

## Article

# A New Class of Bertrand Curves in Euclidean 4-Space

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**Abstract:** Bertrand curves are a pair of curves that have a common principal normal vector at any point and are related to symmetry properties. In the present paper, we define the notion of  $(1,3)$ - $V$  Bertrand curves in Euclidean 4-space. Then we find the necessary and sufficient conditions for curves in Euclidean 4-space to be  $(1,3)$ - $V$  Bertrand curves. Some related examples are given.

**Keywords:** Bertrand curves; curvatures; Euclidean space

**MSC:** 53A04



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## 1. Introduction

Many studies have focused on the general theorem of curves and surfaces in Euclidean space (or more generally in a Riemannian manifold). In light of these studies, we have a deeper understanding of its local geometry and its global geometry. One of the most important problems in Euclidean space is the characterization of curves. There are two well-known methods for solving this problem. One of them is to point out the relevance for Frenet vectors for the curves [1] and the other is to determine the shape and scale of a regular curve by taking advantage of its curvatures  $k_1$  (or  $\kappa$ ) and  $k_2$  (or  $\tau$ ).

Bertrand curves have a long history and could possibly be traced back to Saint Venant [2]. Two space curves  $L$  and  $L^*$  with common principal normals are called a Bertrand curve. Let  $\kappa_1$  be the curvature of  $L$  and  $\kappa_2$  be the torsion of  $L$ . Then, the necessary and sufficient condition for  $L$  and  $L^*$  to conjugate is that

$$\alpha\kappa_1 \sin \omega + \alpha\kappa_2 \cos \omega = \sin \omega$$

is true, where  $\omega$  is the angle of tangent vectors of  $L$  and  $L^*$  and  $\alpha$  is a constant. They were introduced by J. Bertrand in 1850 [3]. The Bertrand curve is also known as the associate curve or the conjugate curve [1]. It has widespread applications in many fields, such as classical differential geometry, differential equations, physics, biology, etc. Therefore, these kinds of curves are related to symmetry. We could combine geometry and symmetry as a basic tool of scientific investigations to study Bertrand Curves.

When considering the properties of Bertrand curves in Euclidean  $n$ -space, either  $k_2$  or  $k_3$  turns out to be zero, which denotes that the Bertrand curves in  $\mathbb{E}^n$  ( $n > 3$ ) are degenerate curves [4]. In [5], Matsuda and Yoroze studied these results. In  $\mathbb{E}^n$  ( $n > 3$ ), they found that there are no special Bertrand curves and considered a new type of Bertrand curve; that is  $(1,3)$ -type Bertrand curves in  $\mathbb{E}^4$ . Research related to Bertrand curves and their topological and geometrical properties and characterizations was considered by many researchers in Minkowski space-time, especially in Minkowski 3-space as well as in Euclidean space (see [6–34]). The technique and methods used in those papers consider

inserting the study of Bertrand curves and  $V$ -Bertrand curves in Euclidean 3-space and  $(1, 3)$ - $V$  Bertrand curve in Euclidean 4-space etc. This will help us find more new results relevant to symmetry properties in different spaces. In [25], Camci, Ç et al. showed a new approach for Bertrand curves in Euclidean 3-space. Taking advantage of this method, they give the necessary and sufficient conditions for the curve to be a Bertrand curve in Euclidean 3-space. Furthermore, in [24], the author defined  $V$ -Bertrand curves in Euclidean 3-space. Because of the higher dimension of Euclidean 4-space, the objects in Euclidean 4-space could rotate easily and be represented conveniently. Therefore, there are many applications for Euclidean 4-space, especially for its use in image processing. In this paper, we give the notion of a  $(1, 3)$ - $V$  Bertrand curve, which is the extension of  $(1, 3)$ -Bertrand curves, in Euclidean 4-space,  $\mathbb{E}^4$ . Then, we find the necessary and sufficient conditions of the curves in  $\mathbb{E}^4$  to be  $(1, 3)$ - $V$  Bertrand curves. In special cases, we give the definitions of  $(1, 3)$ - $T$  Bertrand,  $(1, 3)$ - $N$  Bertrand,  $(1, 3)$ - $B_1$  Bertrand,  $(1, 3)$ - $B_2$  Bertrand curves. Some related examples are given. In the future work, we are going to study some applications of  $(1, 3)$ - $V$  Bertrand curves in Euclidean 4-space combine with singularity theory, submanifold theory and quasiconformal harmonic mappings etc. in [35–55] to obtain more new results and theorems relate with symmetry.

