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Article

Some Fixed-Point Theorems in Proximity Spaces with Applications

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Abstract: Considering the ω-distance function defined by Kostić in proximity space, we prove the Matkowski and Boyd–Wong fixed-point theorems in proximity space using ω-distance, and provide some examples to explain the novelty of our work. Moreover, we characterize Edelstein-type fixed-point theorem in compact proximity space. Finally, we investigate an existence and uniqueness result for solution of a kind of second-order boundary value problem via obtained Matkowski-type fixed-point results under some suitable conditions.

Keywords: fixed point; proximity spaces; ϕ-contraction; ω-distance

MSC: 54H25; 47H10

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1. Introduction and Preliminaries

The basic concepts of proximity spaces were initially developed by Frigyes Riesz [1] in 1908, and later on this theory was revived and axiomatized by Efremovich [2] in 1934, which was published in 1951. Over the years, a lot of studies on proximity spaces have been carried out [3–6]. Smirnov [5] established the link between the proximity relation and topological spaces. In addition, he was the first to introduce the relationship between proximities and uniformities.

Let δ be a relation on a set 2^{Ω}. Then, the pair (Ω, δ) is said to a proximity space if the following hold: for all $U, V, W \in 2^{\Omega}$, where 2^{Ω} the power set of Ω

(Pr1) $U\delta V$ implies $V\delta U$;
(Pr2) $U\delta V$ implies $U, V \neq \emptyset$;
(Pr3) $U\delta(V \cup W)$ iff $U\delta V$ or $U\delta W$;
(Pr4) $U \cap V \neq \emptyset$ implies $U\delta V$;
(Pr5) For all $Y \subseteq \Omega$, $U\delta Y$ or $V\delta(\Omega - Y)$ implies $U\delta V$;

For all $\xi \in \Omega$ and $U \subseteq \Omega$, we will use the notation $\xi\delta U$ and $U\delta\xi$ instead of $\{\xi\}\delta U$ and $U\delta\{\xi\}$, respectively. A proximity space (Ω, δ) is called separated if $\xi\delta\eta$ implies $\xi = \eta$ for all $\xi, \eta \in \Omega$. The properties of proximity spaces are generalizations of uniform properties and continuity properties of metric and topology, respectively. Every proximity relation on a nonempty set Ω induces a topology τ_δ via the Kuratowski closure operator. The Kuratowski closure operator using proximity relation can be defined by $cl(U) = \{\xi \in \Omega : \xi\delta U\}$ for all $U \subseteq \Omega$. In this case, the topology τ_δ is always completely regular, further, it is Tychonoff if (Ω, δ) is separated. If (Ω, τ) is a topological space and δ is a proximity on Ω such that $\tau_\delta = \tau$, then τ and δ are said to be compatible. Every completely regular topology on a nonempty set Ω has a compatible proximity. Additionally, if a sequence $\{\xi_n\}$ converges to a point

$\xi \in \Omega$ with respect to induced topology τ_δ, then we obtain $\xi\delta\{\xi_n\}$. Furthermore, every uniform space $(\Omega, \mathcal{U})$ has an associated proximity structure defined by for all $U, V \subseteq \Omega$, $U\delta V$ if $(U \times V) \cap W \neq \emptyset$ for all $W \in \mathcal{U}$. For further information, see [7,8].

Now we state some examples of proximity spaces.

Example 1. *Let (Ω, ρ) be a metric space. Consider the relation δ on 2^Ω as*

$$U\delta V \Leftrightarrow \rho(U, V) = 0,$$

where $\rho(U, V) = \inf\{\rho(u, v) : u \in U, v \in V\}$. Then δ is a proximity on Ω. In addition, the metric topology τ_ρ and δ are compatible.

Example 2. *Let (Ω, τ) be a T_4 (both normal and T_1) topological space. Consider the relation δ on 2^Ω as*

$$U\delta V \Leftrightarrow \overline{U} \cap \overline{V} \neq \emptyset.$$

Then δ is a proximity on Ω. In addition, τ and δ are compatible.

Example 3. *Let (Ω, τ) be a completely regular topological space. Consider the relation δ on 2^Ω as $U\delta V$ if and only if there does not exist a continuous function $f : \Omega \to [0, 1]$ such that $f(U) = 0$ and $f(V) = 1$. Then δ is a proximity on Ω. In addition, τ and δ are compatible.*

Considering that fixed-point theory on proximity space will bring a new direction and interesting results, Kostić [9] defined the concept of ω-distance and ω_0-distance inspired by [10] (see [11] for more information about ω-distance) in proximity space as follows and obtained the proximity space version of Banach fixed-point theorem.

