paper. Our technique allows us to obtain a scale of characterization which was impossible before; moreover, we obtain the previous characterizations of inequality (1.2) as a special case (more, precisely for $\beta = 0$, see Theorem 1.1).

We would like to point out that the characterization of Hardy inequality involving non-decreasing functions, that is,

$$\left(\int_0^\infty \left(\int_0^x f(s)u(s)ds\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}, \quad f \in \mathfrak{M}^\uparrow(0,\infty), \tag{1.4}$$

is obtained directly from inequality (1.2) without any further work (see the proof of Theorem 1.2) as a direct outcome of our main theorem (see Theorem 1.1).

We should also mention that in [11, 12], using reduction techniques, inequality (1.4) is reduced to inequality (1.3). However, as we have already mentioned, the characterizations of the reduced inequalities were not known at that point in time. Corollary 3.2 from [12] characterizes inequality (1.4), but the result is non-standard. The earlier works on inequality (1.4) can be found in [13, 14, 16, 26].

We should note that Theorem 1.1 allowed us to give a scale of characterizations of (1.4), and we would like to provide the characterization here to integrate all relevant parameter choices into a single theorem for the reader's convenience (see Theorem 1.2).

As one can see in Section 3, the discretization method transforms the inequality at hand equivalently to discrete inequalities that involve local characterizations of inequalities having low-order iterations. For this very reason, our aim in this paper is to revisit inequality (1.2) on (a, b), where $-\infty \leq a < b \leq \infty$.

Let $-\infty \le a < b \le \infty$ and weight be a positive, measurable function on (a,b). The principal goal of this study is to determine the scale of necessary and sufficient conditions on weights u, v, w on (a,b) for which

$$\left(\int_a^b \left(\int_a^x \left(\int_a^t f(s)ds\right)^q u(t)dt\right)^{\frac{r}{q}} w(x)dx\right)^{\frac{1}{r}} \le C\left(\int_a^b f(x)^p v(x)dx\right)^{\frac{1}{p}} \tag{1.5}$$

holds for $f \in \mathfrak{M}^+(a,b)$, with exponents $1 \le p < \infty$ and $0 < q, r < \infty$. It is worth noting that if $p < 1$, inequality (1.5) only holds for trivial functions.

Let us first review the essential notations and conventions before presenting our main results. The left and right sides of the inequality numbered by $(*)$ are denoted by LHS$(*)$ and RHS$(*)$, respectively. We put $0.\infty = \infty/\infty = 0/0 = 0$. The symbol $A \lesssim B$ means a constant $c > 0$ exists such that $A \le cB$, where c depends only on the parameters $p, q,$ and r. If both $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

For $1 \le p < \infty$, non-negative measurable functions v and w on (a,b), and $x, y \in [a,b]$, denote by

$$V_p(x,y) := \begin{cases} \left(\int_x^y v^{-\frac{1}{p-1}}\right)^{\frac{p-1}{p}}, & 1 < p < \infty, \\ \operatorname*{ess\,sup}_{s\in(x,y)} v(s)^{-1}, & p = 1, \end{cases} \tag{1.6}$$

and

$$\mathcal{W}(x) = \int_x^b w(s)\,ds.$$

Now, we are ready to formulate our main results.

Theorem 1.1. *Let $1 \le p < \infty$, $0 < q, r < \infty$, $-\infty < \beta < 1$ and let u, v, w be weights on (a,b) such that $0 < \mathcal{W}(t) < \infty$ for all $t \in (a,b)$. Then, inequality (1.5) holds for all $f \in \mathfrak{M}^+(a,b)$ if and only if*

(i) $p \le r$, $p \le q$ and

$$C_1 := \operatorname*{ess\,sup}_{x\in(a,b)} \left(\int_x^b \mathcal{W}(t)^{-\beta} w(t) \left(\int_x^t \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}} dt\right)^{\frac{1}{r}} V_p(a,x) < \infty. \tag{1.7}$$

Moreover, the best constant C in inequality (1.5) satisfies $C \approx C_1$.

(ii) $r < p \le q$,

$$C_2 := \left(\int_a^b \mathcal{W}(x)^{\frac{(1-\beta)p}{p-r}-1} w(x) \operatorname*{ess\,sup}_{t\in(a,x)} \left(\int_t^x \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{rp}{q(p-r)}} V_p(a,t)^{\frac{pr}{p-r}} dx\right)^{\frac{p-r}{pr}} < \infty,$$

and

$$C_3 := \left(\int_a^b \left(\int_x^b \mathcal{W}(s)^{-\beta} w(s) \left(\int_x^s \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}} ds\right)^{\frac{r}{p-r}} \mathcal{W}(x)^{-\beta} w(x) \operatorname*{ess\,sup}_{t\in(a,x)} \left(\int_t^x \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}} dx\right)^{\frac{p-r}{pr}} < \infty. \tag{1.8}$$

Moreover, the best constant C in inequality (1.5) satisfies $C \approx C_2 + C_3$.

(iii) $q < p \le r$, $C_1 < \infty$ *and*

$$C_4 := \sup_{x\in(a,b)} \mathcal{W}(x)^{\frac{1-\beta}{r}} \left(\int_a^x \left(\int_t^x \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(a,t)^{\frac{pq}{p-q}} dt \right)^{\frac{p-q}{pq}} < \infty,$$

where C_1 is defined in Equation (1.7). Moreover, the best constant C in inequality (1.5) satisfies $C \approx C_1 + C_4$.

(iv) $r < p$, $q < p$, $C_3 < \infty$ *and*

$$C_5 := \left(\int_a^b \mathcal{W}(x)^{\frac{(1-\beta)p}{p-r}-1} w(x) \left(\int_a^x \left(\int_t^x \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(a,t)^{\frac{pq}{p-q}} dt \right)^{\frac{r(p-q)}{q(p-r)}} dx \right)^{\frac{p-r}{pr}} < \infty,$$

where C_3 is defined in Equation (1.8). Moreover, the best constant C in inequality (1.5) satisfies $C \approx C_3 + C_5$.

Theorem 1.2. *Let $0 < p, q < \infty$, $-\infty < \beta < 1$ and u, v, w be weights on (a,b) such that $0 < \mathcal{W}(t) < \infty$ for all $t \in (a,b)$. Then, inequality*

$$\left(\int_a^b \left(\int_a^x f(s)u(s)ds \right)^q w(x)dx \right)^{\frac{1}{q}} \le C \left(\int_a^b f(x)^p v(x)dx \right)^{\frac{1}{p}} \tag{1.9}$$

holds for all $f \in \mathfrak{M}^{\uparrow}(a,b)$ if and only if

(i) $p \le q$, $p \le 1$ *and*

$$C_1 := \operatorname*{ess\,sup}_{x\in(a,b)} \left(\int_x^b \mathcal{W}(t)^{-\beta} w(t) \left(\int_x^t \mathcal{W}^{\frac{\beta}{q}} u \right)^q dt \right)^{\frac{1}{q}} \left(\int_x^b v \right)^{-\frac{1}{p}} < \infty. \tag{1.10}$$

Moreover, the best constant C in inequality (1.9) satisfies $C \approx C_1$.

(ii) $q < p \le 1$,

$$C_2 := \left(\int_a^b \mathcal{W}(x)^{\frac{(1-\beta)p}{p-q}-1} w(x) \operatorname*{ess\,sup}_{t\in(a,x)} \left(\int_t^x \mathcal{W}^{\frac{\beta}{q}} u \right)^{\frac{pq}{p-q}} \left(\int_t^b v \right)^{-\frac{q}{p-q}} dx \right)^{\frac{p-q}{pq}} < \infty,$$

and

$$C_3 := \left(\int_a^b \left(\int_x^b \mathcal{W}(s)^{-\beta} w(s) \left(\int_x^s \mathcal{W}^{\frac{\beta}{q}} u \right)^q ds \right)^{\frac{q}{p-q}} \mathcal{W}(x)^{-\beta} w(x) \operatorname*{ess\,sup}_{t\in(a,x)} \left(\int_t^x \mathcal{W}^{\frac{\beta}{q}} u \right)^q \left(\int_t^b v \right)^{-\frac{q}{p-q}} dx \right)^{\frac{p-q}{pq}} < \infty. \tag{1.11}$$

Moreover, the best constant C in inequality (1.9) satisfies $C \approx C_2 + C_3$.

(iii) $1 < p \le q$, $C_1 < \infty$ *and*

$$C_4 := \sup_{x\in(a,b)} \mathcal{W}(x)^{\frac{1-\beta}{q}} \left(\int_a^x \left(\int_t^x \mathcal{W}^{\frac{\beta}{q}} u \right)^{\frac{1}{p-1}} \mathcal{W}(t)^{\frac{\beta}{q}} u(t) \left(\int_t^b v \right)^{-\frac{1}{p-1}} dt \right)^{\frac{p-1}{p}} < \infty,$$

where C_1 is defined in Equation (1.10). Moreover, the best constant C in inequality (1.9) satisfies $C \approx C_1 + C_4$.

(iv) $q < p$, $1 < p$, $C_3 < \infty$ *and*

$$C_5 := \left(\int_a^b \mathcal{W}^{\frac{(1-\beta)p}{p-q}-1} w(x) \left(\int_a^x \left(\int_t^x \mathcal{W}^{\frac{\beta}{q}} u \right)^{\frac{1}{p-1}} \mathcal{W}(t)^{\frac{\beta}{q}} u(t) \left(\int_t^b v \right)^{-\frac{1}{p-1}} dt \right)^{\frac{q(p-1)}{p-q}} dx \right)^{\frac{p-q}{pq}} < \infty,$$

where C_3 is defined in Equation (1.11). Moreover, the best constant C in inequality (1.9) satisfies $C \approx C_3 + C_5$.

Proofs of Theorems 1.1 and 1.2 will be given in Section 4.

2 | PRELIMINARY RESULTS

In this section, we cover the foundations of discretization and several new results that will be employed frequently throughout the proof of the main theorem.

Definition 2.1. Let $N \in \mathbb{Z} \cup \{-\infty\}$, $M \in e\mathbb{Z} \cup \{+\infty\}$, $N < M$, and $\{a_k\}_{k=N}^M$ be a sequence of positive numbers. We say that $\{a_k\}_{k=N}^M$ is *geometrically decreasing* if

$$\sup \left\{ \frac{a_{k+1}}{a_k}, \quad N \le k \le M \right\} < 1.$$

Lemma 2.2 ([9]). *Let $\alpha > 0$ and $n \in \mathbb{Z} \cup \{-\infty\}$. If $\{\tau_k\}_{k=n}^\infty$ is a geometrically decreasing sequence, then,*

$$\sup_{n\le k<\infty} \tau_k \sum_{i=n}^k a_i \approx \sup_{n\le k<\infty} \tau_k a_k, \tag{2.1}$$

$$\sum_{k=n}^\infty \tau_k \left(\sum_{i=n}^k a_i \right)^\alpha \approx \sum_{k=n}^\infty \tau_k a_k^\alpha, \tag{2.2}$$

and

$$\sum_{k=n}^\infty \tau_k \sup_{n\le i\le k} a_i \approx \sum_{k=n}^\infty \tau_k a_k, \tag{2.3}$$

for all non-negative sequences $\{a_k\}_{k=n}^\infty$.