## 2. Preliminaries

In this section, we present some basic notions that we need in the paper; for more details, refer to [7,28]. Let  $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be a curve in Euclidean 4-space. Recall that the curve  $\beta$  is said to be of unit speed (or parameterized by arc length  $s$ ) if  $\langle \beta'(s), \beta'(s) \rangle = 1$ , where  $\langle \cdot, \cdot \rangle$  is the standard scalar product of  $\mathbb{E}^4$  given by

$$\langle X, Y \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4,$$

where  $X = (x_1, x_2, x_3, x_4)$ ,  $Y = (y_1, y_2, y_3, y_4) \in \mathbb{E}^4$ . In particular, the norm of a vector  $X \in \mathbb{E}^4$  is given by  $\|X\| = \sqrt{\langle X, X \rangle}$ .

Let  $\{T, N, B_1, B_2\}$  be orthonormal Frenet frame along the unit speed curve  $\beta$ , where  $T$ ,  $N$ ,  $B_1$  and  $B_2$  denote the tangent, the principal normal, the first binormal and the second binormal vector field, respectively. Then, the Frenet formulae are given by

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ 0 & 0 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}. \quad (1)$$

The functions  $\kappa_1$ ,  $\kappa_2$  and  $\kappa_3$  are called the first curvature, the second curvature (torsion) and the third curvature (torsion) of the curve  $\beta$ , respectively [28]. This formula could apply to the study of Eigenproblem of Alignment [56].

## 3. On $(1, 3)$ - $V$ Bertrand Curves in $\mathbb{E}^4$

In this section, we give the definition of the  $(1, 3)$ - $V$  Bertrand curve in  $\mathbb{E}^4$  and we find the necessary and sufficient conditions for curves in  $\mathbb{E}^4$  to be  $(1, 3)$ - $V$  Bertrand curves.

Let  $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be a curve parametrized by arc length  $s$  with Frenet frame  $\{T, N, B_1, B_2\}$  and curvatures  $\kappa_1, \kappa_2, \kappa_3$ . Consider a vector field  $V$  given by

$$V(s) = a(s)T(s) + b(s)N(s) + c(s)B_1(s) + d(s)B_2(s)$$

where  $a(s), b(s), c(s), d(s)$  functions on  $I$  satisfying  $a^2(s) + b^2(s) + c^2(s) + d^2(s) = 1$ . Then we can define a unit speed curve

$$\gamma(s) = \int_0^s V(u)du,$$

which is called the integral curve of  $V(s)$  ([26]).

**Definition 1.** Let  $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be a curve parametrized by arc length  $s$  with Frenet frame  $\{T, N, B_1, B_2\}$  and curvatures  $\kappa_1, \kappa_2, \kappa_3$ . We can define a curve

$$\beta^*(h(s)) = \int_0^s V(u)du + e(s)N(s) + f(s)B_2(s)$$

with the Frenet frame  $\{T^*, N^*, B_1^*, B_2^*\}$  and curvatures  $\kappa_1^*, \kappa_2^*, \kappa_3^*$ . Then  $\beta$  is called a (1,3)-V Bertrand curve if there is a curve  $\beta^*(h(s))$ , such that the plane spanned by  $\{N, B_2\}$  coincides with the plane spanned by  $\{N^*, B_2^*\}$ . Here the curve  $\beta^*$  is called the (1,3)-V Bertrand mate curve of curve  $\beta$ .

For the special cases of  $a(s), b(s), c(s), d(s)$ , we give the definitions in Table 1.

**Table 1.** Types of (1,3)-V Bertrand curves in  $\mathbb{E}^4$ .

| Bertrand Curve               | $a(s)$ | $b(s)$ | $c(s)$ | $d(s)$ |
|------------------------------|--------|--------|--------|--------|
| (1, 3)-T Bertrand curve      | 1      | 0      | 0      | 0      |
| (1, 3)-N Bertrand curve      | 0      | 1      | 0      | 0      |
| (1, 3)- $B_1$ Bertrand curve | 0      | 0      | 1      | 0      |
| (1, 3)- $B_2$ Bertrand curve | 0      | 0      | 0      | 1      |

**Theorem 1.** Let  $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be a curve parametrized by arc length  $s$  with Frenet frame  $\{T, N, B_1, B_2\}$  and non-zero curvatures  $\kappa_1, \kappa_2, \kappa_3$ . Then, the curve  $\beta$  is a (1,3)-V Bertrand curve if—and only if—it satisfies one of the following conditions.