Definition 1. *Let (Ω, δ) be proximity space and $\omega : \Omega \times \Omega \longrightarrow [0, \infty)$ be a function. If ω satisfies*

(w1) , $\omega(\zeta, U) = 0$ and $\omega(\zeta, V) = 0$ implies $U\delta V$ for all $\zeta \in \Omega$ and $U, V \subseteq \Omega$,

then ω is said to be a ω-distance on Ω, where

$$\omega(\zeta, U) = \inf\{\omega(\zeta, \xi) : \xi \in U\}.$$

In addition, a ω-distance on a proximity space (Ω, δ) is called ω_0-distance if the following axioms hold:

(w2) $\omega(\xi, \eta) \leq \omega(\xi, \zeta) + \omega(\zeta, \eta)$ for all $\xi, \eta, \zeta \in \Omega$,
(w3) ω is lower semicontinuous in both variables with respect to τ_δ, i.e., for all $\xi, \eta \in \Omega$, we have

$$\omega(\xi, \eta) \leq \liminf_{\xi' \to \xi} \omega(\xi', \eta) = \sup_{V \in \mathcal{U}_\xi} \inf_{\xi' \in V} \omega(\xi', \eta)$$

and

$$\omega(\eta, \xi) \leq \liminf_{\xi' \to \xi} \omega(\eta, \xi') = \sup_{V \in \mathcal{U}_\xi} \inf_{\xi' \in V} \omega(\eta, \xi'),$$

where $\mathcal{U}_\xi$ is a base of neighborhoods of the point $\xi \in \Omega$.

Remark 1. *It is clear that if ω is lower semicontinuous in both variables with respect to τ_δ, then we have $\omega(\xi, \eta) \leq \liminf_{n\to\infty} \omega(\xi_n, \eta)$ and $\omega(\eta, \xi) \leq \liminf_{n\to\infty} \omega(\eta, \xi_n)$ for every sequence $\{\xi_n\}$ converging to ξ with respect to τ_δ.*

Example 4. *Let $\Omega = \mathbb{R}$ be endowed with the usual metric and the proximity δ defined in Example 1. Define $\omega_1, \omega_2 : \Omega \times \Omega \longrightarrow [0, \infty)$ by*

$$\omega_1(\xi, \eta) = \max\{|\xi|, |\eta|\}$$

and

$$\omega_2(\xi,\eta)=\frac{|\xi|+|\eta|}{2},$$

then both ω_1 and ω_2 are ω_0-distance on Ω.

Example 5. *Let $\Omega=[0,\infty)$ be endowed with the lower limit topology τ_l and the proximity δ defined in Example 2. It is well-known that $(\mathbb{R},\tau_l)$ is not metrizable but normal topological space. Define $\omega:\Omega\times\Omega\longrightarrow[0,\infty)$ by*

$$\omega(\xi,\eta)=\eta,$$

then ω is ω_0-distance on Ω.

Example 6. *Let $\Omega=C[0,2]$ be endowed with the metric ρ defined by*

$$\rho(f,g)=\int_0^2|f(\xi)-g(\xi)|d\xi$$

and the proximity δ defined in Example 1. Define $\omega:\Omega\times\Omega\longrightarrow[0,\infty)$ by

$$\omega(f,g)=\int_0^2|g(\xi)|d\xi,$$

then ω is ω_0-distance on Ω.

Proof. Let $U,V\subseteq C[0,2]$, $h\in C[0,2]$ and

$$\omega(h,U)=0=\omega(h,V). \tag{1}$$

Let $\varepsilon>0$. Then by (1) there exist $f\in U$ and $g\in V$ such that

$$\omega(h,f)=\int_0^2|f(\xi)|d\xi<\frac{\varepsilon}{2}\text{ and }\omega(h,g)=\int_0^2|g(\xi)|d\xi<\frac{\varepsilon}{2}.$$

Hence we have

$$\begin{aligned}\rho(f,g)&=\int_0^2|f(\xi)-g(\xi)|d\xi\\&\le\omega(h,f)+\omega(h,g)\\&<\varepsilon.\end{aligned}$$

Thus we have

$$\rho(U,V)=\inf\{\rho(f,g):f\in U,g\in V\}=0$$

and hence $U\delta V$. Therefore, (*w1*) holds, and so ω is a ω-distance on Ω. It is clear that (*w2*) holds. Now, let $f,g\in\Omega$, then we have

$$\omega(f,g)=\int_0^2|g(\xi)|d\xi=\liminf_{f'\to f}\omega(f',g)=\sup_{V\in\mathcal{U}_f}\inf_{f'\in V}\omega(f',g),$$

that is, ω is lower semicontinuous in the first variable. On the other hand, let $V\in\mathcal{U}_f$ and $f'\in V$, then there exists $\varepsilon>0$ such that $\rho(f,f')<\varepsilon$, that is $\sup_{V\in\mathcal{U}_f}\inf_{f'\in V}\rho(f,f')=0$. Hence, we have

$$\begin{aligned}\omega(g,f)&=\int_0^2|f(\xi)|d\xi\\&\le\int_0^2|f(\xi)-f'(\xi)|d\xi+\int_0^2|f'(\xi)|d\xi\\&\le\varepsilon+\omega(g,f')\end{aligned}$$

and so we obtain

$$\omega(g,f) \leq \sup_{V\in\mathcal{U}_f} \inf_{f'\in V} \omega(g,f'),$$

that is, ω is lower semicontinuous in second variable. $\square$

Example 7. *Let $\Omega = C[0,1]$ be endowed with the metric ρ defined by*

$$\rho(f,g) = \sup\{|f(\xi) - g(\xi)| : \xi \in [0,1]\}$$

and the proximity δ defined in Example 1. Define $\omega_1, \omega_2 : \Omega \times \Omega \longrightarrow [0,\infty)$ by

$$\omega_1(f,g) = \sup\{|g(\xi)| : \xi \in [0,1]\}$$

and

$$\omega_2(f,g) = \sup\{|f(\xi)| + |g(\xi)| : \xi \in [0,1]\},$$

then ω_1 and ω_2 are ω_0-distance on Ω.