Lemma 2.3. *Let $\alpha > 0$ and $n \in \mathbb{Z} \cup \{-\infty\}$. Assume that $\{x_k\}_{k=n}^{\infty} \subset (a, b)$ is a strictly increasing sequence. If $\{\tau_k\}_{k=n+1}^{\infty}$ is a geometrically decreasing sequence, then,*

$$\sup_{n+1 \le k < \infty} \tau_k \left(\int_{x_n}^{x_k} g \right) \approx \sup_{n+1 \le k < \infty} \tau_k \left(\int_{x_{k-1}}^{x_k} g \right), \tag{2.4}$$

$$\sum_{k=n+1}^{\infty} \tau_k \left(\int_{x_n}^{x_k} g \right)^{\alpha} \approx \sum_{k=n+1}^{\infty} \tau_k \left(\int_{x_{k-1}}^{x_k} g \right)^{\alpha}, \tag{2.5}$$

and

$$\sum_{k=n+1}^{\infty} \tau_k \operatorname*{ess\,sup}_{s \in (x_n, x_k)} g(s) \approx \sum_{k=n+1}^{\infty} \tau_k \operatorname*{ess\,sup}_{s \in (x_{k-1}, x_k)} g(s), \tag{2.6}$$

for all non-negative measurable g on (a, b).

Proof. For each $n \in \mathbb{Z} \cup \{-\infty\}$, we can write

$$\int_{x_n}^{x_k} g = \sum_{i=n+1}^{k} \int_{x_{i-1}}^{x_i} g.$$

Then, Equations (2.4) and (2.5) are direct consequences of Equations (2.1) and (2.2), respectively.

Similarly, for each $n \in \mathbb{Z} \cup \{-\infty\}$, we have

$$\operatorname*{ess\,sup}_{s \in (x_n, x_k)} g(s) = \sup_{n+1 \le i \le k} \operatorname*{ess\,sup}_{s \in (x_{i-1}, x_i)} g(s),$$

so that applying Equation (2.3), we obtain Equation (2.6). $\square$

Lemma 2.4. *Let $\alpha > 0$ and $n \in \mathbb{Z} \cup \{-\infty\}$. Assume that $\{x_k\}_{k=n}^{\infty} \subset (a, b)$ is a strictly increasing sequence, $\{\tau_k\}_{k=n+1}^{\infty}$ is a geometrically decreasing sequence, and $\{\sigma_k\}_{k=n}^{\infty}$ is a positive non-decreasing sequence. Then*

$$\sup_{n+1 \le k < \infty} \tau_k \sup_{n \le i < k} \left(\int_{x_i}^{x_k} g \right)^{\alpha} \sigma_i \approx \sup_{n+1 \le k < \infty} \tau_k \left(\int_{x_{k-1}}^{x_k} g \right)^{\alpha} \sigma_{k-1} \tag{2.7}$$

and

$$\sum_{k=n+1}^{\infty} \tau_k \sup_{n \le i < k} \left(\int_{x_i}^{x_k} g \right)^{\alpha} \sigma_i \approx \sum_{k=n+1}^{\infty} \tau_k \left(\int_{x_{k-1}}^{x_k} g \right)^{\alpha} \sigma_{k-1} \tag{2.8}$$

hold for all non-negative measurable g on (a, b).

Proof. Let us start with the equivalency (2.7). Since $\{\tau_k\}_{k=n+1}^{\infty}$ is a geometrically decreasing sequence, interchanging supremum and Equation (2.4) give

$$\mathrm{LHS}(2.7) = \sup_{n+1 \le i < \infty} \sigma_i \sup_{i+1 \le k < \infty} \tau_k \left(\int_{x_i}^{x_k} g \right)^{\alpha} \approx \sup_{n+1 \le i < \infty} \sigma_i \sup_{i+1 \le k < \infty} \tau_k \left(\int_{x_{k-1}}^{x_k} g \right)^{\alpha}.$$

Interchanging supremum once again and monotonicity of $\{\sigma_k\}_{k=n}^{\infty}$ results in

$$\mathrm{LHS}(2.7) \approx \sup_{n+1\le k<\infty} \tau_k \left(\int_{x_{k-1}}^{x_k} g\right)^{\alpha} \sup_{n\le i\le k-1} \sigma_i = RHS(2.7).$$

Let us now tackle Equation (2.8). Monotonicity of $\{\sigma_k\}_{k=n}^{\infty}$ gives that

$$\mathrm{LHS}(2.8) \le \sum_{k=n+1}^{\infty} \tau_k \sup_{n\le i<k} \left(\sum_{j=i}^{k-1} \sigma_j^{\frac{1}{\alpha}} \int_{x_j}^{x_{j+1}} g\right)^{\alpha} = \sum_{k=n+1}^{\infty} \tau_k \left(\sum_{j=n}^{k-1} \sigma_j^{\frac{1}{\alpha}} \int_{x_j}^{x_{j+1}} g\right)^{\alpha}.$$

Then, using Equation (2.2), we have the following upper estimate:

$$\mathrm{LHS}(2.8) \le \sum_{k=n+1}^{\infty} \tau_k \left(\sum_{j=n+1}^{k} \sigma_{j-1}^{\frac{1}{\alpha}} \int_{x_{j-1}}^{x_j} g\right)^{\alpha} \approx \mathrm{RHS}(2.8).$$

On the other hand, the reverse estimate is clear, and the proof is complete. □

Definition 2.5. Let w be a non-negative measurable function on (a, b) such that $0 < \mathcal{W}(t) < \infty$ for all $t \in (a, b)$. A strictly increasing sequence $\{x_k\}_{k=N}^{\infty} \subset [a, b]$ is said to be a discretizing sequence of the function $\mathcal{W}$, if it satisfies $\mathcal{W}(x_k) \approx 2^{-k}$, $N \le k < \infty$. If $N > -\infty$ then $x_N := a$, otherwise $x_{-\infty} := \lim_{k\to-\infty} x_k = a$.

Remark 2.6. Let w be a non-negative measurable function on (a, b) such that $0 < \mathcal{W}(t) < \infty$ for all $t \in (a, b)$. The Darboux property of continuous functions implies that the discretizing sequence exists for $\mathcal{W}$. We can define the discretizing sequence in the following way:

$$N = \sup\left\{k \in \mathbb{Z} : 2^{-k} \ge \mathcal{W}(t) \text{ for every } t \in (a, b)\right\}.$$

If the set is empty then $N = -\infty$. If $\lim_{t\to a+} \mathcal{W}(t) < \infty$, then $N > -\infty$ and $x_N = a$ and for $k > N$

$$x_k = \sup\left\{t \in (a, b) : 2^{-k} \le \mathcal{W}(t)\right\}.$$

It is worth noting that if $N = -\infty$, then $N + 1$ is also $-\infty$.

Lemma 2.7. *Let $\alpha > 0$ and $N \in \mathbb{Z} \cup \{-\infty\}$. Assume that w is a weight on (a, b) such that $0 < \mathcal{W}(t) < \infty$ for all $t \in (a, b)$, and $\{x_k\}_{k=N}^{\infty}$ is a discretizing sequence of the function $\mathcal{W}$. Then for any $n : N \le n$,*

$$\int_{x_n}^{b} \mathcal{W}(x)^{\alpha-1} w(x) h(x) dx \approx \sum_{k=n+1}^{\infty} 2^{-k\alpha} h(x_k) \tag{2.9}$$

and

$$\operatorname*{ess\,sup}_{x\in(x_n,b)} \mathcal{W}(x)^{\alpha} h(x) \approx \sup_{n+1\le k} 2^{-k\alpha} h(x_k) \tag{2.10}$$

hold for all non-negative and non-decreasing h on (a, b).

Proof. Monotonicity of h and properties of the discretizing sequence $\{x_k\}_{k=N}^{\infty}$ yield

$$\mathrm{LHS}(2.9) = \sum_{k=n+1}^{\infty} \int_{x_{k-1}}^{x_k} h(x)\mathcal{W}(x)^{\alpha-1}w(x)dx \lesssim \sum_{k=n+1}^{\infty} h(x_k) \int_{x_{k-1}}^{x_k} d[-\mathcal{W}(x)^{\alpha}] \approx \sum_{k=n+1}^{\infty} 2^{-k\alpha} h(x_k) = RHS(2.9),$$

and, conversely

$$\mathrm{LHS}(2.9) \geq \sum_{k=n+1}^{\infty} \int_{x_k}^{x_{k+1}} h(x)\mathcal{W}(x)^{\alpha-1}w(x)dx \gtrsim \sum_{k=n+1}^{\infty} h(x_k) \int_{x_k}^{x_{k+1}} d[-\mathcal{W}(x)^{\alpha}] \approx \sum_{k=n+1}^{\infty} 2^{-k\alpha} h(x_k) = \mathrm{RHS}(2.9).$$

Thus, Equation (2.9) holds.

On the other hand, similarly,

$$\mathrm{LHS}(2.10) = \sup_{n+1\leq k<\infty} \operatorname*{ess\,sup}_{x\in(x_{k-1},x_k)} \mathcal{W}(x)^{\alpha} h(x) \approx \sup_{n+1\leq k<\infty} 2^{-k\alpha} \operatorname*{ess\,sup}_{x\in(x_{k-1},x_k)} h(x) = \mathrm{RHS}(2.10)$$

holds. □

Theorem 2.8 ([10, Theorem 4.5]). *Let $p, q \in (0,\infty)$, $N, M \in \mathbb{Z} \cup \{-\infty,\infty\}$, $N < M$. Assume that $\{w_k\}_{k=N}^{M}$ and $\{v_k\}_{k=N}^{M}$ be two sequences of positive real numbers. Then, inequality*

$$\left(\sum_{k=N}^{M} a_k^q v_k^q\right)^{\frac{1}{q}} \leq C\left(\sum_{k=N}^{M} a_k^p w_k^p\right)^{\frac{1}{p}} \tag{2.11}$$

holds for every sequence $\{a_k\}_{k=N}^{M}$ of non-negative real numbers if and only if either

(i) $p \leq q$ and

$$L_1 = \sup_{N\leq k\leq M} \{v_k w_k^{-1}\} < \infty,$$

or

(ii) $p > q$ and

$$L_2 = \left(\sum_{k=N}^{M} v_k^{\frac{pq}{p-q}} w_k^{-\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, if we denote by C the optimal embedding constant in Equation (2.11), then

$$C = \begin{cases} L_1 & \text{in case (i),} \\ L_2 & \text{in case (ii).} \end{cases}$$

Theorem 2.9 ([10, Theorem 4.6, 15, Theorem 9.2]). *Assume that $0 < p, q < \infty$, $N, M \in \mathbb{Z} \cup \{-\infty,\infty\}$, $N < M$, $\{a_k\}_{k=N}^{M}$ and $\{b_k\}_{k=N}^{M}$ are sequences of non-negative real numbers. Then, the inequality*

$$\left(\sum_{k=N}^{M} \left(\sum_{i=N}^{k} x_i b_i\right)^{q} a_k\right)^{\frac{1}{q}} \leq C\left(\sum_{k=N}^{M} x_k^p\right)^{\frac{1}{p}} \tag{2.12}$$

holds for every sequence $\{x_k\}_{k=N}^{M}$ of non-negative real numbers if and only if *(i) either $p \leq 1$, $p \leq q$ and*

$$H_1 = \sup_{N \leq k \leq M} \left\{ \left(\sum_{i=k}^{M} a_i \right)^{\frac{1}{q}} b_k \right\} < \infty,$$

(ii) or $q < p \leq 1$ and

$$H_2 = \left(\sum_{k=N}^{M} a_k \left(\sum_{i=k}^{M} a_i \right)^{\frac{q}{p-q}} \sup_{N \leq i \leq k} b_i^{\frac{qp}{p-q}} \right)^{\frac{p-q}{pq}} < \infty,$$

(iii) or $1 < p$, $q < p$ and

$$H_3 = \left(\sum_{k=N}^{M} a_k \left(\sum_{i=k}^{M} a_i \right)^{\frac{q}{p-q}} \left(\sum_{i=N}^{k} b_i^{\frac{p}{p-1}} \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{pq}} < \infty,$$

(iv) or $1 < p \leq q$ and

$$H_4 = \sup_{N \leq k \leq M} \left\{ \left(\sum_{i=k}^{M} a_i \right)^{\frac{1}{q}} \left(\sum_{i=N}^{k} b_i^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \right\} < \infty.$$

Moreover, if we denote by C the best constant in Equation (2.12), then

$$C \approx \begin{cases} H_1 & \textit{in case } \text{(i)}, \\ H_2 & \textit{in case } \text{(ii)}, \\ H_3 & \textit{in case } \text{(iii)}, \\ H_4 & \textit{in case } \text{(iv)}. \end{cases}$$

Moreover, the multiplicative constants in all the equivalences above depend only on p, q.