(i) There exist functions  $a, b, c, d, e$ , and  $f$ , satisfying

$$b + e' = 0, \quad d + f' = 0, \quad c + e\kappa_2 - f\kappa_3 = 0, \quad a - e\kappa_1 \neq 0. \quad (2)$$

(ii) There exist functions  $a, b, c, d, e, f$  and constant real numbers  $v, \mu$ , satisfying

$$b + e' = 0, \quad d + f' = 0, \quad c + e\kappa_2 - f\kappa_3 \neq 0, \quad (3)$$

$$a - e\kappa_1 = v(c + e\kappa_2 - f\kappa_3), \quad (4)$$

$$\mu\kappa_3 = v\kappa_1 - \kappa_2, \quad (5)$$

$$\kappa_1\kappa_2(1 - v^2) + v(-\kappa_1^2 + \kappa_2^2 + \kappa_3^2) \neq 0. \quad (6)$$

**Proof.** We assume that  $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  is a (1,3)-V Bertrand curve parametrized by arc length  $s$  with Frenet frame  $\{T, N, B_1, B_2\}$  and non-zero curvatures  $\kappa_1, \kappa_2, \kappa_3$ , and  $\beta^*$  is the (1,3)-V Bertrand mate curve of  $\beta$  parametrized by arc-length  $s^*$  with the Frenet frame  $\{T^*, N^*, B_1^*, B_2^*\}$  and curvatures  $\kappa_1^*, \kappa_2^*, \kappa_3^*$ . Then, we can write the curve  $\beta^*$  as follows

$$\begin{aligned} \beta^*(s^*) &= \beta^*(h(s)) = \int (a(s)T(s) + b(s)N(s) + c(s)B_1(s) + d(s)B_2(s))ds \\ &\quad + e(s)N(s) + f(s)B_2(s) \end{aligned} \quad (7)$$

for all  $s \in I$  where  $a(s), b(s), c(s), d(s), e(s)$  and  $f(s)$  are  $C^\infty$ —functions on  $I$ . Differentiating (7) as for  $s$  and taking advantage of the Frenet Formulae (1), we get

$$T^*h' = (a - e\kappa_1)T + (b + e')N + (c + e\kappa_2 - f\kappa_3)B_1 + (d + f')B_2. \quad (8)$$

By taking advantage of the scalar product of (8) with  $N$  and  $B_2$ , respectively, we have

$$b + e' = 0 \quad \text{and} \quad d + f' = 0. \quad (9)$$

Substituting (9) in (8), we find

$$T^*h' = (a - e\kappa_1)T + (c + e\kappa_2 - f\kappa_3)B_1. \quad (10)$$

By taking advantage of the scalar product of (10) with itself, we obtain

$$(h')^2 = (a - e\kappa_1)^2 + (c + e\kappa_2 - f\kappa_3)^2. \quad (11)$$

If we denote

$$\delta = \frac{a - e\kappa_1}{h'} \quad \text{and} \quad \gamma = \frac{c + e\kappa_2 - f\kappa_3}{h'}, \quad (12)$$

we get

$$T^* = \delta T + \gamma B_1 \quad (13)$$

Differentiating (13) as for  $s$  and taking advantage of the Frenet Formulae (1), we have

$$h'\kappa_1^*N^* = \delta'T + (\delta\kappa_1 - \gamma\kappa_2)N + \gamma'B_1 + \gamma\kappa_3B_2. \quad (14)$$

By taking advantage of the scalar product of (14) with  $T$  and  $B_1$ , respectively, we get

$$\delta' = 0 \quad \text{and} \quad \gamma' = 0. \quad (15)$$

**Case 1.** Assume that  $\gamma = 0$ . Then, we obtain

$$c + e\kappa_2 - f\kappa_3 = 0 \quad \text{and} \quad a - e\kappa_1 \neq 0.$$

**Case 2.** Assume that  $\gamma \neq 0$ . Then, we find

$$c + e\kappa_2 - f\kappa_3 \neq 0 \quad (16)$$

and

$$a - e\kappa_1 = v(c + e\kappa_2 - f\kappa_3) \quad (17)$$

where  $v = \delta/\gamma$ . Substituting (15) in (14), we get

$$h'\kappa_1^*N^* = (\delta\kappa_1 - \gamma\kappa_2)N + \gamma\kappa_3B_2. \quad (18)$$

By taking advantage of the scalar product of (18) with itself, we obtain

$$(h')^2(\kappa_1^*)^2 = (\delta\kappa_1 - \gamma\kappa_2)^2 + \gamma^2\kappa_3^2. \quad (19)$$