Lemma 1 ([9,10])**.** *Let (Ω,δ) be a proximity space with ω-distance ω. Then, the following properties hold:*

(i) *If (Ω,δ) is separated, then $\omega(\zeta,\xi) = 0$ and $\omega(\zeta,\eta) = 0$ implies $\xi = \eta$.*
(ii) *If $\omega(\zeta,\xi) = 0$ and $\omega(\zeta,\xi_n) \to 0$ as $n \to \infty$, then $\{\xi_n\}$ subsequently converges to ξ with respect to τ_δ.*

In [9], Kostić introduced the Banach contraction principle in proximity space using ω_0-distance ω as follows:

Theorem 1. *Let (Ω,δ) be a separated proximity space with ω_0-distance ω. Suppose that the mapping $T : \Omega \longrightarrow \Omega$ satisfies the following:*

(i) *There exists $k \in (0,1)$ such that*

$$\omega(T\xi, T\eta) \leq k\omega(\xi,\eta)$$

for all $\xi,\eta \in \Omega$;

(ii) *For all $\xi \in \Omega$, any iterative sequence $\{T^n\xi\}$ has a convergent subsequence with respect to τ_δ.*

Then, there exists $\zeta \in \Omega$, such that $\zeta = T\zeta$ and $\omega(\zeta,\zeta) = 0$.

In this study, we will use some auxiliary functions in the contraction inequality to generalize Kostić's theorem. Thus, we will obtain equivalents in proximity space of the famous Boyd–Wong [12] and Matkowski [13] fixed-point theorems known in metric fixed-point theory. In addition, for the equivalent of Edelstein's theorem [14], in this space, we will consider that the space is compact according to the topology induced by proximity.

Let $\phi : [0,\infty) \to [0,\infty)$ be a function. Next, we will consider the below properties for ϕ:

(ϕ_1) ϕ is non-decreasing;
(ϕ_2) For all $t \geq 0$, $\lim\limits_{n\to\infty} \phi^n(t) = 0$;
(ϕ_3) For all $t > 0$, $\phi(t) < t$;
(ϕ_4) ϕ is upper semicontinuous from the right.

We will symbolize the set of all functions with properties (ϕ_1–ϕ_2) and (ϕ_3–ϕ_4), by Φ and Ψ respectively. It is easy to see that both $\Phi\backslash\Psi$ and $\Psi\backslash\Phi$ are nonempty and also every function in Φ satisfies the property (ϕ_3). Some examples of the functions belonging both Φ and Ψ are $\phi_1(t) = Lt$, where $0 \leq L < 1$ and $\phi_2(t) = \frac{t}{1+t}$.

2. Main Result

Here, we present our main theorems.

Theorem 2. *Let (Ω, δ) be a separated proximity space with ω_0-distance ω and $T : \Omega \longrightarrow \Omega$ be a mapping satisfying the followings:*

(i) *There exists $\phi \in \Phi$, satisfying*

$$\omega(T\xi, T\eta) \leq \phi(\omega(\xi, \eta))$$

for all $\xi, \eta \in \Omega$;

(ii) *For all $\xi \in \Omega$, any iterative sequence $\{T^n \xi\}$ has a convergent subsequence with respect to τ_δ.*

Then there exists $\zeta \in \Omega$, such that $\zeta = T\zeta$ and $\omega(\zeta, \zeta) = 0$.

Proof. Let $\xi_0 \in \Omega$ be arbitrary. Consider the corresponding Picard sequence $\{\xi_n\}$ constructed by

$$\xi_n = T^n \xi_0 = T\xi_{n-1}.$$

Since ϕ is non-decreasing, and by (i), we have

$$\begin{aligned} \omega(\xi_n, \xi_{n+1}) &= \omega(T\xi_{n-1}, T\xi_n)) \\ &\leq \phi(\omega(\xi_{n-1}, \xi_n)) \\ &\vdots \\ &\leq \phi^n(\omega(\xi_0, \xi_1)). \end{aligned}$$

Similarly, we can obtain

$$\omega(\xi_{n+1}, \xi_n) \leq \phi^n(\omega(\xi_1, \xi_0)).$$

By (ϕ_2), we have

$$\lim_{n \to \infty} \omega(\xi_n, \xi_{n+1}) = 0 \tag{2}$$

and

$$\lim_{n \to \infty} \omega(\xi_{n+1}, \xi_n) = 0. \tag{3}$$

On the other hand, since $\phi \in \Phi$, we have $\phi(\epsilon) < \epsilon$ for all $\epsilon > 0$. Hence by (2), there exists a natural number N satisfying

$$\omega(\xi_N, \xi_{N+1}) < \epsilon - \phi(\epsilon)$$

for all $\epsilon > 0$.

