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Research Article

A Suzuki Type Fixed-Point Theorem

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We present a fixed-point theorem for a single-valued map in a complete metric space using implicit relation, which is a generalization of several previously stated results including that of Suziki (2008).

1. Introduction

There are a lot of generalizations of Banach fixed-point principle in the literature. See [1–5]. One of the most interesting generalizations is that given by Suzuki [6]. This interesting fixed-point result is as follows.

Theorem 1.1. *Let (X, d) be a complete metric space, and let T be a mapping on X. Define a nonincreasing function θ from $[0, 1)$ into $(1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1, & 0 \le r \le \dfrac{\sqrt{5}-1}{2}, \\ \dfrac{1-r}{r^2}, & \dfrac{\sqrt{5}-1}{2} \le r \le \dfrac{1}{\sqrt{2}}, \\ \dfrac{1}{1+r}, & \dfrac{1}{\sqrt{2}} \le r < 1. \end{cases} \tag{1.1}$$

Assume that there exists $r \in [0, 1)$, such that

$$\theta(r)d(x, Tx) \le d(x, y) \quad \text{implies } d(Tx, Ty) \le rd(x, y), \tag{1.2}$$

for all $x, y \in X$, then there exists a unique fixed-point z of T. Moreover, $\lim_n T^n x = z$ for all $x \in X$.

Like other generalizations mentioned above in this paper, the Banach contraction principle does not characterize the metric completeness of X. However, Theorem 1.1 does characterize the metric completeness as follows.

Theorem 1.2. *Define a nonincreasing function θ as in Theorem 1.1, then for a metric space (X, d) the following are equivalent:*

(i) *X is complete,*

(ii) *Every mapping T on X satisfying* (1.2) *has a fixed point.*

In addition to the above results, Kikkawa and Suzuki [7] provide a Kannan type version of the theorems mentioned before. In [8], it is provided a Chatterjea type version. Popescu [9] gives a Ciric type version. Recently, Kikkawa and Suzuki also provide multivalued versions which can be found in [10, 11]. Some fixed-point theorems related to Theorems 1.1 and 1.2 have also been proven in [12, 13].

The aim of this paper is to generalize the above results using the implicit relation technique in such a way that

$$F\big(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big) \leq 0, \tag{1.3}$$

for $x, y \in X$, where $F : [0, \infty)^6 \to \mathbb{R}$ is a function as given in Section 2.

2. Implicit Relation

Implicit relations on metric spaces have been used in many papers. See [1, 14–16].

Let $\mathbb{R}_+$ denote the nonnegative real numbers, and let Ψ be the set of all continuous functions $F : [0, \infty)^6 \to \mathbb{R}$ satisfying the following conditions:

F_1: $F(t_1, \ldots, t_6)$ is nonincreasing in variables $t_2, \ldots, t_6$,

F_2: there exists $r \in [0, 1)$, such that

$$F(u, v, v, u, u + v, 0) \leq 0 \tag{2.1}$$

or

$$F(u, v, 0, u + v, u, v) \leq 0 \tag{2.2}$$

or

$$F(u, v, v, v, v, v) \leq 0 \tag{2.3}$$

implies $u \leq rv$,

F_3: $F(u, 0, 0, u, u, 0) > 0$, for all $u > 0$.

Example 2.1. $F(t_1, \ldots, t_6) = t_1 - rt_2$, where $r \in [0, 1)$. It is clear that $F \in \Psi$.

Example 2.2. $F(t_1,\ldots,t_6) = t_1 - \alpha[t_3 + t_4]$, where $\alpha \in [0,1/2)$.

Let $F(u,v,v,u,u+v,0) = u-\alpha[u+v] \le 0$, then we have $u \le (\alpha/(1-\alpha))v$. Similarly, let $F(u,v,0,u+v,u,v) \le 0$, then we have $u \le (\alpha/(1-\alpha))v$. Again, let $F(u,v,v,v,v,v) \le 0$, then $u \le 2\alpha v$. Since $\alpha/(1-\alpha) \le 2\alpha < 1$, F_2 is satisfied with $r = 2\alpha$. Also $F(u,0,0,u,u,0) = (1-\alpha)u > 0$, for all $u > 0$. Therefore, $F \in \Psi$.

Example 2.3. $F(t_1,\ldots,t_6) = t_1 - \alpha \max\{t_3,t_4\}$, where $\alpha \in [0,1/2)$.