Substituting (12) in (19), we find

$$(h')^2(\kappa_1^*)^2 = \frac{(c + e\kappa_2 - f\kappa_3)^2}{(h')^2}[(v\kappa_1 - \kappa_2)^2 + \kappa_3^2]. \quad (20)$$

Substituting (17) in (11), we have

$$(h')^2 = (c + e\kappa_2 - f\kappa_3)^2[v^2 + 1]. \quad (21)$$

Substituting (21) in (20), we get

$$(h')^2(\kappa_1^*)^2 = \frac{1}{v^2 + 1}[(v\kappa_1 - \kappa_2)^2 + \kappa_3^2]. \quad (22)$$

If we denote

$$\lambda_1 = \frac{(\delta\kappa_1 - \gamma\kappa_2)}{h'\kappa_1^*} \quad \text{and} \quad \lambda_2 = \frac{\gamma\kappa_3}{h'\kappa_1^*}, \quad (23)$$

We get

$$N^* = \lambda_1 N + \lambda_2 B_2. \quad (24)$$

Differentiating (24) as for  $s$  and taking advantage of the Frenet Formulae (1), we find

$$-h'\kappa_1^* T^* + h'\kappa_2^* B_1^* = -\kappa_1 \lambda_1 T + \lambda_1' N + (\lambda_1 \kappa_2 + \lambda_2 \kappa_3) B_1 + \lambda_2' B_2. \quad (25)$$

By taking advantage of the scalar product of (25) with  $N$  and  $B_2$ , respectively, we obtain

$$\lambda_1' = 0 \text{ and } \lambda_2' = 0. \quad (26)$$

Substituting (12) in (23), we get

$$\lambda_1 = \frac{(c + e\kappa_2 - f\kappa_3)(v\kappa_1 - \kappa_2)}{(h')^2 \kappa_1^*} \quad (27)$$

and

$$\lambda_2 = \frac{\kappa_3(c + e\kappa_2 - f\kappa_3)}{(h')^2 \kappa_1^*} \quad (28)$$

From (27) and (28), since  $\lambda_2 \neq 0$ , we have (5)

$$\mu\kappa_3 = v\kappa_1 - \kappa_2 \quad (29)$$

where  $\mu = \lambda_1/\lambda_2$  is constant. Substituting (26) in (25), we find

$$-h'\kappa_1^* T^* + h'\kappa_2^* B_1^* = -\kappa_1 \lambda_1 T + (\lambda_1 \kappa_2 - \lambda_2 \kappa_3) B_1. \quad (30)$$

From (10) and (30), we obtain

$$h'\kappa_2^* B_1^* = A(s)T + B(s)B_1 \quad (31)$$

where

$$A(s) = \frac{(c + e\kappa_2 - f\kappa_3)}{(h')^2 \kappa_1^* (v^2 + 1)} [\kappa_1 \kappa_2 (1 - v^2) + v(-\kappa_1^2 + \kappa_2^2 + \kappa_3^2)] \quad (32)$$

and

$$B(s) = \frac{-v(c + e\kappa_2 - f\kappa_3)}{(h')^2 \kappa_1^* (v^2 + 1)} [\kappa_1 \kappa_2 (1 - v^2) + v(-\kappa_1^2 + \kappa_2^2 + \kappa_3^2)]. \quad (33)$$

Since  $f'\kappa_2^* B_1^* \neq 0$ , we get

$$\kappa_1 \kappa_2 (1 - v^2) + v(-\kappa_1^2 + \kappa_2^2 + \kappa_3^2) \neq 0. \quad (34)$$

Conversely, assume that  $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  is a curve parametrized by arc length  $s$  with Frenet frame  $\{T, N, B_1, B_2\}$  and non-zero curvatures  $\kappa_1, \kappa_1, \kappa_3$ . Firstly, assume that the condition (i) holds for functions  $a, b, c, d, e, f$ . Then, we can define a curve  $\beta^*$  as

$$\begin{aligned} \beta^*(s^*) &= \int (a(s)T(s) + b(s)N(s) + c(s)B_1(s) + d(s)B_2(s))ds \\ &\quad + e(s)N(s) + f(s)B_2(s). \end{aligned} \quad (35)$$