Since ω is a ω_0-distance and ϕ is non-decreasing, then we have

$$\begin{aligned} \omega(\xi_N, \xi_{N+2}) &\leq \omega(\xi_N, \xi_{N+1}) + \omega(\xi_{N+1}, \xi_{N+2}) \\ &< \epsilon - \phi(\epsilon) + \omega(T\xi_N, T\xi_{N+1}) \\ &\leq \epsilon - \phi(\epsilon) + \phi(\omega(\xi_N, \xi_{N+1})) \\ &< \epsilon - \phi(\epsilon) + \phi(\epsilon - \phi(\epsilon)) \\ &\leq \epsilon - \phi(\epsilon) + \phi(\epsilon) \\ &= \epsilon. \end{aligned}$$

Similarly,

$$\begin{aligned}
\omega(\xi_N, \xi_{N+3}) &\leq \omega(\xi_N, \xi_{N+1}) + \omega(\xi_{N+1}, \xi_{N+3}) \\
&< \epsilon - \phi(\epsilon) + \omega(T\xi_N, T\xi_{N+2}) \\
&\leq \epsilon - \phi(\epsilon) + \phi(\omega(\xi_N, \xi_{N+2})) \\
&< \epsilon - \phi(\epsilon) + \phi(\epsilon) \\
&= \epsilon.
\end{aligned}$$

Continuing this process, we obtain $\omega(\xi_N, \xi_{N+k}) < \epsilon$ for all $k = 1, 2, \cdots$ and similarly by (3) we obtain $\omega(\xi_{N+k}, \xi_N) < \epsilon$ for all $k = 1, 2, \cdots$. Thus, for all $m, n > N$, we have

$$\omega(\xi_n, \xi_m) \leq \omega(\xi_n, \xi_N) + \omega(\xi_N, \xi_m) < 2\epsilon. \tag{4}$$

By (*ii*), there exists a subsequence $\{\xi_{n_k}\}$ of the sequence $\{\xi_n\}$ which is convergent with respect to τ_δ to some $\zeta \in \Omega$, and by $(w3)$, Remark 1 and Inequality (4), we have

$$\omega(\zeta, \xi_{n_k}) \leq \liminf_{l \to \infty} \omega(\xi_{n_l}, \xi_{n_k}) \leq 2\epsilon, \tag{5}$$

and symmetrically, we obtain

$$\omega(\xi_{n_k}, \zeta) \leq 2\epsilon \tag{6}$$

for all $n_k > N$. By Inequality (6), we have

$$\omega(T\xi_{n_k}, T\zeta) \leq \phi(\omega(\xi_{n_k}, \zeta)) \leq \phi(2\epsilon) < 2\epsilon \tag{7}$$

for all $n_k > N$. Finally, by Inequalities (5) and (7), we have

$$\begin{aligned}
\omega(\zeta, T\zeta) &\leq \omega(\zeta, \xi_{n_k}) + \omega(\xi_{n_k}, T\xi_{n_k}) + \omega(T\xi_{n_k}, T\zeta) \\
&< 2\epsilon + \phi^{n_k}(\omega(\xi_0, \xi_1)) + 2\epsilon \\
&= 4\epsilon + \phi^{n_k}(\omega(\xi_0, \xi_1))
\end{aligned}$$

for all $n_k > N$. Since, by (ϕ_2), $\phi^{n_k}(\omega(\xi_0, \xi_1)) \to 0$ as $k \to \infty$, it follows that $\omega(\zeta, T\zeta) = 0$. On the other hand, by $(w3)$ and Remark 1, we have

$$\omega(\zeta, \zeta) \leq \liminf_{l \to \infty} \omega(\zeta, \xi_{n_l}) \leq 2\epsilon,$$

and consequently, $\omega(\zeta, \zeta) = 0$. Thus, by Lemma 1 (*i*), we have $\zeta = T\zeta$. To check the uniqueness, let $\varsigma \in \Omega$ be a different fixed point of T. Then, $\omega(\zeta, \varsigma) > 0$, because $\zeta \neq \varsigma$ and $\omega(\zeta, \zeta) = 0$. By (*i*), we obtain

$$0 < \omega(\zeta, \varsigma) = \omega(T\zeta, T\varsigma) \leq \phi(\omega(\zeta, \varsigma)) < \omega(\zeta, \varsigma),$$

a contradiction. Hence, the fixed point of T is unique. □

Remark 2. *Note that for $\phi(t) = kt$, $0 \leq k < 1$, Theorem 2 reduces to Theorem 1.*

Example 8. *Let (Ω, δ) be the proximity space and ω be the ω_0-distance given in Example 5. Clearly (Ω, δ) is separated proximity space. Define a mapping $T : \Omega \to \Omega$ by $T\xi = \frac{\xi}{1+\xi}$ and consider the function $\phi \in \Phi$ as $\phi(t) = \frac{t}{1+t}$. In this case, we have*

$$\omega(T\xi, T\eta) = T\eta = \frac{\eta}{1+\eta} = \phi(\eta) = \phi(\omega(\xi, \eta))$$

for all $\xi, \eta \in \Omega$. In addition, for all $\xi \in \Omega$, $T^n\,\xi = \frac{\xi}{1+n\xi}$ is convergent with respect to τ_δ. Hence, all conditions of Theorem 2 hold. Therefore, T has a unique fixed point. Note that, since

$$\sup_{\eta \in \Omega} \frac{\omega(T\xi, T\eta)}{\omega(\xi, \eta)} = 1,$$

we can not find a constant $k \in (0,1)$ satisfying

$$\omega(T\xi, T\eta) \leq k\omega(\xi, \eta).$$

Hence, Theorem 1 cannot be applied.

