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ON GAUSS–WEIERSTRASS TYPE INTEGRAL OPERATORS

Abstract. In this paper, we introduce a generalization of Gauss–Weierstrass operators based on q-integers using the q-integral and we call them q-Gauss–Weierstrass integral operators. For these operators, we obtain a convergence property in a weighted function space using Korovkin theory. Then we estimate the rate of convergence of these operators in terms of a weighted modulus of continuity. We also prove optimal global smoothness preservation property of these operators.

1. Introduction

Recently, in [2] a q-generalization of Gauss–Weierstrass and Picard singular integral operators was introduced by using the q-analogue of the Euler Gamma integral. In [1], we gave a different generalization of q-Picard singular integral operators by using the nonisotropic β-distance.

In this paper, we introduce a q-generalization of Gauss–Weierstrass singular integral operators by using the q-integral. In 1910, Jackson [5] defined and studied the q-integral. He also was the first to develop q-calculus in a systematic way. Nowadays there is a significant increase of activity in the area of the q-calculus due to its applications in mathematics and physics.

The purpose of this paper is to obtain the weighted approximation error of the new type Gauss–Weierstrass singular integral operators for functions of polynomial growth. This estimate will be in terms of a weighted modulus of continuity that we give below. Also we give a direct approximation result for these functions. We finally establish the optimal global smoothness of these operators by using the usual modulus of continuity.

Next we provide a summary of the mathematical notations and definitions used in this paper. All of the results can be found in [4] and [6]. Throughout this paper, we fix $q \in (0, 1)$.

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For $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$,

$$[n]_q = \frac{1-q^n}{1-q},$$

$$(\alpha;q)_n = \prod_{k=0}^{n-1}\left(1-\alpha q^k\right), \quad n = 1,2,\ldots, \ (-x;q)_\infty = \prod_{k=0}^{\infty}\left(1+xq^k\right),$$

$$[n]_q! = \begin{cases} [n]_q\,[n-1]_q \cdots [1]_q\,, & n = 1,\, 2, \ldots \\ 1 & n = 0. \end{cases}.$$

The q-derivative $D_q f$ of a real valued function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)\,x}, \quad \text{if } x \neq 0, \tag{1.1}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

The q-Jackson integrals and the q-improper integrals of a real valued function are defined as (see [5] and [9])

$$\int_0^a f(x)\, d_q x = (1-q)\, a \sum_{n=0}^{\infty} f(aq^n)\, q^n, \ a \in \mathbb{R},$$

and

$$\int_0^{\frac{\infty}{A}} f(x)\, d_q x = (1-q) \sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A}, \quad A > 0, \tag{1.2}$$

provided the sums converge absolutely.

One can define the Jackson integral in a generic interval $[a,b]$ as

$$\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x.$$

There are two important q-analogues of the exponential function:

$$E_q(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2}\frac{x^n}{[n]_q!} = (-(1-q)\,x;q)_\infty \tag{1.3}$$

and

$$e_q(x) = \sum_{n=0}^{\infty}\frac{x^n}{[n]_q!} = \frac{1}{((1-q)\,x;q)_\infty}. \tag{1.4}$$

Note that for $q \in (0,1)$ the series expansion of $e_q(x)$ has radius of convergence $\frac{1}{1-q}$. On the contrary, the series expansion of $E_q(x)$ converges for every real x.

The q-gamma integral is defined by [9]

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q(-qx)\, d_q x, \qquad t > 0 \tag{1.5}$$

which satisfies the following functional equation:

$$\Gamma_q(t+1) = [t]_q \, \Gamma_q(t),$$

where $[t]_q = \frac{1-q^t}{1-q}$ and $\Gamma_q(1) = 1$.