3 | DISCRETE CHARACTERIZATION

We begin this section by observing that inequality (1.5) is equivalent to two other discrete inequalities, and we present the characterization in discrete form, which is noteworthy on its own.

Let us start with the discretization of inequality (1.5). To this end we need the following notation, denote by $M(x_{k-1}, x_k)$ the best constant of weighted Hardy inequality, that is,

$$B(x_{k-1}, x_k) := \sup_{h \in \mathfrak{M}^+(x_{k-1}, x_k)} \frac{\left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} h(s) ds \right)^q u(t) dt \right)^{\frac{1}{q}}}{\left(\int_{x_{k-1}}^{x_k} h(t)^p v(t) dt \right)^{\frac{1}{p}}}$$

and using the classical characterizations of weighted Hardy inequalities (see, [20, 30]), we have for $1 \le p < \infty, 0 < q < \infty$

$$B(x_{k-1}, x_k) \approx \begin{cases} \operatorname*{ess\,sup}_{t\in(x_{k-1},x_k)} \left(\int_t^{x_k} u\right)^{\frac{1}{q}} V_p(x_{k-1}, t) & \text{if} \quad p \le q, \\ \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} u\right)^{\frac{q}{p-q}} u(t) V_p(x_{k-1}, t)^{\frac{pq}{p-q}} dt\right)^{\frac{p-q}{pq}} & \text{if} \quad q < p. \end{cases} \tag{3.1}$$

Furthermore, observe that $V_p(x_{k-1}, x_k)$, $N+1 \le k$ defined in Equation (1.6) can be expressed as

$$\sup_{g\in\mathfrak{M}^+(x_{k-1},x_k)} \frac{\int_{x_{k-1}}^{x_k} g(t)dt}{\left(\int_{x_{k-1}}^{x_k} g(t)^p v(t)dt\right)^{\frac{1}{p}}} = V_p(x_{k-1}, x_k).$$

Lemma 3.1. *Let $1 \le p < \infty$, $0 < q, r < \infty$ and let u, v, w be weights on (a, b) such that $0 < \mathcal{W}(t) < \infty$ for all $t \in (a, b)$, and $\{x_k\}_{k=N}^{\infty} \subset [a, b]$ is a discretizing sequence of the function $\mathcal{W}$. Then, there exists a positive constant C such that inequality (1.5) holds for all $f \in \mathfrak{M}^+(a, b)$ if and only if there exist positive constants C' and C'' such that*

$$\left(\sum_{k=N+1}^{\infty} 2^{-k}\left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} f\right)^q u(t)dt\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \le C'\left(\sum_{k=N+1}^{\infty} \int_{x_{k-1}}^{x_k} f^p v\right)^{\frac{1}{p}} \tag{3.2}$$

and

$$\left(\sum_{k=N+1}^{\infty} 2^{-k}\left(\int_a^{x_k} f\right)^r \left(\int_{x_k}^{x_{k+1}} u\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \le C''\left(\sum_{k=N+1}^{\infty} \int_{x_{k-1}}^{x_k} f^p v\right)^{\frac{1}{p}} \tag{3.3}$$

hold for all $f \in \mathfrak{M}^+(a, b)$. Moreover, $C \approx C' + C''$.

Proof. Applying Equation (2.9) with $\alpha = 1$ and $h(x) = \left(\int_a^x \left(\int_a^t f\right)^q u(t)dt\right)^{\frac{r}{q}}$, we have that

$$\mathrm{LHS}(1.5) \approx \left(\sum_{k=N+1}^{\infty} 2^{-k}\left(\int_a^{x_k} \left(\int_a^t f\right)^q u(t)dt\right)^{\frac{r}{q}}\right)^{\frac{1}{r}}.$$

Since, $\{2^{-k}\}$ is geometrically decreasing, using Equation (2.5), we obtain that

$$\begin{aligned} \mathrm{LHS}(1.5) &\approx \left(\sum_{k=N+1}^{\infty} 2^{-k}\left(\int_{x_{k-1}}^{x_k} \left(\int_a^t f\right)^q u(t)dt\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \\ &\approx \left(\sum_{k=N+1}^{\infty} 2^{-k}\left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^t f\right)^q u(t)dt\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} + \left(\sum_{k=N+2}^{\infty} 2^{-k}\left(\int_a^{x_{k-1}} f\right)^r \left(\int_{x_{k-1}}^{x_k} u\right)^{\frac{r}{q}}\right)^{\frac{1}{r}}. \end{aligned}$$

Then, it is clear that there exists a positive constant C such that inequality (1.5) holds for all $f \in \mathfrak{M}^+(a,b)$ if and only if there exist positive constants C' and C'' such that Equations (3.2) and (3.3) hold for all $f \in \mathfrak{M}^+(a,b)$. Moreover, $C \approx C' + C''$. □

Lemma 3.2. *Let $1 \le p < \infty$, $0 < q, r < \infty$ and let u, v be weights on (a,b). Assume that $N \in \mathbb{Z} \cup \{-\infty\}$ and $\{x_k\}_{k=N}^{\infty}$ is a strictly increasing sequence in $[a,b]$. Then, there exists a positive constant C' such that inequality (3.2) holds for all $f \in \mathfrak{M}^+(a,b)$ if and only if there exists a positive constant C' such that*

$$\left(\sum_{k=N+1}^{\infty} 2^{-k} a_k^r B(x_{k-1},x_k)^r\right)^{\frac{1}{r}} \le C'\left(\sum_{k=N+1}^{\infty} a_k^p\right)^{\frac{1}{p}} \tag{3.4}$$

holds for every sequence of non-negative numbers $\{a_k\}_{k=N+1}^{\infty}$. Moreover, the best constants C' and C', respectively, in Equations (3.2) and (3.4) satisfy $C' \approx C'$.

Proof. Assume that Equation (3.2) holds. By the definition of $B(x_{k-1},x_k)$, there exist non-negative measurable functions h_k, $N+1 \le k$ on (a,b) such that

$$\operatorname{supp} h_k \subset [x_{k-1},x_k], \quad \int_{x_{k-1}}^{x_k} h_k^p v = 1, \quad \left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t} h_k\right)^q u(t)dt\right)^{\frac{1}{q}} \gtrsim B(x_{k-1},x_k).$$

Thus, inserting $f = \sum_{m=N+1}^{\infty} a_m h_m$, where $\{a_m\}_{m=N+1}^{\infty}$ is any sequence of non-negative numbers, into Equations (3.2) and (3.4) follows. Moreover, $C' \lesssim C'$.

Conversely, observe first that for each $h \in \mathfrak{M}^+(x_{k-1},x_k)$,

$$B(x_{k-1},x_k) \ge \left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t} h(s)ds\right)^q u(t)dt\right)^{\frac{1}{q}}\left(\int_{x_{k-1}}^{x_k} h(t)^p v(t)dt\right)^{-\frac{1}{p}}.$$

Then, Equation (3.2) follows by, inserting $a_k = (\int_{x_{k-1}}^{x_k} h^p v)^{\frac{1}{p}}$ in Equation (3.4) and $C' \le C'$. Further, we have $C' \approx C'$. □

Lemma 3.3. *Let $1 \le p < \infty$, $0 < q, r < \infty$ and let u, v be weights on (a,b). Assume that $N \in \mathbb{Z} \cup \{-\infty\}$ and $\{x_k\}_{k=N}^{\infty}$ is a strictly increasing sequence in $[a,b]$. Then, there exists a positive constant C'' such that inequality (3.3) holds for all $f \in \mathfrak{M}^+(a,b)$ if and only if there exists a positive constant C'' such that*

$$\left(\sum_{k=N+1}^{\infty} 2^{-k}\left(\int_{x_k}^{x_{k+1}} u\right)^{\frac{r}{q}}\left(\sum_{j=N+1}^{k} a_j V_p(x_{j-1},x_j)\right)^r\right)^{\frac{1}{r}} \le C''\left(\sum_{k=N+1}^{\infty} a_k^p\right)^{\frac{1}{p}} \tag{3.5}$$

holds for every sequence of non-negative numbers $\{a_k\}_{k=N+1}^{\infty}$. Moreover, the best constants C'' and C'', respectively, in Equations (3.3) and (3.5) satisfy $C'' \approx C''$.

Proof. Suppose that Equation (3.3) holds for all $f \in \mathfrak{M}^+(a,b)$. Using Equation (3.5), there exist non-negative measurable functions g_k, $N+1 \le k$ on (x_{k-1},x_k) such that

$$\operatorname{supp} g_k \subset [x_{k-1},x_k], \quad \int_{x_{k-1}}^{x_k} g_k^p v = 1, \quad \int_{x_{k-1}}^{x_k} g_k \gtrsim V_p(x_{k-1},x_k).$$

Thus, inserting $f = \sum_{m=N+1}^{\infty} a_m g_m$, where $\{a_m\}_{m=N+1}^{\infty}$ is any sequence of non-negative numbers, into Equations (3.3), (3.5) follows. Moreover, $C'' \lesssim C''$ holds.