Theorem 3. *Let (Ω, δ) be a separated proximity space with ω_0-distance ω and $T : \Omega \longrightarrow \Omega$ be a mapping satisfying the followings:*

(i) There exists $\phi \in \Psi$ such that $\phi(0) = 0$ and ϕ is right continuous at 0 satisfying $\omega(T\xi, T\eta) \leq \phi(\omega(\xi, \eta))$ for all $\xi, \eta \in \Omega$;
(ii) For all $\xi \in \Omega$, any iterative sequence $\{T^n\xi\}$ has a convergent subsequence with respect to τ_δ.

Then, there exists $\zeta \in \Omega$ such that $\zeta = T\zeta$ and $\omega(\zeta, \zeta) = 0$.

Proof. Let $\xi_0 \in \Omega$ be arbitrary. Consider the corresponding Picard sequence $\{\xi_n\}$ constructed by

$$\xi_n = T^n\xi_0 = T\xi_{n-1}.$$

for all $n \in \mathbb{N}$. For simplicity, call $\omega_n = \omega(\xi_n, \xi_{n+1})$. If there exists $n_0 \in \mathbb{N}$ such that $\omega_{n_0} = \omega_{n_0+1} = 0$, then by triangular inequality we have

$$\begin{aligned} \omega(\xi_{n_0}, \xi_{n_0+2}) &\leq \omega(\xi_{n_0}, \xi_{n_0+1}) + \omega(\xi_{n_0+1}, \xi_{n_0+2}) \\ &= \omega_{n_0} + \omega_{n_0+1} = 0 \end{aligned}$$

and so by Lemma 1 (*i*) we have $\xi_{n_0+1} = \xi_{n_0+2}$. This shows that ξ_{n_0+1} is a fixed point of T. Now assume that any of consecutive terms of the $\{\omega_n\}$ are not zero. In this case, there exists a subsequence $\{\omega_{n_k}\}$ of $\{\omega_n\}$ such that $\omega_{n_k} > 0$ for all $k \in \mathbb{N}$. By (*i*), we have

$$\begin{aligned} \omega_{n_k+1} &= \omega(\xi_{n_k+1}, \xi_{n_k+2}) \\ &= \omega(T\xi_{n_k}, T\xi_{n_k+1}) \\ &\leq \phi(\omega(\xi_{n_k}, \xi_{n_k+1})) \\ &= \phi(\omega_{n_k}) < \omega_{n_k}. \end{aligned} \tag{8}$$

Since the index sequence n_k is strictly increasing, we have $n_{k+1} > n_k$ and so $n_{k+1} \geq n_k + 1$. Therefore, from (8), we have

$$\omega_{n_{k+1}} \leq \omega_{n_k+1} < \omega_{n_k}. \tag{9}$$

It follows that $\{\omega_{n_k}\}$ is a decreasing sequence which is also bounded below. Thus, there exists $\gamma \geq 0$ such that $\lim_{k\to\infty} \omega_{n_k} = \gamma$. If $\gamma > 0$, then by (9), (ϕ_3) and (ϕ_4), we have

$$0 < \gamma = \lim_{k\to\infty} \omega_{n_{k+1}} \leq \lim_{k\to\infty} \omega_{n_k+1} \leq \lim_{k\to\infty} \phi(\omega_{n_k}) \leq \lim_{k\to\infty} \sup \phi(\omega_{n_k}) \leq \phi(\gamma) < \gamma,$$

a contradiction. Therefore, $\gamma = 0$, and consequently,

$$\lim_{k\to\infty} \omega_{n_k} = 0. \tag{10}$$

Similarly, it can be obtained that the limit of all subsequences of $\{\omega_n\}$ whose all terms are greater than zero, is zero. Hence we have

$$\lim_{n\to\infty} \omega_n = \lim_{n\to\infty} \omega(\xi_n, \xi_{n+1}) = 0.$$

Now, if we call $\varpi_n = \omega(\xi_{n+1}, \xi_n)$, similarly we can obtain

$$\lim_{n\to\infty} \varpi_n = \lim_{n\to\infty} \omega(\xi_{n+1}, \xi_n) = 0. \tag{11}$$

Now, we claim that for every $\epsilon > 0$, there exists a natural number N satisfying

$$\omega(\xi_m, \xi_n) < \epsilon$$

for all $m > n > N$. Assume the contrary, then there exist $\epsilon > 0$, $m_k, n_k \in \mathbb{N}$ with $m_k > n_k > k$, $k \in \mathbb{N}$ such that $\omega(\xi_{m_k}, \xi_{n_k}) \geq \epsilon$. Here, we can choose m_k as the smallest positive integer that satisfies $\omega(\xi_{m_k}, \xi_{n_k}) \geq \epsilon$. Therefore, we obtain $\omega(\xi_{m_k-1}, \xi_{n_k}) < \epsilon$, and since ω is a ω_0-distance, we obtain

$$\begin{aligned}
\epsilon &\leq \omega(\xi_{m_k}, \xi_{n_k}) \\
&\leq \omega(\xi_{m_k}, \xi_{m_k-1}) + \omega(\xi_{m_k-1}, \xi_{n_k}) \\
&\leq \omega(\xi_{m_k}, \xi_{m_k-1}) + \epsilon \\
&= \varpi_{m_k-1} + \epsilon.
\end{aligned}$$