The change of variable formula (see [6]) for $u(x) = \gamma x^\beta$ is

$$\int_{u(a)}^{u(b)} f(u)\, d_q u = \int_a^b f(u(x))\, D_{q^{1/\beta}} u(x)\, d_{q^{1/\beta}} x. \tag{1.6}$$

2. Description of the operators

Definition 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For $n \in \mathbb{N}$, $q \in (0,1)$ and $x \in \mathbb{R}$, the q-Gauss–Weierstrass integral of f is

$$\begin{aligned} (2.1) \quad & \mathcal{W}_n(f; q, x) \\ & := \frac{\sqrt{[n]_q}\,(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}} f(x+t)\, E_{q^2}\left(-q^2 [n]_q \frac{t^2}{4}\right) d_q t. \end{aligned}$$

Lemma 1. *The operator $\mathcal{W}_n$ satisfies, for every $k \in \mathbb{N}$,*

$$\mathcal{W}_n\left(t^k; q, x\right) = \sum_{j=0}^{k} \binom{k}{j} \frac{2^j \Gamma_{q^2}\left(\frac{j+1}{2}\right)}{[n]_q^{\frac{j}{2}}\, \Gamma_{q^2}\left(\frac{1}{2}\right)} x^{k-j}.$$

Proof. From (2.1) we get

$$\begin{aligned} (2.2) \quad & \mathcal{W}_n\left(t^k; q, x\right) \\ & = \frac{\sqrt{[n]_q}\,(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}} (t+x)^k E_{q^2}\left(-q^2 [n]_q \frac{t^2}{4}\right) d_q t \\ & = \frac{\sqrt{[n]_q}\,(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \sum_{j=0}^{k} \binom{k}{j} x^{k-j} \int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}} t^j E_{q^2}\left(-q^2 [n]_q \frac{t^2}{4}\right) d_q t. \end{aligned}$$

Using (1.1) we can write the q-derivative of the equality $t=2\frac{\sqrt{u}}{\sqrt{[n]_q}}$ as

$$D_{q^2}(t)=\frac{2}{\sqrt{[n]_q}}\frac{\sqrt{u}-\sqrt{q^2u}}{(1-q^2)\,u}=\frac{2}{(q+1)\sqrt{[n]_q}\sqrt{u}}. \tag{2.3}$$

Also, using the change of variable formula (1.6) for q-integral with $\beta=\frac{1}{2}$, then from (2.3) and (1.5) we have

$$\int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}} t^j E_{q^2}\left(-q^2[n]_q\frac{t^2}{4}\right)d_qt=\frac{2^{j+1}}{(q+1)[n]_q^{\frac{j+1}{2}}}\int_0^{\frac{1}{1-q^2}} u^{\frac{j-1}{2}}E_{q^2}\left(-q^2u\right)d_{q^2}u$$
$$=\frac{2^{j+1}\Gamma_{q^2}\left(\frac{j+1}{2}\right)}{(q+1)[n]_q^{\frac{j+1}{2}}},$$

for $j=0,\ldots,k$. From (2.2) we have desired result. ∎

Remark 1. Note that q-Gauss–Weierstrass operators $\mathcal{W}_n$ given by (2.1) can be rewritten via an improper integral by using definition (1.2). From (1.3) we can easily see that $E_q\left(-\frac{q^n}{1-q}\right)=0$ for $n\le 0$. Thus we can write

$$\mathcal{W}_n(f;q,x)=\frac{\sqrt{[n]_q}\,(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\infty}{\sqrt{[n]_q}\sqrt{1-q^2}}\Big/2} f(x+t)\,E_{q^2}\left(-q^2[n]_q\frac{t^2}{4}\right)d_qt.$$

3. Approximation properties in a weighted space

In this section, by using a Bohman-Korovkin type theorem proved in [3], we present the direct approximation property of the operator $\mathcal{W}_n$ given by (2.1).

Let us denote by $B_2(\mathbb{R})$ the weighted space of real-valued functions f defined on $\mathbb{R}$ with the property $|f(x)|\le M_f\left(1+x^2\right)$ for all $x\in\mathbb{R}$, where M_f is a constant depending on the function f. We consider the weighted subspace $C_2(\mathbb{R})$ of $B_2(\mathbb{R})$ given by

$$C_2(\mathbb{R})=\{f\in B_2(\mathbb{R}): f \text{ continuous on } \mathbb{R}\}.$$

We also consider the space of functions $C_2^k(\mathbb{R})=\{f\in C_2(\mathbb{R}): \lim_{x\to\infty}\frac{f(x)}{1+x^2}=k\in\mathbb{R}\}$ equipped with the norm $\|f\|_2=\sup_{x\in\mathbb{R}}\frac{|f(x)|}{1+x^2}$.