Conversely, taking $a_k = (\int_{x_{k-1}}^{x_k} f^p v)^{\frac{1}{p}}$ in Equation (3.5) and using the estimate

$$V_p(x_{k-1}, x_k) \geq \int_{x_{k-1}}^{x_k} f(t)dt \left(\int_{x_{k-1}}^{x_k} f(t)^p v(t)dt \right)^{-\frac{1}{p}}$$

gives Equation (3.3). Additionally, $C'' \leq C''$ holds. Consequently $C'' \approx C''$ follows. □

Theorem 3.4. *Let $1 \leq p < \infty$, $0 < q, r < \infty$, $-\infty < \beta < 1$ and let u, v, w be weights on (a, b) such that $0 < \mathcal{W}(t) < \infty$ for all $t \in (a, b)$, and $\{x_k\}_{k=N+1}^{\infty}$ be the discretizing sequence of $\mathcal{W}$. Then, inequality (3.4) holds for every sequence of non-negative numbers $\{a_k\}_{k=N+1}^{\infty}$ if and only if*

(i) $p \leq r$, $p \leq q$ and

$$\mathcal{A}_1 := \sup_{N+1 \leq k < \infty} 2^{-\frac{k(1-\beta)}{r}} \operatorname*{ess\,sup}_{t \in (x_{k-1}, x_k)} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(x_{k-1}, t) < \infty,$$

(ii) $r < p \leq q$ and

$$\mathcal{A}_2 := \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)\frac{p}{p-r}} \operatorname*{ess\,sup}_{t \in (x_{k-1}, x_k)} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} V_p(x_{k-1}, t)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}} < \infty,$$

(iii) $q < p \leq r$ and

$$\mathcal{A}_3 := \sup_{N+1 \leq k < \infty} 2^{-\frac{k(1-\beta)}{r}} \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(x_{k-1}, t)^{\frac{pq}{p-q}} dt \right)^{\frac{p-q}{pq}} < \infty,$$

(iv) $r < p$, $q < p$ and

$$\mathcal{A}_4 := \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)\frac{p}{p-r}} \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(x_{k-1}, t)^{\frac{pq}{p-q}} dt \right)^{\frac{r(p-q)}{q(p-r)}} \right)^{\frac{p-r}{pr}} < \infty.$$

Moreover, the best constant C' in inequality (3.4) satisfies

$$C' = \begin{cases} \mathcal{A}_1 & \textit{in case } \text{(i)}, \\ \mathcal{A}_2 & \textit{in case } \text{(ii)}, \\ \mathcal{A}_3 & \textit{in case } \text{(iii)}, \\ \mathcal{A}_4 & \textit{in case } \text{(iv)}. \end{cases}$$

Proof. The result follows easily by combining Theorem 2.8 for suitable parameters and weights with Equation (3.1) and the fact that $\mathcal{W}(t) \approx 2^{-k}$, for every $t \in [x_{k-1}, x_k]$. □

Theorem 3.5. *Let $1 \leq p < \infty$, $0 < q, r < \infty$, $-\infty < \beta < 1$ and let u, v, w be weights on (a, b) such that $0 < \mathcal{W}(t) < \infty$ for all $t \in (a, b)$, and $\{x_k\}_{k=N+1}^{\infty}$ be the discretizing sequence of $\mathcal{W}$. Then, inequality (3.5) holds for every sequence of non-negative numbers $\{a_k\}_{k=N+1}^{\infty}$ if and only if* *(i)* $p \le r$ *and*

$$\mathcal{B}_1 := \sup_{N+1\le k<\infty}\left(\sum_{i=k}^{\infty} 2^{-i(1-\beta)}\left(\int_{x_i}^{x_{i+1}} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} V_p(a,x_k) < \infty,$$

(ii) $r < p$ *and*

$$\mathcal{B}_2 := \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)}\left(\int_{x_k}^{x_{k+1}} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}}\left(\sum_{i=k}^{\infty} 2^{-i(1-\beta)}\left(\int_{x_i}^{x_{i+1}} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}}\right)^{\frac{r}{p-r}} V_p(a,x_k)^{\frac{pr}{p-r}}\right)^{\frac{p-r}{pr}} < \infty.$$

Moreover, the best constant C'' in inequality (3.5) satisfies

$$C'' = \begin{cases} \mathcal{B}_1 & \text{in case (i)},\\ \mathcal{B}_2 & \text{in case (ii)}.\end{cases}$$

Proof. We will apply Theorem 2.9 with suitable parameters and weights $a_k = 2^{-k}\left(\int_{x_k}^{x_{k+1}} u\right)^{\frac{r}{q}}$ and $b_k = V_p(x_{k-1},x_k)$. We need to treat the cases when $p > 1$ and $p = 1$, separately. Note that when $p > 1$, since $\{x_k\}_{k=N+1}^{\infty}$ is the discretizing sequence of $\mathcal{W}$, we have for each $k : N+1 \le k$ that

$$\sum_{i=N+1}^{\infty} V_p(x_{i-1},x_i)^{\frac{p}{p-1}} = \sum_{i=N+1}^{\infty} \int_{x_{i-1}}^{x_i} v(s)^{-\frac{1}{p-1}} ds = V_p(a,x_k)^{\frac{p-1}{p}}.$$

Thus, applying [Theorem 2.9, (iv)] when $1 < p \le r$ and [Theorem 2.9, (iii)] when $1 < p, r < p$, the result follows.

On the other hand, if $p = 1$, for each $k : N+1 \le k$,

$$\sup_{N+1\le i\le k} V_1(x_{i-1},x_i) = \sup_{N+1\le i\le k}\ \operatorname*{ess\,sup}_{s\in(x_{i-1},x_i)} v(s)^{-1} = V_1(a,x_k).$$

Therefore, applying [Theorem 2.9, (ii)] when $r < p = 1$, we have $C'' \approx \mathcal{B}_2$.

Lastly, if $p = 1 \le r$, then applying [Theorem 2.9, (i)], and the fact that $\mathcal{W}(t) \approx 2^{-i}$, for every $t \in [x_i, x_{i+1}]$ we have

$$C'' \approx \sup_{N+1\le k<\infty}\left(\sum_{i=k}^{\infty} 2^{-i}\left(\int_{x_i}^{x_{i+1}} u\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} V_1(x_{k-1},x_k) \approx \sup_{N+1\le k<\infty}\left(\sum_{i=k}^{\infty} 2^{-i(1-\beta)}\left(\int_{x_i}^{x_{i+1}} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} V_1(x_{k-1},x_k).$$

Finally, interchanging supremum yields that

$$C'' \approx \sup_{N+1\le k<\infty} V_1(x_{k-1},x_k) \sup_{k\le m<\infty}\left(\sum_{i=m}^{\infty} 2^{-i(1-\beta)}\left(\int_{x_i}^{x_{i+1}} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}}\right)^{\frac{1}{r}}$$

$$= \sup_{N+1\le m<\infty}\left(\sum_{i=m}^{\infty} 2^{-i(1-\beta)}\left(\int_{x_i}^{x_{i+1}} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \sup_{N+1\le k\le m} V_1(x_{k-1},x_k) = \mathcal{B}_1.$$

□

We can formulate the discrete characterization of inequality (1.5).

Theorem 3.6. *Let $1 \le p < \infty$, $0 < q, r < \infty$ and let u, v, w be weights on (a,b) such that $0 < \mathcal{W}(t) < \infty$ for all $t \in (a,b)$, and $\{x_k\}_{k=N+1}^{\infty}$ be the discretizing sequence of $\mathcal{W}$. Then, inequality (1.5) holds for all $f \in \mathfrak{M}^+(a,b)$ if and only if*

(i) $p \le r$, $p \le q$ and $\mathcal{A}_1 + \mathcal{B}_1 < \infty$,
(ii) $r < p \le q$ and $\mathcal{A}_2 + \mathcal{B}_2 < \infty$,
(iii) $q < p \le r$ and $\mathcal{A}_3 + \mathcal{B}_1 < \infty$,
(iv) $r < p$, $q < p$ and $\mathcal{A}_4 + \mathcal{B}_2 < \infty$.

Moreover, the best constant C in Equation (1.5) satisfies

$$C = \begin{cases} \mathcal{A}_1 + \mathcal{B}_1 & \text{in case (i)}, \\ \mathcal{A}_2 + \mathcal{B}_2 & \text{in case (ii)}, \\ \mathcal{A}_3 + \mathcal{B}_1 & \text{in case (iii)}, \\ \mathcal{A}_4 + \mathcal{B}_2 & \text{in case (iv)}. \end{cases}$$

Proof. According to Lemmas [3.1–3.3], inequality (1.5) holds if and only if inequalities (3.4) and (3.5) hold. Moreover, the best constant C in Equation (1.5) satisfies $C \approx C' + C''$, where C' and C'' are the best constants in inequalities (3.4) and (3.5), respectively. The result follows from the combination of Theorems 3.4 and 3.5. □

4 | PROOFS

Proof of Theorem 1.1.

(i) Let $1 \le p \le \min\{r, q\}$. We have from [Theorem 3.6, (i)] that $C \approx \mathcal{A}_1 + \mathcal{B}_1$. We will prove that $C_1 \approx \mathcal{A}_1 + \mathcal{B}_1$. First, we will show that $\mathcal{A}_1 + \mathcal{B}_1 \approx A_1 + \mathcal{B}_1$, where

$$A_1 := \sup_{N+1 \le k} 2^{-\frac{k(1-\beta)}{r}} \operatorname*{ess\,sup}_{t \in (a, x_k)} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t).$$

It is clear that $\mathcal{A}_1 \le A_1$. On the other hand, observe that

$$A_1 = \sup_{N+1 \le k} 2^{-\frac{k(1-\beta)}{r}} \sup_{N+1 \le i \le k} \operatorname*{ess\,sup}_{t \in (x_{i-1}, x_i)} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t)$$

$$\approx \sup_{N+1 \le k} 2^{-\frac{k(1-\beta)}{r}} \sup_{N+1 \le i \le k} \operatorname*{ess\,sup}_{t \in (x_{i-1}, x_i)} \left(\int_t^{x_i} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t) + \sup_{N+2 \le k} 2^{-\frac{k(1-\beta)}{r}} \sup_{N+1 \le i < k} \left(\int_{x_i}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a, x_i).$$

Then, interchanging the supremum in the first term and applying Equation (2.7) with $n = N+2$, for the second term, we have that

$$A_1 \approx \sup_{N+1 \le k} 2^{-\frac{k(1-\beta)}{r}} \operatorname*{ess\,sup}_{t \in (x_{k-1}, x_k)} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t) + \sup_{N+2 \le k} 2^{-\frac{k(1-\beta)}{r}} \left(\int_{x_{k-1}}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a, x_{k-1}).$$

Note that, for any $k \ge N+2$, we have

$$V_p(a,t) \approx V_p(a, x_{k-1}) + V_p(x_{k-1}, t), \quad \text{for every} \quad t \in (x_{k-1}, x_k). \tag{4.1}$$

Then, in view of Equation (4.1),

$$A_1 \approx \sup_{N+1\le k} 2^{-\frac{k(1-\beta)}{r}} \operatorname*{ess\,sup}_{t\in(x_{k-1},x_k)} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{1}{q}} V_p(x_{k-1},t) + \sup_{N+2\le k} 2^{-\frac{k(1-\beta)}{r}} \left(\int_{x_{k-1}}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{1}{q}} V_p(a,x_{k-1})$$

$$\lesssim \mathcal{A}_1 + \mathcal{B}_1.$$

Then, we have that $\mathcal{A}_1 + \mathcal{B}_1 \le A_1 + \mathcal{B}_1 \lesssim \mathcal{A}_1 + \mathcal{B}_1$.