By Equation (11), it follows that $\lim_{k\to\infty} \omega(\xi_{m_k}, \xi_{n_k}) = \epsilon$. On other hand,

$$\begin{aligned}
\omega(\xi_{m_k}, \xi_{n_k}) &\leq \omega(\xi_{m_k}, \xi_{m_k+1}) + \omega(\xi_{m_k+1}, \xi_{n_k+1}) + \omega(\xi_{n_k+1}, \xi_{n_k}) \\
&= \omega_{m_k} + \omega(T\xi_{m_k}, T\xi_{n_k}) + \varpi_{n_k} \\
&\leq \omega_{m_k} + \phi(\omega(\xi_{m_k}, \xi_{n_k})) + \varpi_{n_k}.
\end{aligned}$$

Taking limit $k \to \infty$ and by (ϕ_4), we have $\epsilon \leq \phi(\epsilon)$, a contradiction. Thus, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that

$$\omega(\xi_m, \xi_n) < \epsilon \tag{12}$$

for all $m > n > N$. Similarly, we obtain

$$\omega(\xi_n, \xi_m) < \epsilon \tag{13}$$

for all $m > n > N$.

By (*ii*), there exists a subsequence $\{\xi_{n_k}\}$ of sequence $\{\xi_n\}$ which is convergent with respect to τ_δ to some $\zeta \in \Omega$, and by (ω_3) and Inequality (12), we have

$$\omega(\zeta, \xi_{n_k}) \leq \lim_{l\to\infty} \inf \omega(\xi_{n_l}, \xi_{n_k}) \leq \epsilon, \tag{14}$$

for all $n_k \in \mathbb{N}$ and symmetrically by (13), we obtain

$$\omega(\xi_{n_k}, \zeta) \leq \epsilon \tag{15}$$

for all $n_k \in \mathbb{N}$. Then, we have

$$\omega(\zeta, \zeta) \leq \lim_{k\to\infty} \inf \omega(\zeta, \xi_{n_k}) \leq \epsilon,$$

and consequently, $\omega(\zeta, \zeta) = 0$. On the other hand, we have

$$\begin{aligned}
\omega(\zeta, T\zeta) &\leq \omega(\zeta, \xi_{n_k}) + \omega(\xi_{n_k}, T\zeta) \\
&= \omega(\zeta, \xi_{n_k}) + \omega(T\xi_{n_k-1}, T\zeta) \\
&< \epsilon + \phi(\omega(\xi_{n_k-1}, \zeta)).
\end{aligned}$$

Since $\phi(0) = 0$ and ϕ is right continuous at 0, we have $\omega(\zeta, T\zeta) = 0$. Thus, by Lemma 1 (*i*), we have $\zeta = T\zeta$.

To check the uniqueness, let $v \in \Omega$ with $v \neq \zeta$ be another fixed point of mapping T. Since $\omega(\zeta, \zeta) = 0$ it must be $\omega(\zeta, v) > 0$. Hence by (*i*), we obtain

$$\omega(\zeta, v) = \omega(T\zeta, Tv) \leq \phi(\omega(\zeta, v)) < \omega(\zeta, v),$$

a contradiction. Thus, the fixed point of T is unique. □

Finally, in the theoretical part, we present a fixed-point result, taking into account the compactness of the space and a strict contraction inequality. Here, we will use the following well-known lemma.

Lemma 2 ([15]). *Let $f : \Omega \to \mathbb{R}$ a lower semicontinuous function, where Ω is a compact topological space. Then, there exists an element $\xi_0 \in \Omega$ such that*

$$f(\xi_0) = \inf\{f(\xi) : \xi \in \Omega\}.$$

Definition 2. *Let (Ω, δ) be a proximity space with ω-distance ω and $T : \Omega \longrightarrow \Omega$ be a mapping. Then T is said to have 0-property if $\omega(T\xi, T\eta) = 0$ whenever $\omega(\xi, \eta) = 0$ for all $\xi, \eta \in \Omega$.*

Example 9. *Let $\Omega = [0, \infty)$ be endowed with the lower limit topology τ_l and the proximity δ defined in Example 2. Consider the ω-distance on Ω defined by $\omega(\xi, \eta) = \eta$. Then every self-mapping T of Ω has 0-property, provided that $T0 = 0$.*

Theorem 4. *Let (Ω, δ) be a separated proximity space with ω_0-distance ω such that Ω is compact with respect to τ_δ, and $T : \Omega \longrightarrow \Omega$ be a continuous mapping with 0-property. Assume that*

$$\omega(T\xi, T\eta) < \omega(\xi, \eta)$$

for all $\xi, \eta \in \Omega$ with $\omega(\xi, \eta) > 0$. Then, there exists $\varsigma \in \Omega$ such that $\varsigma = T\varsigma$.