Theorem A. *Let T_n be a sequence of linear positive operators mapping $C_2(\mathbb{R})$ into $B_2(\mathbb{R})$ and satisfying the conditions*

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\frac{|\mathcal{T}_n(t^\nu;x)-x^\nu|}{1+x^2}=0,\quad \textit{for } \nu=0,1,2.$$

Then, for any $f \in C_2^k(\mathbb{R})$,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{|\mathcal{T}_n(f;x) - f(x)|}{1+x^2} = 0,$$

and there exists a function $f^* \in C_2(\mathbb{R}) \backslash C_2^k(\mathbb{R})$ *such that*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}_+} \frac{|\mathcal{T}_n(f^*;x) - f^*(x)|}{1+x^2} \geq 1.$$

For $f \in C_2^k(\mathbb{R})$, we will consider the weighted modulus of continuity defined in [10] given by

$$\Omega_2(f,\delta) = \sup_{x\in\mathbb{R}, |h|\leq\delta} \frac{|f(x+h) - f(x)|}{1+(h+x)^2}.$$

This function has the following properties:

(1) $\Omega_2(f,\delta) \leq 2\|f\|_2$,
(2) $\Omega_2(f,m\delta) \leq m\Omega_2(f,\delta)$, $m \in \mathbb{N}$,
(3) $\lim_{\delta\to 0} \Omega_2(f,\delta) = 0$.

Note that, we can not find a rate of convergence in terms of usual first modulus of continuity $\omega_1(f,\delta)$ of the function f because the modulus of continuity $\omega_1(f;\ \delta)$ on the infinite interval does not tend to zero as $\delta \to 0$. For this reason we consider the weighted modulus of continuity $\Omega_2(f,\delta)$.

Remark 2. Since any linear and positive operator is monotone, Lemma 1 guarantee that $\mathcal{W}_n(f) \in C_2(\mathbb{R})$ for each $f \in C_2(\mathbb{R})$.

Note that, if we choose $q = 1$ then the operators $\mathcal{W}_n$ turn out to be the classical Gauss–Weierstrass singular integral operators.

Since for a fixed value of q with $0 < q < 1$,

$$\lim_{n\to\infty} [n]_q = \frac{1}{1-q},$$

to ensure the convergence properties of $\mathcal{W}_n$ we will assume $q = q_n$ as a sequence such that $q_n \to 1$ as $n \to \infty$ for $0 < q_n < 1$ and so that $[n]_{q_n} \to \infty$ as $n \to \infty$. An example of such a sequence is $q_n = 1 - 1/na^n$, where $a > 3$ (see [7]).

Theorem 1. *Let* $q = q_n$ *satisfies* $0 < q_n < 1$ *and let* $q_n \to 1$ *as* $n \to \infty$. *For each* $f \in C_2^k(\mathbb{R})$ *we have*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{|\mathcal{W}_n(f;q_n,x) - f(x)|}{1+x^2} = 0.$$

Proof. Clearly, $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{|\mathcal{W}_n(1;q_n,x)-1|}{1+x^2} = 0$. From Lemma 1 we get

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{|\mathcal{W}_n(t;q_n,x) - x|}{1+x^2} = 0.$$

Also, using Lemma 1 again, we can write

$$\sup_{x\in\mathbb{R}}\frac{\left|\mathcal{W}_n\left(t^2;q_n,x\right)-x^2\right|}{1+x^2}\leq\sup_{x\in\mathbb{R}}\frac{|x|}{1+x^2}\frac{4}{\sqrt{[n]_{q_n}}\Gamma_{q_n^2}\left(\frac{1}{2}\right)}+\frac{4\Gamma_{q_n^2}\left(\frac{3}{2}\right)}{[n]_{q_n}\Gamma_{q_n^2}\left(\frac{1}{2}\right)},$$

which implies that

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\frac{\left|\mathcal{W}_n\left(t^2;q_n,x\right)-x^2\right|}{1+x^2}=0.$$