It remains to show that $A_1 + \mathcal{B}_1 \approx C_1$. Applying Equation (2.10) with $\alpha = \frac{1-\beta}{r}$,

$$A_1 \approx \operatorname*{ess\,sup}_{x\in(a,b)} \left(\int_x^b w\right)^{\frac{1-\beta}{r}} \operatorname*{ess\,sup}_{t\in(a,x)} \left(\int_t^x \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{1}{q}} V_p(a,t)$$

holds, and interchanging supremum gives that

$$\begin{aligned} A_1 &\approx \operatorname*{ess\,sup}_{t\in(a,b)} V_p(a,t) \operatorname*{ess\,sup}_{x\in(t,b)} \left(\int_x^b w\right)^{\frac{1-\beta}{r}} \left(\int_t^x \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{1}{q}} \\ &\le \operatorname*{ess\,sup}_{t\in(a,b)} V_p(a,t) \left(\int_t^b \mathcal{W}(s)^{-\beta} w(s) \left(\int_t^s \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}} ds\right)^{\frac{1}{r}} = C_1. \end{aligned} \tag{4.2}$$

On the other hand, applying Equation (2.9) with $\alpha = 1-\beta$, then using Equation (2.5) with $n = k$, we obtain for any $k \ge N$ that

$$\int_{x_k}^b \mathcal{W}(s)^{-\beta} w(s) \left(\int_{x_k}^s \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}} ds \approx \sum_{i=k+1}^{\infty} 2^{-i(1-\beta)} \left(\int_{x_k}^{x_i} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}} \approx \sum_{i=k}^{\infty} 2^{-i(1-\beta)} \left(\int_{x_i}^{x_{i+1}} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}}. \tag{4.3}$$

Therefore, in view of Equation (4.3),

$$\begin{aligned} \mathcal{B}_1 &\approx \sup_{N+1\le k} \left(\int_{x_k}^b \mathcal{W}(s)^{-\beta} w(s) \left(\int_{x_k}^s \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}} ds\right)^{\frac{1}{r}} V_p(a,x_k) \\ &\le \sup_{N+1\le k} \operatorname*{ess\,sup}_{t\in(x_{k-1}\ x_k)} \left(\int_t^b \mathcal{W}(s)^{-\beta} w(s) \left(\int_t^s \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}} ds\right)^{\frac{1}{r}} V_p(a,t) = C_1. \end{aligned} \tag{4.4}$$

Thus, combining Equation (4.2) with Equation (4.4), we have that $A_1 + \mathcal{B}_1 \lesssim C_1$.

Conversely, using Equation (4.1), we have

$$\begin{aligned} C_1 &\approx \sup_{N+1\le k} \left(\int_{x_k}^b \mathcal{W}(s)^{-\beta} w(s) \left(\int_{x_k}^s \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{r}{q}} ds\right)^{\frac{1}{r}} V_p(x_{k-1},x_k) \\ &\quad + \sup_{N+1\le k} 2^{-\frac{k(1-\beta)}{r}} \operatorname*{ess\,sup}_{t\in(x_{k-1},x_k)} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{1}{q}} V_p(x_{k-1},t) \end{aligned}$$

$$+\sup_{N+1\le k}\operatorname*{ess\,sup}_{t\in(x_{k-1},x_k)}\left(\int_t^{x_k}\mathcal{W}(s)^{-\beta}w(s)\left(\int_t^s \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}ds\right)^{\frac{1}{r}}V_p(x_{k-1},t)$$

$$+\sup_{N+2\le k}\left(\int_{x_{k-1}}^b\mathcal{W}(s)^{-\beta}w(s)\left(\int_{x_{k-1}}^s \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}ds\right)^{\frac{1}{r}}V_p(a,x_{k-1}).$$

Then, using Equation (4.3), we arrive at

$$C_1\lesssim \sup_{N+1\le k}2^{-\frac{k(1-\beta)}{r}}\operatorname*{ess\,sup}_{t\in(x_{k-1},x_k)}\left(\int_t^{x_k}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{1}{q}}V_p(a,t)+\sup_{N+1\le k}\left(\sum_{i=k}^{\infty}2^{-i(1-\beta)}\left(\int_{x_i}^{x_{i+1}}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}\right)^{\frac{1}{r}}V_p(a,x_k)$$

$$\le A_1+\mathcal{B}_1. \tag{4.5}$$

As a result, we arrive at the conclusion that the best constant C in Equation (1.5) satisfies $C\approx C_1$.

(ii) Let $r<p\le q$. Then, we have from [Theorem 3.6, (ii)] that the best constant in Equation (1.5) satisfies $C\approx\mathcal{A}_2+\mathcal{B}_2$. We will start by showing that $\mathcal{A}_2+\mathcal{B}_2\approx A_2+B_2$, where

$$A_2:=\left(\sum_{k=N+1}^{\infty}2^{-k(1-\beta)\frac{p}{p-r}}\operatorname*{ess\,sup}_{t\in(a,x_k)}\left(\int_t^{x_k}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{pr}{q(p-r)}}V_p(a,t)^{\frac{pr}{p-r}}\right)^{\frac{p-r}{pr}}$$

and

$$B_2:=\left(\sum_{k=N+1}^{\infty}2^{-k(1-\beta)}\left(\int_{x_k}^{x_{k+1}}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}\left(\sum_{i=k+2}^{\infty}2^{-i(1-\beta)}\left(\int_{x_i}^{x_{i+1}}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}\right)^{\frac{r}{p-r}}V_p(a,x_k)^{\frac{pr}{p-r}}\right)^{\frac{p-r}{pr}}. \tag{4.6}$$

We have $\mathcal{A}_2\le A_2$ by the definitions of $\mathcal{A}_2$ and A_2. On the other hand,

$$\mathcal{B}_2\approx B_2+\left(\sum_{k=N+1}^{\infty}2^{-k(1-\beta)\frac{p}{p-r}}\left(\int_{x_k}^{x_{k+1}}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{pr}{q(p-r)}}V_p(a,x_k)^{\frac{pr}{p-r}}\right)^{\frac{p-r}{pr}}$$

$$+\left(\sum_{k=N+1}^{\infty}2^{-k(1-\beta)\frac{p}{p-r}}\left(\int_{x_k}^{x_{k+1}}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}\left(\int_{x_{k+1}}^{x_{k+2}}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r^2}{q(p-r)}}V_p(a,x_k)^{\frac{pr}{p-r}}\right)^{\frac{p-r}{pr}}$$

$$\lesssim B_2+\left(\sum_{k=N+1}^{\infty}2^{-k(1-\beta)\frac{p}{p-r}}\left(\int_{x_k}^{x_{k+2}}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{pr}{q(p-r)}}V_p(a,x_k)^{\frac{pr}{p-r}}\right)^{\frac{p-r}{pr}}$$

$$\lesssim B_2+\left(\sum_{k=N+1}^{\infty}2^{-k(1-\beta)\frac{p}{p-r}}\operatorname*{ess\,sup}_{t\in(a,x_k)}\left(\int_t^{x_{k+2}}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{pr}{q(p-r)}}V_p(a,t)^{\frac{pr}{p-r}}\right)^{\frac{p-r}{pr}}$$

$$\lesssim B_2+A_2 \tag{4.7}$$

holds, hence $\mathcal{A}_2+\mathcal{B}_2\lesssim A_2+B_2$.

Next, we will show that $A_2 \lesssim \mathcal{A}_2 + \mathcal{B}_2$. Observe that

$$A_2 = \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)\frac{p}{p-r}} \sup_{N+1\le i\le k} \operatorname*{ess\,sup}_{t\in(x_{i-1},x_i)} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} V_p(a,t)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}$$

$$\approx \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)\frac{p}{p-r}} \sup_{N+1\le i\le k} \operatorname*{ess\,sup}_{t\in(x_{i-1},x_i)} \left(\int_t^{x_i} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} V_p(a,t)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}$$

$$+ \left(\sum_{k=N+2}^{\infty} 2^{-k(1-\beta)\frac{p}{p-r}} \sup_{N+1\le i< k} \left(\int_{x_i}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} V_p(a,x_i)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}.$$

Applying Equation (2.3) for the first term and Equation (2.8) for the second term, we obtain that

$$A_2 \approx \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)\frac{p}{p-r}} \operatorname*{ess\,sup}_{t\in(x_{k-1},x_k)} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} V_p(a,t)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}$$

$$+ \left(\sum_{k=N+2}^{\infty} 2^{-k(1-\beta)\frac{p}{p-r}} \left(\int_{x_{k-1}}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} V_p(a,x_{k-1})^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}.$$

Using Equation (4.1) we arrive at

$$A_2 \lesssim \mathcal{A}_2 + \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)\frac{p}{p-r}} \left(\int_{x_k}^{x_{k+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} V_p(a,x_k)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}} \lesssim \mathcal{A}_2 + \mathcal{B}_2.$$

Furthermore, it is clear from the definitions of $\mathcal{B}_2$ and B_2 that $B_2 \le \mathcal{B}_2$. Then, we have $A_2 + B_2 \lesssim \mathcal{A}_2 + \mathcal{B}_2$, as well. Consequently, $C \approx A_2 + B_2$ holds.

Next, we will prove that $A_2 + B_2 \approx C_2 + C_3$. First of all, applying Equation (2.9) with $\alpha = \frac{p(1-\beta)}{p-r}$ and

$$h(x) = \operatorname*{ess\,sup}_{t\in(a,x)} \left(\int_t^{x} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} V_p(a,t)^{\frac{pr}{p-r}}, \quad x \in (a,b),$$

it is clear that

$$A_2 \approx \left(\int_a^b \mathcal{W}(x)^{\frac{p(1-\beta)}{p-r}-1} w(x) \operatorname*{ess\,sup}_{t\in(a,x)} \left(\int_t^{x} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} V_p(a,t)^{\frac{pr}{p-r}} dx \right)^{\frac{p-r}{pr}} = C_2. \tag{4.8}$$

On the other hand, using Equation (4.3),

$$B_2 \approx \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)} \left(\int_{x_{k+2}}^{b} \mathcal{W}(s)^{-\beta} w(s) \left(\int_{x_{k+2}}^{s} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} ds \right)^{\frac{r}{p-r}} \left(\int_{x_k}^{x_{k+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} V_p(a,x_k)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}$$

$$\le \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)} \left(\int_{x_{k+2}}^{b} \mathcal{W}(s)^{-\beta} w(s) \left(\int_{x_{k+2}}^{s} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} ds \right)^{\frac{r}{p-r}} \operatorname*{ess\,sup}_{t\in(a,x_{k+1})} \left(\int_t^{x_{k+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}$$

$$\approx \left(\sum_{k=N+1}^{\infty}\int_{x_{k+1}}^{x_{k+2}} \mathcal{W}(x)^{-\beta}w(x)dx\left(\int_{x_{k+2}}^{b} \mathcal{W}(s)^{-\beta}w(s)\left(\int_{x_{k+2}}^{s} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}ds\right)^{\frac{r}{p-r}} \operatorname*{ess\,sup}_{t\in(a,x_{k+1})}\left(\int_t^{x_{k+1}} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}}\right)^{\frac{p-r}{pr}}$$

$$\leq \left(\sum_{k=N+1}^{\infty}\int_{x_{k+1}}^{x_{k+2}} \mathcal{W}(x)^{-\beta}w(x)\left(\int_{x}^{b} \mathcal{W}(s)^{-\beta}w(s)\left(\int_{x}^{s} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}ds\right)^{\frac{r}{p-r}} \operatorname*{ess\,sup}_{t\in(a,x)}\left(\int_t^{x} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}}dx\right)^{\frac{p-r}{pr}}$$

$$\leq C_3. \tag{4.9}$$

Combination of Equations (4.8) and (4.9) yield that $A_2 + B_2 \lesssim C_2 + C_3$.