Proof. Define $f, g : \Omega \to \mathbb{R}$ by $f(\xi) = \omega(\xi, T\xi)$ and $g(\xi) = \omega(T\xi, \xi)$. Since ω is lower semicontinuous in both variables and T is continuous, then f and g are lower semicontinuous. Since Ω is compact with respect to τ_δ, then from Lemma 2 there exist $\zeta, \varsigma \in \Omega$, such that

$$f(\zeta) = \inf\{f(\xi) : \xi \in \Omega\}$$

and

$$g(\varsigma) = \inf\{g(\xi) : \xi \in \Omega\}.$$

In this case, if $\omega(\zeta, T\zeta) > 0$, then we obtain

$$f(T\zeta) = \omega(T\zeta, TT\zeta) < \omega(\zeta, T\zeta) = f(\zeta),$$

which is a contradiction. Hence, we obtain $f(\zeta) = \omega(\zeta, T\zeta) = 0$. Similarly, we obtain $\omega(T\varsigma, \varsigma) = 0$.

Now, if $\omega(\zeta,\varsigma) > 0$, then we obtain

$$\begin{aligned} \omega(\zeta,\varsigma) &\leq \omega(\zeta,T\zeta)+\omega(T\zeta,T\varsigma)+\omega(T\varsigma,\varsigma) \\ &= \omega(T\zeta,T\varsigma) \\ &< \omega(\zeta,\varsigma), \end{aligned}$$

which is a contradiction. Hence, we have $\omega(\zeta,\varsigma) = 0$. Therefore, from Lemma 1 (*i*), we have $\varsigma = T\zeta$ and so $\omega(T\varsigma, T\zeta) = 0$. Now,

$$\omega(T\varsigma,T\varsigma) \leq \omega(T\varsigma,T\zeta)+\omega(T\zeta,T\varsigma) = 0,$$

and so from Lemma 1 (*i*), we have $\varsigma = T\varsigma$. Additionally,

$$\omega(\varsigma,\varsigma) = \omega(T\zeta,T\varsigma) = 0.$$

Now let v be another fixed point of T. Then, we have $v = Tv$ and $\omega(\varsigma, v) > 0$. Therefore, we obtain

$$\omega(\varsigma,v) = \omega(T\varsigma,Tv) < \omega(\varsigma,v),$$

which is a contradiction. □

3. Application

Now, by considering Theorem 2, we give existence and uniqueness results about the solution of the second-order boundary value problem (BVP) as follows:

$$\begin{cases} -\frac{d^2\xi}{dt^2} = F(t,\xi(t)),\ t \in [0,1] \\ \xi(0) = \xi(1) = 0 \end{cases}, \tag{16}$$

where $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function. By taking into account some certain conditions F, some existence theorems were recently presented for problem (16) in the literature (see [16–18]). Here, we will consider some different conditions on F, and we provide a new theorem. We can see that the problem (16) is equivalent to the integral equation

$$\xi(t) = \int_0^1 G(t,s)F(s,\xi(s))ds,\ t \in [0,1], \tag{17}$$

where $G(t,s)$ is associated Green's function defined as

$$G(t,s) = \begin{cases} t(1-s) & , \quad 0 \leq t \leq s \leq 1 \\ \\ s(1-t) & , \quad 0 \leq s \leq t \leq 1 \end{cases}.$$

Therefore, $\xi \in C^2[0,1]$ is a solution of (16) if and only if it is a solution of (17). It is clear that

$$\int_0^1 G(t,s)ds = \frac{t(1-t)}{2}.$$

Let (Ω,δ) be the proximity space, where $\Omega = C[0,1]$ and δ is induced by the uniform metric ρ_∞

$$\rho_\infty(\xi,\eta) = \sup\{|\xi(t)-\eta(t)| : t \in [0,1]\}.$$

In this case, (Ω,δ) is separated proximity space. Consider the following ω_0-distances ω_α on Ω defined by

$$\omega_\alpha(\xi,\eta) = \sup\{e^{-\alpha t}|\xi(t)-\eta(t)| : t \in [0,1]\},$$

where $\alpha > 0$ is a constant.

Theorem 5. *The second-order BVP given by (16) has a unique solution under the following assumptions:*

(a) There exists a non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ such that

$$|F(s,\xi) - F(s,\eta)| \leq \varphi(e^{-\alpha s}|\xi - \eta|)$$

for all $s \in [0,1]$.

(b) $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \geq 0$, where $\phi = K_\alpha \varphi$ and

$$K_\alpha = \sup\left\{e^{-\alpha t}\frac{t(1-t)}{2} : t \in [0,1]\right\}.$$

Proof. Consider the operator $T : C[0,1] \times C[0,1] \to C[0,1]$ defined by

$$T\xi(t) = \int_0^1 G(t,s)F(s,\xi(s))ds$$

Then, for any $\xi, \eta \in C[0,1]$ and $t \in [0,1]$, we have

$$\begin{aligned}
|T\xi(t) - T\eta(t)| &= \left|\int_0^1 G(t,s)F(s,\xi(s))ds - \int_0^1 G(t,s)F(s,\eta(s))ds\right| \\
&\leq \int_0^1 G(t,s)|F(s,\xi(s)) - F(s,\eta(s))|ds \\
&\leq \int_0^1 G(t,s)\varphi(e^{-\alpha s}|\xi(s) - \eta(s)|)ds \\
&\leq \varphi(\omega_\alpha(\xi,\eta))\int_0^1 G(t,s)ds \\
&= \varphi(\omega_\alpha(\xi,\eta))\frac{t(1-t)}{2}
\end{aligned}$$

and then we obtain

$$e^{-\alpha t}|T\xi(t) - T\eta(t)| \leq \varphi(\omega_\alpha(\xi,\eta))e^{-\alpha t}\frac{t(1-t)}{2}.$$