Differentiating (35) as for  $s$  and taking advantage of the Frenet Formula (1), we find

$$\frac{d\beta^*}{ds} = (a - e\kappa_1)T. \quad (36)$$

From (36), we have

$$h' = \frac{ds^*}{ds} = \left\| \frac{d\beta^*}{ds} \right\| = m_1(a - e\kappa_1) > 0$$

where  $m_1 = \text{sgn}(a - e\kappa_1)$ . Then, we can easily obtain

$$T^* = m_1 T, \quad N^* = m_1 m_2 N, \quad B_1^* = m_1 m_2 m_3 B_1, \quad B_2^* = m_1 m_2 m_3 m_4 B_2$$

and

$$\kappa_1^* = \frac{m_2 \kappa_1}{h'}, \quad \kappa_2^* = \frac{m_3 \kappa_2}{h'}, \quad \kappa_3^* = \frac{m_4 \kappa_3}{h'},$$

where  $m_i = \text{sgn}(\kappa_{i-1})$  where  $i \in \{2, 3, 4\}$ . Therefore the curve  $\beta$  is a  $(1, 3)$ -V Bertrand curve.

Now assume that the condition (ii) holds. Then, we can define a curve  $\beta^*$  as

$$\begin{aligned} \beta^*(s^*) = & \int (a(s)T(s) + b(s)N(s) + c(s)B_1(s) + d(s)B_2(s))ds \\ & + e(s)N(s) + f(s)B_2(s). \end{aligned} \quad (37)$$

Differentiating (37) as for  $s$  and taking advantage of the Frenet Formula (1), we find

$$\frac{d\beta^*}{ds} = (a - e\kappa_1)T + (c + e\kappa_2 - f\kappa_3)B_1. \quad (38)$$

From (38) and (4), we get

$$\frac{d\beta^*}{ds} = (c + e\kappa_2 - f\kappa_3)[vT + B_1]. \quad (39)$$

From (39), we have

$$h' = \frac{ds^*}{ds} = \left\| \frac{d\beta^*}{ds} \right\| = \varepsilon_1 (c + e\kappa_2 - f\kappa_3) \sqrt{v^2 + 1} > 0 \quad (40)$$

where  $\varepsilon_1 = \text{sgn}(c + e\kappa_2 - f\kappa_3)$ . Now, by rewriting (39), we obtain

$$T^* h' = (c + e\kappa_2 - f\kappa_3)[vT + B_1]. \quad (41)$$

Substituting (40) in (41), we find

$$T^* = \varepsilon_1 (v^2 + 1)^{-\frac{1}{2}} [vT + B_1]. \quad (42)$$

From (42), we get  $\langle T^*, T^* \rangle = 1$ . Differentiating (42) as for  $s$  and taking advantage of the Frenet Formulae (1), we find

$$\frac{dT^*}{ds^*} = \frac{\varepsilon_1 (v^2 + 1)^{-\frac{1}{2}}}{h'} [(v\kappa_1 - \kappa_2)N + \kappa_3 B_2]. \quad (43)$$

Taking advantage of (43), we have

$$\kappa_1^* = \left\| \frac{dT^*}{ds^*} \right\| = \frac{\sqrt{(v\kappa_1 - \kappa_2)^2 + \kappa_3^2}}{h' \sqrt{v^2 + 1}}. \quad (44)$$

From (43) and (44), we have

$$N^* = \frac{1}{\kappa_1^*} \frac{dT^*}{ds^*} = \frac{\varepsilon_1}{\sqrt{(v\kappa_1 - \kappa_2)^2 + \kappa_3^2}} [(v\kappa_1 - \kappa_2)N + \kappa_3 B_2]. \quad (45)$$

taking advantage of (45), we get  $\langle N^*, N^* \rangle = 1$ . If we denote

$$\delta_1 = \frac{\varepsilon_1 (v\kappa_1 - \kappa_2)}{\sqrt{(v\kappa_1 - \kappa_2)^2 + \kappa_3^2}} \quad \text{and} \quad \gamma_1 = \frac{\varepsilon_1 \kappa_3}{\sqrt{(v\kappa_1 - \kappa_2)^2 + \kappa_3^2}}, \quad (46)$$

we obtain

$$N^* = \delta_1 N + \gamma_1 B_2. \quad (47)$$

Differentiating (47) as for  $s$  and taking advantage of the Frenet Formulae (1), we find

$$h' \frac{dN^*}{ds^*} = -\delta_1 \kappa_1 T + \delta_1' N + (\delta_1 \kappa_2 - \gamma_1 \kappa_3) B_1 + \gamma_1' B_2. \quad (48)$$

From (5), we have

$$\mu = \frac{v\kappa_1 - \kappa_2}{\kappa_3}.$$

Differentiating the above equation as for  $s$ , we have

$$(v\kappa_1 - \kappa_2)' \kappa_3 - (v\kappa_1 - \kappa_2) \kappa_3' = 0. \quad (49)$$