Conversely,

$$C_3 = \left(\sum_{k=N+1}^{\infty}\int_{x_{k-1}}^{x_k}\left(\int_{x}^{b} \mathcal{W}(s)^{-\beta}w(s)\left(\int_{x}^{s} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}ds\right)^{\frac{r}{p-r}} \mathcal{W}(x)^{-\beta}w(x)\operatorname*{ess\,sup}_{t\in(a,x)}\left(\int_t^{x} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}}dx\right)^{\frac{p-r}{pr}}$$

$$\approx \left(\sum_{k=N+1}^{\infty}\int_{x_{k-1}}^{x_k}\left(\int_{x}^{x_k} \mathcal{W}(s)^{-\beta}w(s)\left(\int_{x}^{s} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}ds\right)^{\frac{r}{p-r}} \mathcal{W}(x)^{-\beta}w(x)\operatorname*{ess\,sup}_{t\in(a,x)}\left(\int_t^{x} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}}dx\right)^{\frac{p-r}{pr}}$$

$$+ \left(\sum_{k=N+1}^{\infty}\int_{x_{k-1}}^{x_k}\left(\int_{x_k}^{b} \mathcal{W}(s)^{-\beta}w(s)\left(\int_{x}^{s} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}ds\right)^{\frac{r}{p-r}} \mathcal{W}(x)^{-\beta}w(x)\operatorname*{ess\,sup}_{t\in(a,x)}\left(\int_t^{x} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}}dx\right)^{\frac{p-r}{pr}}$$

$$\approx \left(\sum_{k=N+1}^{\infty}\int_{x_{k-1}}^{x_k}\left(\int_{x}^{x_k} \mathcal{W}(s)^{-\beta}w(s)\left(\int_{x}^{s} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}ds\right)^{\frac{r}{p-r}} \mathcal{W}(x)^{-\beta}w(x)\operatorname*{ess\,sup}_{t\in(a,x)}\left(\int_t^{x} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}}dx\right)^{\frac{p-r}{pr}}$$

$$+ \left(\sum_{k=N+1}^{\infty}\left(\int_{x_k}^{b} \mathcal{W}(s)^{-\beta}w(s)ds\right)^{\frac{r}{p-r}}\int_{x_{k-1}}^{x_k}\left(\int_{x}^{x_k} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r^2}{q(p-r)}} \mathcal{W}(x)^{-\beta}w(x)\operatorname*{ess\,sup}_{t\in(a,x)}\left(\int_t^{x} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}}dx\right)^{\frac{p-r}{pr}}$$

$$+ \left(\sum_{k=N+1}^{\infty}\int_{x_{k-1}}^{x_k}\left(\int_{x_k}^{b} \mathcal{W}(s)^{-\beta}w(s)\left(\int_{x_k}^{s} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}ds\right)^{\frac{r}{p-r}} \mathcal{W}(x)^{-\beta}w(x)\operatorname*{ess\,sup}_{t\in(a,x)}\left(\int_t^{x} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}}dx\right)^{\frac{p-r}{pr}}.$$

Since

$$\left(\int_{x_k}^{b} \mathcal{W}(s)^{-\beta}w(s)ds\right)^{\frac{r}{p-r}} \approx 2^{-k(1-\beta)\frac{r}{p-r}},$$

$$C_3 \approx \left(\sum_{k=N+1}^{\infty}\int_{x_{k-1}}^{x_k}\left(\int_{x}^{x_k} \mathcal{W}(s)^{-\beta}w(s)\left(\int_{x}^{s} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}ds\right)^{\frac{r}{p-r}} \mathcal{W}(x)^{-\beta}w(x)\operatorname*{ess\,sup}_{t\in(a,x)}\left(\int_t^{x} \mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}}dx\right)^{\frac{p-r}{pr}}$$

$$+\left(\sum_{k=N+1}^{\infty}2^{-k\frac{(1-\beta)r}{p-r}}\int_{x_{k-1}}^{x_k}\left(\int_x^{x_k}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r^2}{q(p-r)}}\mathcal{W}(x)^{-\beta}w(x)\operatorname*{ess\,sup}_{t\in(a,x)}\left(\int_t^x\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}V_p(a,t)^{\frac{pr}{p-r}}dx\right)^{\frac{p-r}{pr}}$$

$$+\left(\sum_{k=N+1}^{\infty}\left(\int_{x_k}^{b}\mathcal{W}(s)^{-\beta}w(s)\left(\int_{x_k}^{s}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}ds\right)^{\frac{r}{p-r}}\int_{x_{k-1}}^{x_k}\mathcal{W}(x)^{-\beta}w(x)\operatorname*{ess\,sup}_{t\in(a,x)}\left(\int_t^x\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}V_p(a,t)^{\frac{pr}{p-r}}dx\right)^{\frac{p-r}{pr}}$$

$$=: C_{3,1}+C_{3,2}+C_{3,3}.$$

Note that

$$\int_{x_{k-1}}^{x_k}\left(\int_x^{x_k}\mathcal{W}^{-\beta}w\right)^{\frac{r}{p-r}}\mathcal{W}(x)^{-\beta}w(x)dx\approx\left(\int_{x_{k-1}}^{x_k}\mathcal{W}^{-\beta}w\right)^{\frac{p}{p-r}}\approx 2^{-k(1-\beta)\frac{p}{p-r}}. \tag{4.10}$$

In view of Equation (4.10), we get

$$C_{3,1}\le\left(\sum_{k=N+1}^{\infty}\int_{x_{k-1}}^{x_k}\left(\int_x^{x_k}\mathcal{W}^{-\beta}w\right)^{\frac{r}{p-r}}\mathcal{W}(x)^{-\beta}w(x)\operatorname*{ess\,sup}_{t\in(a,x)}\left(\int_t^{x_k}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{pr}{q(p-r)}}V_p(a,t)^{\frac{pr}{p-r}}dx\right)^{\frac{p-r}{pr}}$$

$$\le\left(\sum_{k=N+1}^{\infty}\int_{x_{k-1}}^{x_k}\left(\int_x^{x_k}\mathcal{W}^{-\beta}w\right)^{\frac{r}{p-r}}\mathcal{W}(x)^{-\beta}w(x)dx\operatorname*{ess\,sup}_{t\in(a,x_k)}\left(\int_t^{x_k}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{pr}{q(p-r)}}V_p(a,t)^{\frac{pr}{p-r}}\right)^{\frac{p-r}{pr}}$$

$$\approx A_2$$

and

$$C_{3,2}\le\left(\sum_{k=N+1}^{\infty}2^{-k\frac{(1-\beta)r}{p-r}}\int_{x_{k-1}}^{x_k}\mathcal{W}(x)^{-\beta}w(x)\operatorname*{ess\,sup}_{t\in(a,x)}\left(\int_t^{x_k}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r^2}{q(p-r)}}\left(\int_t^{x}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{r}{q}}V_p(a,t)^{\frac{pr}{p-r}}dx\right)^{\frac{p-r}{pr}}$$

$$\le\left(\sum_{k=N+1}^{\infty}2^{-k\frac{(1-\beta)r}{p-r}}\int_{x_{k-1}}^{x_k}\mathcal{W}(x)^{-\beta}w(x)\operatorname*{ess\,sup}_{t\in(a,x)}\left(\int_t^{x_k}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{pr}{q(p-r)}}V_p(a,t)^{\frac{pr}{p-r}}dx\right)^{\frac{p-r}{pr}}$$

$$\le\left(\sum_{k=N+1}^{\infty}2^{-k\frac{(1-\beta)r}{p-r}}\int_{x_{k-1}}^{x_k}\mathcal{W}(x)^{-\beta}w(x)dx\operatorname*{ess\,sup}_{t\in(a,x_k)}\left(\int_t^{x_k}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{pr}{q(p-r)}}V_p(a,t)^{\frac{pr}{p-r}}\right)^{\frac{p-r}{pr}}$$

$$\approx\left(\sum_{k=N+1}^{\infty}2^{-k\frac{(1-\beta)p}{p-r}}\operatorname*{ess\,sup}_{t\in(a,x_k)}\left(\int_t^{x_k}\mathcal{W}^{\frac{\beta q}{r}}u\right)^{\frac{pr}{q(p-r)}}V_p(a,t)^{\frac{pr}{p-r}}\right)^{\frac{p-r}{pr}}$$

$$=A_2.$$

Furthermore, using Equation (4.10) once more,

$$C_{3,3} \lesssim \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)} \left(\int_{x_k}^{b} \mathcal{W}(s)^{-\beta} w(s) \left(\int_{x_k}^{s} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} ds \right)^{\frac{r}{p-r}} \operatorname*{ess\,sup}_{t\in(a,x_k)} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}$$

$$= \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)} \left(\int_{x_k}^{b} \mathcal{W}(s)^{-\beta} w(s) \left(\int_{x_k}^{s} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} ds \right)^{\frac{r}{p-r}} \sup_{N+1\le i\le k} \operatorname*{ess\,sup}_{t\in(x_{i-1},x_i)} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}$$

$$\approx \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)} \left(\int_{x_k}^{b} \mathcal{W}(s)^{-\beta} w(s) \left(\int_{x_k}^{s} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} ds \right)^{\frac{r}{p-r}} \sup_{N+1\le i\le k} \operatorname*{ess\,sup}_{t\in(x_{i-1},x_i)} \left(\int_t^{x_i} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}$$

$$+ \left(\sum_{k=N+2}^{\infty} 2^{-k(1-\beta)} \left(\int_{x_k}^{b} \mathcal{W}(s)^{-\beta} w(s) \left(\int_{x_k}^{s} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} ds \right)^{\frac{r}{p-r}} \sup_{N+1\le i< k} \left(\int_{x_i}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} V_p(a,x_i)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}$$

$$=: I + II.$$

Since, the sequence $\{a_k\}_{k=N+1}^{\infty}$, with

$$a_k =: 2^{-k(1-\beta)} \left(\int_{x_k}^{b} \mathcal{W}(s)^{-\beta} w(s) \left(\int_{x_k}^{s} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} ds \right)^{\frac{r}{p-r}}$$

is geometrically decreasing, Equation (2.3) yields that

$$I \approx \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)} \left(\int_{x_k}^{b} \mathcal{W}(s)^{-\beta} w(s) \left(\int_{x_k}^{s} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} ds \right)^{\frac{r}{p-r}} \operatorname*{ess\,sup}_{t\in(x_{k-1},x_k)} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}.$$

Let $y_k \in [x_{k-1}, x_k]$, $N+1 \le k$ be such that

$$\operatorname*{ess\,sup}_{t\in(x_{k-1},x_k)} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} V_p(a,t)^{\frac{pr}{p-r}} \lesssim \left(\int_{y_k}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} V_p(a,y_k)^{\frac{pr}{p-r}}.$$

Then, we have

$$I \lesssim \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)} \left(\int_{x_k}^{b} \mathcal{W}(s)^{-\beta} w(s) \left(\int_{x_k}^{s} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} ds \right)^{\frac{r}{p-r}} \left(\int_{y_k}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} V_p(a,y_k)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}.$$

Thus, Equation (4.3) ensures that

$$I \lesssim \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)} \left(\sum_{i=k}^{\infty} 2^{-i(1-\beta)} \left(\int_{x_i}^{x_{i+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} \right)^{\frac{r}{p-r}} \left(\int_{y_k}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} V_p(a, y_k)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}$$

$$\leq \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)} \left(\sum_{i=k}^{\infty} 2^{-i(1-\beta)} \left(\int_{y_i}^{y_{i+2}} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} \right)^{\frac{r}{p-r}} \left(\int_{y_k}^{y_{k+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} V_p(a, y_k)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}.$$

Moreover,

$$2^{-k} \approx \int_{x_k}^{b} w \leq \int_{y_k}^{b} w \leq \int_{x_{k-1}}^{b} w \approx 2^{-(k-1)}, \quad N+1 \leq k.$$

As a result, $\{y_k\}_{k=N+1}^{\infty}$ is a discretizing sequence of $\mathcal{W}$, as well. This fact together with Equation (4.7) yield $I \lesssim \mathcal{B}_2 \lesssim A_2 + B_2$.