Let $F(u,v,v,u,u+v,0) = u - \alpha \max\{u,v\} \le 0$, then we have $u \le \alpha v \le (\alpha/(1-\alpha))v$. Similarly, let $F(u,v,0,u+v,u,v) \le 0$, then we have $u \le (\alpha/(1-\alpha))v$. Again, let $F(u,v,v,v,v,v) \le 0$, then $u \le \alpha v \le (\alpha/(1-\alpha))v$. Thus, F_2 is satisfied with $r = \alpha/(1-\alpha)$. Also $F(u,0,0,u,u,0) = (1-\alpha)u > 0$, for all $u > 0$. Therefore, $F \in \Psi$.

Example 2.4. $F(t_1,\ldots,t_6) = t_1 - \alpha[t_5 + t_6]$, where $\alpha \in [0,1/2)$.

Let $F(u,v,v,u,u+v,0) = u-\alpha[u+v] \le 0$, then we have $u \le (\alpha/(1-\alpha))v$. Similarly, let $F(u,v,0,u+v,u,v) \le 0$, then we have $u \le (\alpha/(1-\alpha))v$. Again, let $F(u,v,v,v,v,v) \le 0$, then $u \le 2\alpha v$. Since $\alpha/(1-\alpha) \le 2\alpha < 1$, F_2 is satisfied with $r = 2\alpha$. Also $F(u,0,0,u,u,0) = (1-\alpha)u > 0$, for all $u > 0$. Therefore, $F \in \Psi$.

Example 2.5. $F(t_1,\ldots,t_6) = t_1 - at_3 - bt_4$, where $a,b \in [0,1/2)$.

Let $F(u,v,v,u,u+v,0) = u - av - bu \le 0$, then we have $u \le (a/(1-b))v$. Similarly, let $F(u,v,0,u+v,u,v) \le 0$, then we have $u \le (b/(1-b))v$. Again, let $F(u,v,v,v,v,v) \le 0$, then $u \le (a+b)v$. Thus, F_2 is satisfied with $r = \max\{a/(1-b), b/(1-b), a+b\}$. Also $F(u,0,0,u,u,0) = (1-b)u > 0$, for all $u > 0$. Therefore, $F \in \Psi$.

3. Main Result

Theorem 3.1. *Let (X,d) be a complete metric space, and let T be a mapping on X. Define a nonincreasing function θ from $[0,1)$ into $(1/2,1]$ as in Theorem 1.1. Assume that there exists $F \in \Psi$, such that $\theta(r)d(x,Tx) \le d(x,y)$ implies*

$$F(d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)) \le 0, \tag{3.1}$$

for all $x,y \in X$, then T has a unique fixed-point z and $\lim_n T^n x = z$ holds for every $x \in X$.

Proof. Since $\theta(r) \le 1$, $\theta(r)d(x,Tx) \le d(x,Tx)$ holds for every $x \in X$, by hypotheses, we have

$$F\left(d\left(Tx,T^2x\right), d(x,Tx), d(x,Tx), d\left(Tx,T^2x\right), d\left(x,T^2x\right), 0\right) \le 0, \tag{3.2}$$

and so from (F_1),

$$F\left(d\left(Tx,T^2x\right), d(x,Tx), d(x,Tx), d\left(Tx,T^2x\right), d(x,Tx) + d\left(Tx,T^2x\right), 0\right) \le 0. \tag{3.3}$$

By (F_2), we have

$$d\left(Tx,T^2x\right) \le rd(x,Tx), \tag{3.4}$$

for all $x \in X$. Now fix $u \in X$ and define a sequence $\{u_n\}$ in X by $u_n = T^n u$. Then from (3.4), we have

$$d(u_n, u_{n+1}) = d\left(Tu_{n-1}, T^2 u_{n-1}\right) \le r d(u_{n-1}, Tu_{n-1}) \le \cdots \le r^n d(u, Tu). \tag{3.5}$$