Differentiating (46) as for  $s$  and taking advantage of (49), we get

$$\delta_1' = 0 \quad \text{and} \quad \gamma_1' = 0. \quad (50)$$

Substituting (46) and (50) in (48), we obtain

$$\frac{dN^*}{ds^*} = \frac{-\varepsilon_1 \kappa_1 (v\kappa_1 - \kappa_2)}{h' \sqrt{(v\kappa_1 - \kappa_2)^2 + \kappa_3^2}} T + \frac{\varepsilon_1 (\kappa_2 (v\kappa_1 - \kappa_2) - \kappa_3^2)}{h' \sqrt{(v\kappa_1 - \kappa_2)^2 + \kappa_3^2}} B_1. \quad (51)$$

From (42) and (44), we find

$$\kappa_1^* T^* = \frac{\varepsilon_1 ((v\kappa_1 - \kappa_2)^2 + \kappa_3^2)}{h' (v^2 + 1) \sqrt{(v\kappa_1 - \kappa_2)^2 + \kappa_3^2}} [vT + B_1]. \quad (52)$$

From (51) and (52), we get

$$\frac{dN^*}{ds^*} + \kappa_1^* T^* = \frac{P(s)}{R(s)} T + \frac{Q(s)}{R(s)} B_1 \quad (53)$$

where

$$\begin{aligned} P(s) &= \varepsilon_1 [\kappa_1 \kappa_2 (1 - v^2) + v(-\kappa_1^2 + \kappa_2^2 + \kappa_3^2)] \neq 0, \\ Q(s) &= -\varepsilon_1 v [\kappa_1 \kappa_2 (1 - v^2) + v(-\kappa_1^2 + \kappa_2^2 + \kappa_3^2)] \neq 0, \\ R(s) &= h' (v^2 + 1) \sqrt{(v\kappa_1 - \kappa_2)^2 + \kappa_3^2} \neq 0. \end{aligned} \quad (54)$$

Now we can define  $\kappa_2^*$  as

$$\kappa_2^* = \frac{|\kappa_1 \kappa_2 (1 - v^2) + v(-\kappa_1^2 + \kappa_2^2 + \kappa_3^2)|}{h' \sqrt{v^2 + 1} \sqrt{(v\kappa_1 - \kappa_2)^2 + \kappa_3^2}} > 0. \quad (55)$$

Next, we can define a unit vector  $B_1^*$  as

$$B_1^* = \frac{1}{\kappa_2^*} \left[ \frac{dN^*}{ds^*} + \kappa_1^* T^* \right] = \frac{\varepsilon_1 \varepsilon_2}{\sqrt{v^2 + 1}} [T - vB_1]. \quad (56)$$

where  $\varepsilon_1 = \text{sgn}(c + e\kappa_2 - f\kappa_3)$  and  $\varepsilon_2 = \text{sgn}(\kappa_1 \kappa_2 (1 - v^2) + v(-\kappa_1^2 + \kappa_2^2 + \kappa_3^2))$ . From (56), we have  $\langle B_1^*, B_1^* \rangle = 1$ . Thus, we can define a unit vector  $B_2^*$  as  $B_2^* = -\gamma_1 N + \delta B_2$ ; that is,

$$B_2^* = \frac{-\varepsilon_1 \kappa_3}{\sqrt{(v\kappa_1 - \kappa_2)^2 + \kappa_3^2}} N + \frac{\varepsilon_1 (v\kappa_1 - \kappa_2)}{\sqrt{(v\kappa_1 - \kappa_2)^2 + \kappa_3^2}} B_2. \quad (57)$$

From (57), we get  $\langle B_2^*, B_2^* \rangle = 1$ . Lastly, we can define  $\kappa_3^*$  as

$$\kappa_3^* = \left\langle \frac{dB_1^*}{ds^*}, B_2^* \right\rangle = \frac{-\varepsilon_2 \kappa_1 \kappa_3 (v^2 + 1)}{h' \sqrt{v^2 + 1} \sqrt{(v\kappa_1 - \kappa_2)^2 + \kappa_3^2}} \neq 0.$$

Then, we get that  $\text{span}\{N, B_2\} = \text{span}\{N^*, B_2^*\}$ . Thus, the curve  $\beta$  is a  $(1, 3)$ -V Bertrand curve.  $\square$

As a result of the above theorem, we get the following corollaries.