Since the conditions of Theorem A are satisfied, we obtain for any $f\in C_2^k(\mathbb{R})$

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\frac{\left|\mathcal{W}_n\left(f;q_n,x\right)-f(x)\right|}{1+x^2}=0. \blacksquare$$

Theorem 2. *For $f\in C_2^k(\mathbb{R})$, $n\in\mathbb{N}$ we have*

$$\sup_{x\in\mathbb{R}}\frac{\left|\mathcal{W}_n\left(f;q,x\right)-f(x)\right|}{1+x^2}\leq\left(1+\frac{12}{\Gamma_{q^2}\left(\frac{1}{2}\right)}+\frac{8\Gamma_{q^2}\left(\frac{3}{2}\right)}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\right)\Omega_2\left(f,\frac{1}{\sqrt{[n]_q}}\right).$$

Proof. From the properties of Ω_2 it is obvious that for any $\lambda>0$,

$$\Omega_2(f,\lambda\delta)\leq(\lambda+1)\,\Omega_2(f,\delta).$$

For $\delta>0$, if we use the definition of Ω_2 and the last inequality with $\lambda=\frac{t}{\delta}$ we have

$$\begin{aligned}|f(x+t)-f(x)|&\leq\left(1+(t+x)^2\right)\Omega_2(f,t)\\&\leq\left(1+(t+x)^2\right)\left(1+\frac{t}{\delta}\right)\Omega_2(f,\delta).\end{aligned}$$

By the linearity and monotonicity of $\mathcal{W}_n$ applied to last inequality we obtain

$$\begin{aligned}&\left|\mathcal{W}_n\left(f;q,x\right)-f(x)\right|\\&\leq\frac{\sqrt{[n]_q}(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}}\left(1+(t+x)^2\right)\left(1+\frac{t}{\delta}\right)E_{q^2}\left(-q^2[n]_q\frac{t^2}{4}\right)d_qt\;\Omega_2(f,\delta).\end{aligned}$$

We can use the identity $\left(1+(x+t)^2\right)\left(1+\frac{t}{\delta}\right)=\left(1-\frac{x}{\delta}\right)\left(1+(t+x)^2\right)+\frac{1}{\delta}(t+x)+\frac{1}{\delta}(t+x)^3$, to rewrite the RHS above as follows

$$\left(\left(1-\frac{x}{\delta}\right)\left(1+\mathcal{W}_n\left(t^2;q,x\right)\right)+\frac{1}{\delta}\mathcal{W}_n(t;q,x)+\frac{1}{\delta}\mathcal{W}_n\left(t^3;q,x\right)\right)\Omega_2(f,\delta).$$

Using Lemma 1 and simple algebraic manipulations, the above expression becomes

$$\left\{(1+x^2)\left(1+\frac{2}{\delta\sqrt{[n]_q}\Gamma_{q^2}\left(\frac{1}{2}\right)}\right)+4x\left(\frac{1}{\sqrt{[n]_q}_q\Gamma_{q^2}\left(\frac{1}{2}\right)}+\frac{2\Gamma_{q^2}\left(\frac{3}{2}\right)}{\delta\,[n]_q\,\Gamma_{q^2}\left(\frac{1}{2}\right)}\right)\right.$$
$$\left.+\frac{4\Gamma_{q^2}\left(\frac{3}{2}\right)}{[n]_q\,\Gamma_{q^2}\left(\frac{1}{2}\right)}+\frac{8}{\delta\,[n]_q^{3/2}\,\Gamma_{q^2}\left(\frac{1}{2}\right)}\right\}\Omega_2(f,\delta).$$