On the other hand, applying Equation (2.8) with

$$\tau_k = 2^{-k(1-\beta)} \left(\int_{x_k}^{b} \mathcal{W}(s)^{-\beta} w(s) \left(\int_{x_k}^{s} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} ds \right)^{\frac{r}{p-r}}, \quad \sigma_k = V_p(a, x_k)^{\frac{pr}{p-r}}, \quad \alpha = \frac{r}{q},$$

gives

$$II \approx \left(\sum_{k=N+2}^{\infty} 2^{-k(1-\beta)} \left(\int_{x_k}^{b} \mathcal{W}(s)^{-\beta} w(s) \left(\int_{x_k}^{s} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} ds \right)^{\frac{r}{p-r}} \left(\int_{x_{k-1}}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} V_p(a, x_{k-1})^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}.$$

Lastly, using Equations (4.3) and (4.7), we obtain that

$$II \approx \left(\sum_{k=N+2}^{\infty} 2^{-k(1-\beta)} \left(\sum_{i=k}^{\infty} 2^{-i(1-\beta)} \left(\int_{x_i}^{x_{i+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} \right)^{\frac{r}{1-r}} \left(\int_{x_{k-1}}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} V_p(a, x_{k-1})^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}$$

$$\leq A_2 + B_2.$$

Therefore,

$$C_3 \lesssim A_2 + B_2. \tag{4.11}$$

Finally, a combination of Equations (4.8) and (4.11) yield $C_2 + C_3 \approx A_2 + B_2$. Accordingly, the best constant C in Equation (1.5) satisfies $C \approx C_2 + C_3$.

(iii) Let $q < p \leq r$. According to [Theorem 3.6, (iii)], the best constant in Equation (1.5) satisfies $C \approx \mathcal{A}_3 + \mathcal{B}_1$. We will begin our proof by showing that $\mathcal{A}_3 + \mathcal{B}_1 \approx A_3 + \mathcal{B}_1$, where

$$A_3 := \sup_{N+1 \leq k < \infty} 2^{-\frac{k(1-\beta)}{r}} \left(\int_a^{x_k} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(a,t)^{\frac{pq}{p-q}} dt \right)^{\frac{p-q}{pq}}.$$

It is clear that $\mathcal{A}_3 \leq A_3$, the proof of this part is complete if we show that $A_3 \lesssim \mathcal{A}_3 + \mathcal{B}_1$. Assume that $\max\{\mathcal{A}_3, \mathcal{B}_1\} < \infty$. Then,

$$\left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(a,t)^{\frac{pq}{p-q}} dt \right)^{\frac{p-q}{pq}} < \infty, \quad k \geq N+1,$$

holds. Also, for each $t \in [x_{k-1}, x_k]$, $k \geq N+1$, we have

$$\left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t) \lesssim \left(\int_t^{x_k} \left(\int_s^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{q}{p-q}} \mathcal{W}(s)^{\frac{\beta q}{r}} u(s) V_p(a,s)^{\frac{pq}{p-q}} ds \right)^{\frac{p-q}{pq}}.$$

Therefore, we have

$$\lim_{t \to x_k -} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t) = 0. \tag{4.12}$$

In that case, integration by parts gives

$$\begin{aligned}
&\left(\int_a^{x_k} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(a,t)^{\frac{pq}{p-q}} dt \right)^{\frac{p-q}{pq}} \\
&\approx \left(\int_a^{x_k} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{p}{p-q}} d\left[V_p(a,t)^{\frac{pq}{p-q}} \right] \right)^{\frac{p-q}{pq}} + \lim_{t \to a+} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t) \\
&\approx \left(\sum_{i=N+1}^{k} \int_{x_{i-1}}^{x_i} \left(\int_t^{x_i} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{p}{p-q}} d\left[V_p(a,t)^{\frac{pq}{p-q}} \right] \right)^{\frac{p-q}{pq}} + \left(\sum_{i=N+1}^{k-1} \left(\int_{x_i}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{p}{p-q}} \int_{x_{i-1}}^{x_i} d\left[V_p(a,t)^{\frac{pq}{p-q}} \right] \right)^{\frac{p-q}{pq}} \\
&\quad + \lim_{t \to a+} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t).
\end{aligned}$$

Moreover, Minkowski's inequality with $\frac{p}{p-q} > 1$ yields that

$$\left(\sum_{i=N+1}^{k-1} \left(\int_{x_i}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{p}{p-q}} \int_{x_{i-1}}^{x_i} d\left[V_p(a,t)^{\frac{pq}{p-q}} \right] \right)^{\frac{p-q}{pq}} = \left(\sum_{i=N+1}^{k-1} \left(\sum_{j=i}^{k-1} \int_{x_j}^{x_{j+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{p}{p-q}} \int_{x_{i-1}}^{x_i} d\left[V_p(a,t)^{\frac{pq}{p-q}} \right] \right)^{\frac{p-q}{pq}}$$

$$\le \left(\sum_{j=N+1}^{k-1} \left(\int_{x_j}^{x_{j+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right) \left(\sum_{i=N+1}^{j} \int_{x_{i-1}}^{x_i} d\left[V_p(a,t)^{\frac{pq}{p-q}} \right] \right)^{\frac{p-q}{p}} \right)^{\frac{1}{q}}$$

$$= \left(\sum_{j=N+1}^{k-1} \left(\int_{x_j}^{x_{j+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right) \left(\int_{a}^{x_j} d\left[V_p(a,t)^{\frac{pq}{p-q}} \right] \right)^{\frac{p-q}{p}} \right)^{\frac{1}{q}}.$$

Then, we arrive at

$$\left(\int_a^{x_k} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(a,t)^{\frac{pq}{p-q}} dt \right)^{\frac{p-q}{pq}}$$
$$\lesssim \left(\sum_{i=N+1}^{k} \int_{x_{i-1}}^{x_i} \left(\int_t^{x_i} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{p}{p-q}} d\left[V_p(a,t)^{\frac{pq}{p-q}} \right] \right)^{\frac{p-q}{pq}} + \left(\sum_{j=N+1}^{k-1} \left(\int_{x_j}^{x_{j+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right) \left(\int_a^{x_j} d\left[V_p(a,t)^{\frac{pq}{p-q}} \right] \right)^{\frac{p-q}{p}} \right)^{\frac{1}{q}}$$
$$+ \lim_{t\to a+} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t). \tag{4.13}$$

Now, we are in a position to find the upper estimate for A_3. Using Equation (4.13), we have that

$$A_3 \lesssim \sup_{N+1\le k<\infty} 2^{-\frac{k(1-\beta)}{r}} \left(\sum_{i=N+1}^{k} \int_{x_{i-1}}^{x_i} \left(\int_t^{x_i} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{p}{p-q}} d\left[V_p(a,t)^{\frac{pq}{p-q}} \right] \right)^{\frac{p-q}{pq}}$$
$$+ \sup_{N+2\le k<\infty} 2^{-\frac{k(1-\beta)}{r}} \left(\sum_{j=N+1}^{k-1} \left(\int_{x_j}^{x_{j+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right) \left(\int_a^{x_j} d\left[V_p(a,t)^{\frac{pq}{p-q}} \right] \right)^{\frac{p-q}{p}} \right)^{\frac{1}{q}}$$
$$+ \sup_{N+1\le k<\infty} 2^{-\frac{k(1-\beta)}{r}} \lim_{t\to a+} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t).$$

Further, Equation (2.1) yields

$$A_3 \lesssim \sup_{N+1\le k<\infty} 2^{-\frac{k(1-\beta)}{r}} \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{p}{p-q}} d\left[V_p(a,t)^{\frac{pq}{p-q}} \right] \right)^{\frac{p-q}{pq}}$$
$$+ \sup_{N+1\le k<\infty} 2^{-\frac{k(1-\beta)}{r}} \left(\int_{x_k}^{x_{k+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} \left(\int_a^{x_k} d\left[V_p(a,t)^{\frac{pq}{p-q}} \right] \right)^{\frac{p-q}{pq}}$$
$$+ \sup_{N+1\le k<\infty} 2^{-\frac{k(1-\beta)}{r}} \lim_{t\to a+} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t)$$
$$=: A_{3,1} + A_{3,2} + A_{3,3}.$$

Integrating by parts again, we have that

$$A_{3,1} \lesssim \sup_{N+1\le k<\infty} 2^{-\frac{k(1-\beta)}{r}} \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(a,t)^{\frac{pq}{p-q}} dt \right)^{\frac{p-q}{pq}}.$$

Thus, Equation (4.1) gives

$$\begin{aligned} A_{3,1} &\lesssim \sup_{N+1\le k<\infty} 2^{-\frac{k(1-\beta)}{r}} \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(x_{k-1},t)^{\frac{pq}{p-q}} dt \right)^{\frac{p-q}{pq}} \\ &\quad + \sup_{N+2\le k<\infty} 2^{-\frac{k(1-\beta)}{r}} \left(\int_{x_{k-1}}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a, x_{k-1}) \\ &\lesssim \mathcal{A}_3 + \mathcal{B}_1. \end{aligned} \tag{4.14}$$

Additionally,

$$A_{3,2} \lesssim \sup_{N+1\le k<\infty} 2^{-\frac{k(1-\beta)}{r}} \left(\int_{x_k}^{x_{k+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a, x_k) \le \mathcal{B}_1. \tag{4.15}$$

Lastly, we will find a suitable upper estimate for $A_{3,3}$. To this end, we will treat the cases $N = -\infty$ and $N > -\infty$, separately. Observe that, if $N = -\infty$, since $x_i \to a$ if $i \to -\infty$, we have for any k,

$$\lim_{t\to a+} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t) = \lim_{i\to-\infty} \left(\int_{x_i}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,x_i) \le \sup_{i<k} \left(\int_{x_i}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,x_i). \tag{4.16}$$

Then, then using Equation (4.16) together with Equation (2.7), we get

$$A_{3,3} \lesssim \sup_{k\in\mathbb{Z}} 2^{-\frac{k(1-\beta)}{r}} \sup_{i<k} \left(\int_{x_i}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,x_i) \approx \sup_{k\in\mathbb{Z}} 2^{-\frac{k(1-\beta)}{r}} \left(\int_{x_{k-1}}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,x_{k-1}) \lesssim \mathcal{B}_1.$$

On the other hand, if $N > -\infty$

$$\begin{aligned} \lim_{t\to a+} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t) &\le \operatorname*{ess\,sup}_{t\in(a,x_{N+1})} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t) \\ &\approx \operatorname*{ess\,sup}_{t\in(a,x_{N+1})} \left(\int_t^{x_{N+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t) + \left(\int_{x_{N+1}}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a, x_{N+1}). \end{aligned} \tag{4.17}$$

Additionally, it is easy to see that

$$\operatorname*{ess\,sup}_{\tau\in(x,y)} \left(\int_\tau^y \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(x,\tau) \le \left[\int_x^y \left(\int_t^y \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(x,t)^{\frac{pq}{p-q}} dt \right]^{\frac{p-q}{pq}}. \tag{4.18}$$