**Corollary 1.** Let  $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be a curve parametrized by arc length  $s$  with Frenet frame  $\{T, N, B_1, B_2\}$  and non-zero curvatures  $\kappa_1, \kappa_2, \kappa_3$ . Then, the curve  $\beta$  is  $(1, 3)$ -T Bertrand curve if—and only if—it satisfies one of the following conditions

(i) There are constant real numbers  $e, f$  satisfying

$$e\kappa_2 - f\kappa_3 = 0, \quad 1 - e\kappa_1 \neq 0.$$

(ii) There are constant real numbers  $e, f, v, \mu$  satisfying

$$e\kappa_2 - f\kappa_3 \neq 0, \quad 1 - e\kappa_1 = v(e\kappa_2 - f\kappa_3), \quad \mu\kappa_3 = v\kappa_1 - \kappa_2, \\ \kappa_1\kappa_2(1 - v^2) + v(-\kappa_1^2 + \kappa_2^2 + \kappa_3^2) \neq 0.$$

**Corollary 2.** Let  $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be a curve parametrized by arc length  $s$  with Frenet frame  $\{T, N, B_1, B_2\}$  and non-zero curvatures  $\kappa_1, \kappa_2, \kappa_3$ . Then, the curve  $\beta$  is  $(1, 3)$ -N Bertrand curve if—and only if—it satisfies one of the following conditions

(i) There is a constant real number  $f$ , satisfying  $s\kappa_2 + f\kappa_3 = 0$  and  $\kappa_1 \neq 0$ .

(ii) There are constant real numbers  $f, v, \mu$ , satisfying

$$s\kappa_2 + f\kappa_3 \neq 0, \quad s\kappa_1 = -v(s\kappa_2 + f\kappa_3), \quad \mu\kappa_3 = v\kappa_1 - \kappa_2, \\ \kappa_1\kappa_2(1 - v^2) + v(-\kappa_1^2 + \kappa_2^2 + \kappa_3^2) \neq 0.$$

**Example 1.** Let us choose  $f = v = 1, \mu = 0, \kappa_1 = \kappa_2 = 1, \kappa_3 = -2s$  where  $s < 0$ . Then, it is clear that the conditions of Corollary 2 are satisfied. As a result, the curve with curvatures  $\kappa_1 = \kappa_2 = 1, \kappa_3 = -2s$ , is a  $(1, 3)$ -N Bertrand curve in  $\mathbb{E}^4$ .

**Corollary 3.** Let  $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be a curve parametrized by arc length  $s$  with Frenet frame  $\{T, N, B_1, B_2\}$  and non-zero curvatures  $\kappa_1, \kappa_2, \kappa_3$ . Then, the curve  $\beta$  is a  $(1, 3)$ -B<sub>1</sub> Bertrand curve if—and only if—it satisfies one of the following conditions.

(i) There are constant real numbers  $e \neq 0, f$ , satisfying  $1 + e\kappa_2 - f\kappa_3 = 0$ .

(ii) There are constant real numbers  $e, f, v, \mu$ , satisfying

$$1 + e\kappa_2 - f\kappa_3 \neq 0, \quad -e\kappa_1 = v(1 + e\kappa_2 - f\kappa_3), \quad \mu\kappa_3 = v\kappa_1 - \kappa_2, \\ \kappa_1\kappa_2(1 - v^2) + v(-\kappa_1^2 + \kappa_2^2 + \kappa_3^2) \neq 0.$$

**Example 2.** For  $A = \frac{1}{\sqrt{2+\sqrt{2}}}, B = \frac{1}{\sqrt{2-\sqrt{2}}}, C = \frac{1}{\sqrt{4+2\sqrt{2}}}, D = \frac{1}{\sqrt{4-2\sqrt{2}}}$ , let us consider the curve in  $\mathbb{E}^4$  with the equation

$$\beta(s) = (B \sin(As), -B \cos(As), A \sin(Bs), -A \cos(Bs))$$

with the curvatures  $\kappa_1 = 1, \kappa_2 = \kappa_3 = \frac{1}{\sqrt{2}}$  and the Frenet frame

$$\begin{aligned} T &= \left( \frac{1}{\sqrt{2}} \cos(As), \frac{1}{\sqrt{2}} \sin(As), \frac{1}{\sqrt{2}} \cos(Bs), \frac{1}{\sqrt{2}} \sin(Bs) \right), \\ N &= (-C \sin(As), C \cos(As), -D \sin(Bs), D \cos(Bs)), \\ B_1 &= \left( \frac{1}{\sqrt{2}} \cos(As), \frac{1}{\sqrt{2}} \sin(As), -\frac{1}{\sqrt{2}} \cos(Bs), -\frac{1}{\sqrt{2}} \sin(Bs) \right), \\ B_2 &= (-D \sin(As), D \cos(As), C \sin(Bs), -C \cos(Bs)). \end{aligned}$$