By taking supremum over $t \in [0,1]$, we have

$$\omega_\alpha(T\xi, T\eta) \leq K_\alpha \varphi(\omega_\alpha(\xi,\eta)) = \phi(\omega_\alpha(\xi,\eta)). \tag{18}$$

Therefore, condition (i) of Theorem 2 is satisfied. Now let $\xi \in C[0,1]$ be an arbitrary function. Define a sequence of functions $\{\xi_n\}$ as $\xi_n = T^n\xi$. As in the proof of Theorem 2 and by (18), we have

$$\omega_\alpha(\xi_n, \xi_m) \to 0$$

as $m, n \to \infty$. On the other hand, since

$$e^{-\alpha}\rho_\infty(\xi,\eta) \leq \omega_\alpha(\xi,\eta) \leq \rho_\infty(\xi,\eta)$$

for all $\xi, \eta \in C[0,1]$, we have

$$\rho_\infty(\xi_n, \xi_m) \to 0$$

as $m, n \to \infty$. That is, the sequence $\{\xi_n\}$ is Cauchy and so has a convergent subsequence with respect to ρ_∞ since (Ω, ρ_∞) is complete. Therefore, condition (ii) of Theorem 2 is satisfied. Consequently, there exists unique $\zeta \in C[0,1]$ which is a fixed point of the operator T, moreover $\omega_\alpha(\zeta,\zeta) = 0$. Hence, the (16) has a unique solution. □

4. Conclusions

In this paper, following Kostić's, idea we present some fixed-point theorems for single-valued mappings on proximity space by using ω-distance and some auxiliary functions. Then, we support our theoretical results with some examples and an existence and uniqueness theorem.

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References

1. Riesz, F. Stetigkeit und abstrakte Mengenlehre. *Rom. 4 Math. Kongr.* **1908**, *2*, 18–24.
2. Efremovich, V.A. Infinitesimal spaces. *Dokl. Akad. Nauk. SSSR* **1951**, *76*, 341–343.
3. Kula, M.; Özkan, S.; Maraşlı, T. Pre-Hausdorff and Hausdorff proximity spaces. *Filomat* **2017**, *31*, 3837–3846. [CrossRef]
4. Kula, M.; Özkan, S. Regular and normal objects in the category of proximity spaces. *Kragujev. J. Math.* **2019**, *43*, 127–137.
5. Smirnov, Y.M. On proximity spaces. *Mat. Sb.* **1952**, *31*, 543–574.
6. Smirnov, Y.M. On the completeness of proximity spaces. *Dokl. Akad. Nauk SSSR* **1953**, *88*, 761–764.
7. Naimpally, S.A.; Warrack, B.D. *Proximity Spaces*; Cambridge University Press: Cambridge, UK, 1970.
8. Sharma, P.L. Two examples in proximity spaces. *Proc. Am. Math. Soc.* **1975**, *53*, 202–204. [CrossRef]
9. Kostić, A. ω-distances on proximity spaces and a fixed point theorem. In Proceedings of the International Workshop on Nonlinear Analysis and Its Applications, Niš, Serbia, 13–16 October 2021.
10. Kada, O.; Suzuki, T.; Takahashi, W. Nonconvex minimization theorems and fixed point theorems in complete metric spaces. *Math. Jpn.* **1996**, *44*, 381–391.
11. Babaei, R.; Rahimi, H.; Sen, M.; Rad, G.S. w-b-cone distance and its related results: A survey. *Symmetry* **2020**, *12*, 171. [CrossRef]
12. Boyd, D.W.; Wong, J.S.W. On nonlinear contractions. *Proc. Am. Math. Soc.* **1969**, *20*, 458–464. [CrossRef]
13. Matkowski, J. Fixed point theorems for mappings with a contractive iterate at a point. *Proc. Am. Math. Soc.* **1977**, *62*, 344–348. [CrossRef]
14. Edelstein, M. On fixed and periodic points under contractive mappings. *J. Lond. Math. Soc.* **1962**, *37*, 74–79. [CrossRef]
15. Agarwal, R.P.; O'Regan, D.; Sahu, D.R. *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*; Springer: New York, NY, USA, 2009.
16. Aydi, H.; Karapınar, E. Fixed point results for generalized α-ψ-contractions in metric-like spaces and applications. *Electron. J. Differ. Equ.* **2015**, *133*, 1–15.
17. Saipara, P.; Khammahawong, K.; Kumam, P. Fixed-point theorem for a generalized almost Hardy-Rogers-type F contraction on metric-like spaces. *Math. Meth. Appl. Sci.* **2019**, *42*, 5898–5919. [CrossRef]
18. Vetro, F. Fixed point for α-Θ-Φ -contractions and first-order periodic differential problem. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2019**, *113*, 1823–1837. [CrossRef]