This shows that $\sum_{n=1}^{\infty} d(u_n, u_{n+1}) < \infty$, that is, $\{u_n\}$ is Cauchy sequence. Since X is complete, $\{u_n\}$ converges to some point $z \in X$. Now, we show that

$$d(Tx, z) \le r d(x, z) \quad \forall x \in X \setminus \{z\}. \tag{3.6}$$

For $x \in X \setminus \{z\}$, there exists $n_0 \in \mathbb{N}$, such that $d(u_n, z) \le d(x, z)/3$ for all $n \ge n_0$. Then, we have

$$\begin{aligned} \theta(r) d(u_n, Tu_n) &\le d(u_n, Tu_n) = d(u_n, u_{n+1}) \\ &\le d(u_n, z) + d(z, u_{n+1}) \\ &\le \frac{2}{3} d(x, z) = d(x, z) - \frac{d(x, z)}{3} \\ &\le d(x, z) - d(u_n, z) \le d(u_n, x). \end{aligned} \tag{3.7}$$

Hence, by hypotheses, we have

$$F(d(Tu_n, Tx), d(u_n, x), d(u_n, Tu_n), d(x, Tx), d(u_n, Tx), d(x, Tu_n)) \le 0, \tag{3.8}$$

and so

$$F(d(u_{n+1}, Tx), d(u_n, x), d(u_n, u_{n+1}), d(x, Tx), d(u_n, Tx), d(x, u_{n+1})) \le 0. \tag{3.9}$$

Letting $n \to \infty$, we have

$$F(d(z, Tx), d(z, x), 0, d(x, Tx), d(z, Tx), d(x, z)) \le 0, \tag{3.10}$$

and so

$$F(d(z, Tx), d(z, x), 0, d(x, z) + d(z, Tx), d(z, Tx), d(x, z)) \le 0. \tag{3.11}$$

By (F_2), we have

$$d(z, Tx) \le r d(x, z), \tag{3.12}$$

and this shows that (3.6) is true.

Now, we assume that $T^m z \neq z$ for all $m \in \mathbb{N}$, then from (3.6), we have

$$d\left(T^{m+1}z, z\right) \le r^m d(Tz, z), \tag{3.13}$$

for all $m \in \mathbb{N}$.

Case 1. Let $0 \le r \le (\sqrt{5}-1)/2$. In this case, $\theta(r) = 1$. Now, we show by induction that

$$d(T^n z, Tz) \le r d(z, Tz), \tag{3.14}$$

for $n \ge 2$. From (3.4), (3.14) holds for $n = 2$. Assume that (3.14) holds for some n with $n \ge 2$. Since

$$\begin{aligned} d(z, Tz) &\le d(z, T^n z) + d(T^n z, Tz) \\ &\le d(z, T^n z) + r d(z, Tz), \end{aligned} \tag{3.15}$$

we have

$$d(z, Tz) \le \frac{1}{1-r} d(z, T^n z), \tag{3.16}$$

and so

$$\begin{aligned} \theta(r) d\left(T^n z, T^{n+1} z\right) = d\left(T^n z, T^{n+1} z\right) &\le r^n d(z, Tz) \\ &\le \frac{r^n}{1-r} d(z, T^n z) \le \frac{r^2}{1-r} d(z, T^n z) \\ &\le d(z, T^n z). \end{aligned} \tag{3.17}$$

Therefore, by hypotheses, we have

$$F\left(d\left(T^{n+1}z, Tz\right), d(T^n z, z), d\left(T^n z, T^{n+1}z\right), d(z, Tz), d(T^n z, Tz), d\left(z, T^{n+1}z\right)\right) \le 0, \tag{3.18}$$

and so

$$F\left(d\left(T^{n+1}z, Tz\right), r^{n-1} d(Tz, z), r^n d(z, Tz), d(z, Tz), r d(z, Tz), r^n d(z, Tz)\right) \le 0, \tag{3.19}$$

then

$$F\left(d\left(T^{n+1}z, Tz\right), d(Tz, z), d(z, Tz), d(z, Tz), d(z, Tz), d(z, Tz)\right) \le 0, \tag{3.20}$$

and by (F_2), we have

$$d\left(T^{n+1}z, Tz\right) \leq rd(Tz, z). \tag{3.21}$$

Therefore, (3.14) holds.

Now, from (3.6), we have

$$d\left(T^{n+1}z, z\right) \leq rd(T^n z, z) \leq r^n d(Tz, z). \tag{3.22}$$

This shows that $T^n z \rightarrow z$, which contradicts (3.14).