Taking  $e = -1, f = 1, v = -1 - \sqrt{2}, \mu = -\sqrt{2} - 3$  in corollary 3, for the same curve  $\beta$  in example 2, we obtain

$$\begin{aligned} \beta^*(s) &= \int_0^s B_1(u) du - N(s) + B_2(s) \\ &= \left( \frac{1}{B} \sin(As), -\frac{1}{B} \cos(As), \frac{1}{B} \sin(Bs), -\frac{1}{B} \cos(Bs) \right) \end{aligned}$$

with the curvatures

$$\kappa_1^* = \sqrt{\frac{6+3\sqrt{2}}{8}}, \quad \kappa_2^* = \sqrt{\frac{2+\sqrt{2}}{24}}, \quad \kappa_3^* = \sqrt{\frac{2+\sqrt{2}}{12}}$$

and the Frenet frame

$$\begin{aligned} T^* &= \left( \frac{1}{2B} \cos(As), \frac{1}{2B} \sin(As), \frac{1}{2A} \cos(Bs), \frac{1}{2A} \sin(Bs) \right), \\ N^* &= (-C \sin(As), C \cos(As), -D \sin(Bs), D \cos(Bs)), \\ B_1^* &= \left( \frac{1}{2A} \cos(As), \frac{1}{2A} \sin(As), -\frac{1}{2B} \cos(Bs), -\frac{1}{2B} \sin(Bs) \right), \\ B_2^* &= (-D \sin(As), D \cos(As), C \sin(Bs), -C \cos(Bs)) \end{aligned}$$

where

$$\begin{aligned} V &= \frac{1}{\sqrt{2}}(T + B_1), \quad A = \frac{1}{\sqrt{2+\sqrt{2}}}, \quad B = \frac{1}{\sqrt{2-\sqrt{2}}}, \\ C &= \sqrt{\frac{3-2\sqrt{2}}{6}}, \quad D = \sqrt{\frac{3+2\sqrt{2}}{6}}, \quad E = \sqrt{10+7\sqrt{2}}. \end{aligned}$$

It can be seen that

$$\begin{aligned} N^* &= \sqrt{\frac{10+\sqrt{2}}{12}} N - \sqrt{\frac{2-\sqrt{2}}{12}} B_2, \\ B_2^* &= \sqrt{\frac{2-\sqrt{2}}{12}} N + \sqrt{\frac{10+\sqrt{2}}{12}} B_2 \end{aligned}$$

which implies that  $\beta$  is  $(1,3)$ - $B_1$  Bertrand curve and  $\beta^*$  is  $(1,3)$ - $B_1$  Bertrand mate curve of  $\beta$ .

**Corollary 4.** Let  $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be a curve parametrized by arc length  $s$  with Frenet frame  $\{T, N, B_1, B_2\}$  and non-zero curvatures  $\kappa_1, \kappa_2, \kappa_3$ . Then, the curve  $\beta$  is a  $(1,3)$ - $B_2$  Bertrand curve if—and only if—it satisfies one of the following conditions.

- (i) There is a constant real number  $e \neq 0$  satisfying  $e\kappa_2 + s\kappa_3 = 0$ .

(ii) There are constant real numbers  $e, v, \mu$  satisfying

$$e\kappa_2 + s\kappa_3 \neq 0, \quad -e\kappa_1 = v(e\kappa_2 + s\kappa_3), \quad \mu\kappa_3 = v\kappa_1 - \kappa_2,$$

$$\kappa_1\kappa_2(1 - v^2) + v(-\kappa_1^2 + \kappa_2^2 + \kappa_3^2) \neq 0.$$

**Example 3.** Let us choose  $e = 1, \kappa_1 = \kappa_2 = -s, \kappa_3 = 2, v = 1, \mu = 0$  where  $s < 0$ . Then, it is clear that the conditions of Corollary 4 are satisfied. As a result, the curve with curvatures  $\kappa_1 = \kappa_2 = -s, \kappa_3 = 2$ , is a  $(1, 3)$ - $B_2$  Bertrand curve in  $\mathbb{E}^4$ .

#### 4. Conclusions

In this paper, we presented the concepts of  $(1, 3)$ - $V$  Bertrand curves in Euclidean 4-space. We gave the necessary and sufficient conditions for the curves in Euclidean 4-space to be  $(1, 3)$ - $V$  Bertrand curves. Moreover, we showed some related examples to make the theory more clear.

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