First, using Equation (4.17), we get

$$A_{3,3} \lesssim \sup_{N+1\le k<\infty} 2^{-\frac{k(1-\beta)}{r}} \operatorname*{ess\,sup}_{t\in(a,x_{N+1})} \left(\int_t^{x_{N+1}} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{1}{q}} V_p(a,t) + \sup_{N+2\le k<\infty} 2^{-\frac{k(1-\beta)}{r}} \sup_{N+1\le i<k} \left(\int_{x_i}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{1}{q}} V_p(a,x_i).$$

Then, using Equation (4.18) for the first term and applying Equation (2.7) for the second term, we get

$$A_{3,3} \lesssim \sup_{N+1\le k<\infty} 2^{-\frac{k(1-\beta)}{r}} \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(a,t)^{\frac{pq}{p-q}} dt\right)^{\frac{p-q}{pq}}$$

$$+ \sup_{N+2\le k<\infty} 2^{-\frac{k(1-\beta)}{r}} \left(\int_{x_{k-1}}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{1}{q}} V_p(a,x_{k-1}).$$

Finally, using Equation (4.1), we arrive at

$$A_{3,3} \lesssim \mathcal{A}_3 + \sup_{N+2\le k<\infty} 2^{-\frac{k(1-\beta)}{r}} \left(\int_{x_{k-1}}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{1}{q}} V_p(a,x_{k-1}) \lesssim \mathcal{A}_3 + \mathcal{B}_1.$$

Consequently, we have for any $N \in \mathbb{Z} \cup \{-\infty\}$, $A_{3,3} \lesssim \mathcal{A}_3 + \mathcal{B}_1$. Combining the last estimate with Equations (4.14) and (4.15), we arrive at $A_3 \lesssim \mathcal{A}_3 + \mathcal{B}_1$, hence, $C \approx A_3 + \mathcal{B}_1$.

Let us now continue the proof by showing $A_3 + \mathcal{B}_1 \approx C_1 + C_4$.

To this end, taking $\alpha = \frac{1-\beta}{r}$ and

$$h(x) = \left(\int_a^x \left(\int_t^x \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(a,t)^{\frac{pq}{p-q}} dt\right)^{\frac{p-q}{pq}}, \quad x \in (a,b)$$

in Equation (2.10), we have that $A_3 \approx C_4$. Moreover, we have already shown in Equations (4.4) and (4.5) that $\mathcal{B}_1 \lesssim C_1 \lesssim A_1 + \mathcal{B}_1$. Furthermore, Equation (4.18) yields that $A_1 \lesssim A_3$. Consequently, we have $A_3 + \mathcal{B}_1 \lesssim C_4 + C_1 \lesssim A_3 + \mathcal{B}_1$, which is the desired estimate.

(iv) Let $\max\{r,q\} < p$. Then, using [Theorem 3.6, (iv)], we have that $C \approx \mathcal{B}_2 + \mathcal{A}_4$. First of all, we will show that $\mathcal{B}_2 + \mathcal{A}_4 \approx B_2 + A_4$, where

$$A_4 := \left(\sum_{k=N+1}^{\infty} 2^{-k\frac{p(1-\beta)}{p-r}} \left(\int_a^{x_k} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u\right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(a,t)^{\frac{pq}{p-q}} dt\right)^{\frac{r(p-q)}{q(p-r)}}\right)^{\frac{p-r}{pr}},$$

and B_2 is defined in Equation (4.6).

It is clear that $\mathcal{A}_4 \le A_4$. In Equation (4.7), we have already shown that $\mathcal{B}_2 \lesssim A_2 + B_2$. Moreover, analogously as in the previous proof, using Equation (4.18), one can easily see that $A_2 \lesssim A_4$. Thus, $\mathcal{B}_2 + \mathcal{A}_4 \lesssim B_2 + A_4$ follows.

It remains to prove that $B_2 + A_4 \lesssim \mathcal{B}_2 + \mathcal{A}_4$. Assume that $\max\{\mathcal{A}_4, \mathcal{B}_2\} < \infty$. Then, using the same steps as in the previous case, we can see that Equation (4.12) holds; therefore, Equation (4.13) is true in this case, as well.

Applying Equation (4.13) combined with Equation (2.5), we obtain that

$$
\begin{aligned}
A_4 \lesssim & \left(\sum_{k=N+1}^{\infty} 2^{-k\frac{p(1-\beta)}{p-r}} \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{p}{p-q}} d\left[V_p(a,t)^{\frac{pq}{p-q}} \right] \right)^{\frac{r(p-q)}{q(p-r)}} \right)^{\frac{p-r}{pr}} \\
& + \left(\sum_{k=N+2}^{\infty} 2^{-k\frac{p(1-\beta)}{p-r}} \left(\int_{x_{k-1}}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} \left(\int_a^{x_{k-1}} d\left[V_p(a,t)^{\frac{pq}{p-q}} \right] \right)^{\frac{r(p-q)}{q(p-r)}} \right)^{\frac{p-r}{pr}} \\
& + \left(\sum_{k=N+1}^{\infty} 2^{-k\frac{p(1-\beta)}{p-r}} \left[\lim_{t\to a+} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{1}{q}} V_p(a,t) \right]^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}} \\
=: \ & A_{4,1} + A_{4,2} + A_{4,3}
\end{aligned}
$$

holds.

As in the proof of the previous case, using integration by parts in combination with Equation (4.1), we have that

$$
\begin{aligned}
A_{4,1} & \lesssim \left(\sum_{k=N+1}^{\infty} 2^{-k\frac{p(1-\beta)}{p-r}} \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(a,t)^{\frac{pq}{p-q}} dt \right)^{\frac{r(p-q)}{q(p-r)}} \right)^{\frac{p-r}{pr}} \\
& \approx \mathcal{A}_4 + \left(\sum_{k=N+2}^{\infty} 2^{-k\frac{p(1-\beta)}{p-r}} \left(\int_{x_{k-1}}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} V_p(a,x_{k-1})^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}} \\
& \approx \mathcal{A}_4 + \left(\sum_{k=N+1}^{\infty} 2^{-k(1-\beta)} \left(\int_{x_k}^{x_{k+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r}{q}} 2^{-k\frac{r(1-\beta)}{p-r}} \left(\int_{x_k}^{x_{k+1}} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{r^2}{q(p-r)}} V_p(a,x_k)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}} \\
& \lesssim \mathcal{A}_4 + \mathcal{B}_2.
\end{aligned} \tag{4.19}
$$

On the other hand, it is clear that

$$
A_{4,2} \lesssim \left(\sum_{k=N+2}^{\infty} 2^{-k\frac{p(1-\beta)}{p-r}} \left(\int_{x_{k-1}}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} V_p(a,x_{k-1})^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}} \lesssim \mathcal{B}_2. \tag{4.20}
$$

Furthermore, if $N = -\infty$, using Equation (4.16), and then applying Equation (2.8), we get

$$
A_{4,3} \lesssim \left(\sum_{k=-\infty}^{\infty} 2^{-k\frac{p(1-\beta)}{p-r}} \sup_{i<k} \left(\int_{x_i}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} V_p(a,x_i)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}
$$

$$\approx \left(\sum_{k=-\infty}^{\infty} 2^{-k\frac{p(1-\beta)}{p-r}} \left(\int_{x_{k-1}}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} V_p(a, x_{k-1})^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}} \lesssim \mathcal{B}_2.$$

If $N > -\infty$, Equation (4.17) together with Equation (4.18) yields,

$$A_{4,3} \lesssim \left(\sum_{k=N+1}^{\infty} 2^{-k\frac{p(1-\beta)}{p-r}} \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(a,t)^{\frac{pq}{p-q}} dt \right)^{\frac{r(p-q)}{q(p-r)}} \right)^{\frac{p-r}{pr}}$$

$$+ \left(\sum_{k=N+2}^{\infty} 2^{-k\frac{p(1-\beta)}{p-r}} \sup_{N+1 \le i < k} \left(\int_{x_i}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} V_p(a, x_i)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}}.$$

Applying Equation (4.1) to the first term and Equation (2.8) to the second term, we have

$$A_{4,3} \lesssim \mathcal{A}_4 + \left(\sum_{k=N+2}^{\infty} 2^{-k\frac{p(1-\beta)}{p-r}} \left(\int_{x_{k-1}}^{x_k} \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{pr}{q(p-r)}} V_p(a, x_{k-1})^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}} \lesssim \mathcal{A}_4 + \mathcal{B}_2.$$

Thus, for any $N \in \mathbb{Z} \cup \{-\infty\}$, we arrive at $A_{4,3} \lesssim \mathcal{A}_4 + \mathcal{B}_2$. This fact, combined with Equations (4.19) and (4.20) yields $A_4 \lesssim \mathcal{A}_4 + \mathcal{B}_2$. Since $B_2 \le \mathcal{B}_2$, we have $A_4 + B_2 \lesssim \mathcal{A}_4 + \mathcal{B}_2$ and consequently, $C \approx B_2 + A_4$.

Now, we will show that $B_2 + A_4 \approx C_3 + C_5$. Applying Equation (2.9) with $\alpha = \frac{p(1-\beta)}{p-r}$ and

$$h(x) = \left(\int_a^x \left(\int_t^x \mathcal{W}^{\frac{\beta q}{r}} u \right)^{\frac{q}{p-q}} \mathcal{W}(t)^{\frac{\beta q}{r}} u(t) V_p(a,t)^{\frac{pq}{p-q}} dt \right)^{\frac{r(p-q)}{q(p-r)}}, \quad x \in (a,b),$$

it is clear that $A_4 \approx C_5$. We have also shown in Equation (4.9) that $B_2 \lesssim C_3$. Hence, it remains to show that $C_3 \lesssim A_4 + B_2$. To this end, we can use Equation (4.18), and obtain $A_2 \lesssim A_4$. Moreover, we know from Equation (4.11) that $C_3 \lesssim A_2 + B_2$. Consequently, $C_3 \lesssim A_4 + B_2$ holds and the proof is complete.

□

Proof of Theorem 1.2. We will prove that inequality (1.9) holds for all $f \in \mathfrak{M}^{\uparrow}(a,b)$ if and only if inequality

$$\left(\int_a^b \left(\int_a^x \left(\int_a^t h(\tau) d\tau \right)^{\frac{1}{p}} u(t) dt \right)^q w(x) dx \right)^{\frac{p}{q}} \le C^p \int_a^b h(x) \left(\int_x^b v \right) dx \tag{4.21}$$

holds for all $h \in \mathfrak{M}^+(a,b)$.

Assume that Equation (1.9) holds for all $f \in \mathfrak{M}^{\uparrow}(a,b)$. Substituting $f(x) = \left(\int_a^x h \right)^{\frac{1}{p}}$, $x \in (a,b)$ for $h \in \mathfrak{M}^+(a,b)$ in Equation (1.9) and applying Fubini's theorem on the right-hand side, Equation (4.21) follows.

Conversely, assume that Equation (4.21) holds for all $h \in \mathfrak{M}^+(a,b)$. Since any $f \in \mathfrak{M}^{\uparrow}(a,b)$, even if $f(0) > 0$, can be approximated pointwise from below by a function of the form $f(x)^p = \int_a^x h$, $x \in (a,b)$, then the validity of Equation (4.21) yields Equation (1.9). Therefore, the result follows from Theorem 1.1. □

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CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest to declare relevant to this paper's content.

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