Case 2. Let $(\sqrt{5}-1)/2 \leq r \leq \sqrt{2}/2$. In this case, $\theta(r) = (1-r)/r^2$. Again we want to show that (3.14) is true for $n \geq 2$. From (3.4), (3.14) holds for $n = 2$. Assume that (3.14) holds for some n with $n \geq 2$. Since

$$\begin{aligned} d(z, Tz) &\leq d(z, T^n z) + d(T^n z, Tz) \\ &\leq d(z, T^n z) + rd(z, Tz), \end{aligned} \tag{3.23}$$

we have

$$d(z, Tz) \leq \frac{1}{1-r} d(z, T^n z), \tag{3.24}$$

and so

$$\begin{aligned} \theta(r) d\left(T^n z, T^{n+1}z\right) = \frac{1-r}{r^2} d\left(T^n z, T^{n+1}z\right) &\leq \frac{1-r}{r^n} d\left(T^n z, T^{n+1}z\right) \\ &\leq (1-r) d(z, Tz) \leq d(z, T^n z). \end{aligned} \tag{3.25}$$

Therefore, as in the previous case, we can prove that (3.14) is true for $n \geq 2$. Again from (3.6), we have

$$d\left(T^{n+1}z, z\right) \leq rd(T^n z, z) \leq r^n d(Tz, z). \tag{3.26}$$

This shows that $T^n z \rightarrow z$, which contradicts (3.14).

Case 3. Let $\sqrt{2}/2 \leq r < 1$. In this case, $\theta(r) = 1/(1+r)$. Note that for $x, y \in X$, either

$$\theta(r) d(x, Tx) \leq d(x, y) \tag{3.27}$$

or

$$\theta(r) d\left(Tx, T^2 x\right) \leq d(Tx, y) \tag{3.28}$$

holds. Indeed, if

$$\theta(r)d(x,Tx) > d(x,y),$$
$$\theta(r)d\left(Tx,T^2x\right) > d(Tx,y), \tag{3.29}$$

then we have

$$\begin{aligned} d(x,Tx) &\le d(x,y) + d(Tx,y) < \theta(r)\left[d(x,Tx) + d\left(Tx,T^2x\right)\right] \\ &\le \theta(r)[d(x,Tx) + rd(x,Tx)] = d(x,Tx), \end{aligned} \tag{3.30}$$

which is a contradiction. Therefore, either

$$\theta(r)d(u_{2n},Tu_{2n}) \le d(u_{2n},z) \tag{3.31}$$

or

$$\theta(r)d(u_{2n+1},Tu_{2n+1}) \le d(u_{2n+1},z) \tag{3.32}$$

holds for every $n \in \mathbb{N}$. If

$$\theta(r)d(u_{2n},Tu_{2n}) \le d(u_{2n},z) \tag{3.33}$$

holds, then by hypotheses we have

$$F(d(Tu_{2n},Tz),d(u_{2n},z),d(u_{2n},Tu_{2n}),d(z,Tz),d(u_{2n},Tz),d(z,Tu_{2n})) \le 0, \tag{3.34}$$

and so

$$F(d(u_{2n+1},Tz),d(u_{2n},z),d(u_{2n},u_{2n+1}),d(z,Tz),d(u_{2n},Tz),d(z,u_{2n+1})) \le 0. \tag{3.35}$$

Letting $n \to \infty$, we have

$$F(d(z,Tz),0,0,d(z,Tz),d(z,Tz),0) \le 0, \tag{3.36}$$

which contradicts (F_3). If

$$\theta(r)d(u_{2n+1},Tu_{2n+1}) \le d(u_{2n+1},z) \tag{3.37}$$

holds, then by hypotheses we have

$$F(d(Tu_{2n+1},Tz),d(u_{2n+1},z),d(u_{2n+1},Tu_{2n+1}),d(z,Tz),d(u_{2n+1},Tz),d(z,Tu_{2n+1})) \le 0, \tag{3.38}$$

and so

$$F(d(u_{2n+2}, Tz), d(u_{2n+1}, z), d(u_{2n+1}, u_{2n+2}), d(z, Tz), d(u_{2n+1}, Tz), d(z, u_{2n+2})) \leq 0. \tag{3.39}$$

Letting $n \to \infty$, we have

$$F(d(z, Tz), 0, 0, d(z, Tz), d(z, Tz), 0) \leq 0, \tag{3.40}$$

which contradicts (F_3).