Putting together the above inequalities, we get, after dividing by $\left(1+x^2\right)$ and choosing $\delta=\frac{1}{\sqrt{[n]_q}}$,

$$\frac{|\mathcal{W}_n\,(f;q,\,x)-f\,(x)|}{1+x^2}\leq\left\{1+\frac{2}{\Gamma_{q^2}\left(\frac{1}{2}\right)}+\frac{2}{\sqrt{[n]_q}\Gamma_{q^2}\left(\frac{1}{2}\right)}\left(1+2\Gamma_{q^2}\left(\frac{3}{2}\right)\right)\right.$$
$$\left.+\frac{4}{[n]_q\,\Gamma_{q^2}\left(\frac{1}{2}\right)}\left(\Gamma_{q^2}\left(\frac{3}{2}\right)+2\right)\right\}\Omega_2\left(f,\frac{1}{\sqrt{[n]_q}}\right)$$
$$\leq\left\{1+\frac{12}{\Gamma_{q^2}\left(\frac{1}{2}\right)}+\frac{8\Gamma_{q^2}\left(\frac{3}{2}\right)}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\right\}\Omega_2\left(f,\frac{1}{\sqrt{[n]_q}}\right).$$

This completes the proof. ∎

Remark 3. If $f\in C_2^k(\mathbb{R})$, $n\in\mathbb{N}$, then the weighted convergence rate of the operators of (2.1) to f is $\frac{1}{\sqrt{[n]_{q_n}}}$ for $0<q_n<1$ and $q_n\to 1$ as $n\to\infty$. Also this convergence rate can be made better depending on the choice of q_n and is at least as fast as $\frac{1}{\sqrt{n}}$.

Remark 4. We define the usual modulus of continuity

$$\omega_1\,(f;\ \delta)=\sup_{x\in\mathbb{R},|h|\leq\delta}|f\,(x+h)-f\,(x)|\,,$$

where $f:\mathbb{R}\to\mathbb{R}$.

Then for $f\,(x)=x$ we get trivially that

$$\omega_1\,(x;\ \delta)=\delta.$$

Here we consider $f:\mathbb{R}\to\mathbb{R}$ with $\omega_1\,(f;\ \delta)<\infty$, for any $\delta>0$, and such that $\mathcal{W}_n\,(f;q,\,x)$ exists for any $x\in\mathbb{R}$. We notice that

$$\mathcal{W}_n\,(f;q,\,x+h)-\mathcal{W}_n\,(f;q,\,x)$$
$$=\frac{\sqrt{[n]_q}\,(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}}\left(f\,(x+h+t)-f\,(x+t)\right)E_{q^2}\left(-q^2\,[n]_q\,\frac{t^2}{4}\right)d_qt.$$

Thus it holds

$$
\begin{aligned}
&\left|\mathcal{W}_n\left(f;q,\,x+h\right)-\mathcal{W}_n\left(f;q,\,x\right)\right| \\
&\leq \frac{\sqrt{[n]_q}\,(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}} \left|f\left(x+h+t\right)-f\left(x+t\right)\right| E_{q^2}\left(-q^2\left[n\right]_q\frac{t^2}{4}\right) d_q t \\
&\leq \omega_1\left(f;\ \delta\right) \frac{\sqrt{[n]_q}\,(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}} E_{q^2}\left(-q^2\left[n\right]_q\frac{t^2}{4}\right) d_q t \\
&= \omega_1(f;\ \delta).
\end{aligned}
$$

Therefore

$$
\omega_1\left(\mathcal{W}_n\left(f;q,\,\cdot\right);\ \delta\right)\leq \omega_1(f;\ \delta), \quad \text{for any } \delta>0, \tag{3.1}
$$

proving the global smoothness preservation property of $\mathcal{W}_n$.
We know by Lemma 1 for $k=1$ that

$$
\mathcal{W}_n\left(t;q,\,x\right)=x+\frac{2}{\sqrt{[n]_q}\Gamma_{q^2}\left(\frac{1}{2}\right)},
$$

hence

$$
\omega_1\left(\mathcal{W}_n\left(t;q,\,x\right);\ \delta\right)=\omega_1\left(x;\ \delta\right)=\delta,
$$

proving that (3.1) holds with equality. Hence (3.1) is a sharp inequality.

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