Therefore, in all the cases, there exists $m \in \mathbb{N}$, such that $T^m z = z$. Since $\{T^n z\}$ is Cauchy sequence, we obtain $Tz = z$. That is, z is a fixed point of T. The uniqueness of fixed point follows easily from (3.6). □

Remark 3.2. If we combine Theorem 3.1 with Examples 2.1, 2.2, 2.3, and 2.4, we have Theorem 2 of [6], Theorem 2.2 of [7], Theorem 3.1 of [7], and Theorem 4 of [8], respectively.

Using Example 2.5, we obtain the following result.

Corollary 3.3. *Let (X, d) be a complete metric space, and let T be a mapping on X. Define a nonincreasing function θ from $[0, 1)$ into $(1/2, 1]$ as in Theorem 1.1. Assume that*

$$\theta(r) d(x, Tx) \leq d(x, y) \tag{3.41}$$

implies

$$d(Tx, Ty) \leq a d(x, Tx) + b d(y, Ty), \tag{3.42}$$

for all $x, y \in X$, where $a, b \in [0, 1/2)$, then there exists a unique fixed point of T.

Remark 3.4. We obtain some new results, if we combine Theorem 3.1 with some examples of F.

References

[1] A. Aliouche and V. Popa, "General common fixed point theorems for occasionally weakly compatible hybrid mappings and applications," *Novi Sad Journal of Mathematics*, vol. 39, no. 1, pp. 89–109, 2009.

[2] S. K. Chatterjea, "Fixed-point theorems," *Comptes Rendus de l'Académie Bulgare des Sciences*, vol. 25, pp. 727–730, 1972.

[3] Lj. B. Ćirić, "Generalized contractions and fixed-point theorems," *Publications de l'Institut Mathématique*, vol. 12(26), pp. 19–26, 1971.

[4] R. Kannan, "Some results on fixed points," *Bulletin of the Calcutta Mathematical Society*, vol. 60, pp. 71–76, 1968.

[5] T. Suzuki and M. Kikkawa, "Some remarks on a recent generalization of the Banach contraction principle," in *Fixed Point Theory and Its Applications*, pp. 151–161, Yokohama Publ., Yokohama, Japan, 2008.

[6] T. Suzuki, "A generalized Banach contraction principle that characterizes metric completeness," *Proceedings of the American Mathematical Society*, vol. 136, no. 5, pp. 1861–1869, 2008.

[7] M. Kikkawa and T. Suzuki, "Some similarity between contractions and Kannan mappings," *Fixed Point Theory and Applications*, vol. 2008, Article ID 649749, 8 pages, 2008.
[8] O. Popescu, "Fixed point theorem in metric spaces," *Bulletin of the Transilvania University of Braşov*, vol. 1(50), pp. 479–482, 2008.
[9] O. Popescu, "Two fixed point theorems for generalized contractions with constants in complete metric space," *Central European Journal of Mathematics*, vol. 7, no. 3, pp. 529–538, 2009.
[10] M. Kikkawa and T. Suzuki, "Some notes on fixed point theorems with constants," *Bulletin of the Kyushu Institute of Technology. Pure and Applied Mathematics*, no. 56, pp. 11–18, 2009.
[11] M. Kikkawa and T. Suzuki, "Three fixed point theorems for generalized contractions with constants in complete metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 9, pp. 2942–2949, 2008.
[12] Y. Enjouji, M. Nakanishi, and T. Suzuki, "A generalization of Kannan's fixed point theorem," *Fixed Point Theory and Applications*, vol. 2009, Article ID 192872, 10 pages, 2009.
[13] M. Kikkawa and T. Suzuki, "Some similarity between contractions and Kannan mappings. II," *Bulletin of the Kyushu Institute of Technology. Pure and Applied Mathematics*, no. 55, pp. 1–13, 2008.
[14] I. Altun and D. Turkoglu, "Some fixed point theorems for weakly compatible mappings satisfying an implicit relation," *Taiwanese Journal of Mathematics*, vol. 13, no. 4, pp. 1291–1304, 2009.
[15] M. Imdad and J. Ali, "Common fixed point theorems in symmetric spaces employing a new implicit function and common property (E.A)," *Bulletin of the Belgian Mathematical Society. Simon Stevin*, vol. 16, no. 3, pp. 421–433, 2009.
[16] V. Popa, M. Imdad, and J. Ali, "Using implicit relations to prove unified fixed point theorems in metric and 2-metric spaces," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 33, no. 1, pp. 105–120, 2010.