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Assessment of Shock Models for a Particular Class of Intershock Time Distributions

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Abstract

In this paper, delta and extreme shock models and a mixed shock model which combines delta-shock and extreme shock models are studied. In particular, the interarrival times between successive shocks are assumed to belong to a class of matrix-exponential distributions which is larger than the class of phase-type distributions. The Laplace -Stieltjes transforms of the systems' lifetimes are obtained in a matrix form. Survival functions of the systems are approximated based on the Laplace-Stieltjes transforms. The results are applied for the reliability evaluation of a certain repairable system consisting of two components.

Keywords Matrix-exponential distribution · Reliability · Repairable system · Shock model

Mathematics Subject Classification (2010) Primary: 62N05 Secondary: 62E15

1 Introduction

Shock models are of special importance in applied probability. Various shock models have been introduced and studied in the literature. According to the δ-shock model, the system failure occurs when the times between two consecutive shocks falls below a fixed threshold δ. That is, the system breaks down due to the occurrence of two consecutive shocks which are too close to each other (see, e.g. Jiang (2020), Li (2007), Wen et al. 2017). A mixed shock model which combines δ-shock model by the well-known extreme shock model has been studied by Wang and Zhang (2005). For the latter model, the system fails if either the

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time between two consecutive shocks falls below a fixed threshold δ or the magnitude of a single shock exceeds another critical level γ.

For the δ-shock model, if $X_1, X_2, ...$ are independent and identically distributed (iid) with common cumulative distribution function (cdf) F, then the survival function of the system can be represented as

$$P\{T_\delta > t\} = \sum_{n=1}^{\infty} \left(\bar{F}(\delta)\right)^{n-1} \int_0^\delta P\left\{S_{n-1}^* > t - x\right\} dF(x), \tag{1}$$

where S_n^* is the nth arrival time of a renewal process whose interarrival times have the cdf

$$F_\delta^*(x) = \frac{F(x) - F(\delta)}{1 - F(\delta)},$$

for $x > \delta$ (Eryilmaz and Bayramoglu 2014).

Using the same method in Eryilmaz and Bayramoglu (2014), Parvardeh and Balakrishnan (2015) obtained the following equation for the survival function of the system under the mixed shock model.

$$\begin{aligned} P\left\{T_{\delta,\gamma} > t\right\} = {} & (F(\delta) - F(t))\, I_{[0,\delta)}(t) \\ & + \sum_{n=1}^{\infty} (G(\gamma))^{n-1} \left(\bar{F}(\delta)\right)^{n-1} \int_0^\infty P\left\{S_{n-1}^* > t - x\right\} dF(x) \\ & + \bar{F}(\max\{\delta, t\}) \bar{G}(\gamma) - \sum_{n=2}^{\infty} (G(\gamma))^n \left(\bar{F}(\delta)\right)^n P\left\{S_n^* > t\right\}, \end{aligned} \tag{2}$$

where $G(y) = P\{Y_i \le y\}, i = 1, 2, ...$ is the common cdf of the magnitudes of shocks.

The expressions (1) and (2) are not efficient since they need the distribution of the sum of the random variables that follow the common distribution F_δ^* which is quite difficult to derive even for the case when the common distribution F of intershock times has exponential distribution. Thus, the study of δ-shock models and the derivation of mathematically efficient formulas for the survival functions are of interest at least for a particular class of intershock time distributions.

Throughout the paper, the times $X_1, X_2, ...$ between successive shocks are assumed to follow matrix-exponential distribution with common cumulative distribution function

$$F(x) = P\{X \le x\} = 1 + \boldsymbol{\alpha} \exp(\boldsymbol{S}x)\boldsymbol{S}^{-1}\boldsymbol{s}, \tag{3}$$

where $\boldsymbol{\alpha}$ is a p-dimensional row vector, $\boldsymbol{S}$ is a square matrix of dimension p and $\boldsymbol{s}$ is a p -dimensional column vector. We write $X \sim \text{ME}_p(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$ to represent that the random variable X has matrix-exponential distribution. The density corresponding to Eq. 3 is $f(x) = \boldsymbol{\alpha} \exp(\boldsymbol{S}x)\boldsymbol{s}$.

The Laplace-Stieltjes transform (LST) of X can be computed from

$$\mathscr{L}_X(t) = \boldsymbol{\alpha} \left(t\boldsymbol{I}_p - \boldsymbol{S}\right)^{-1} \boldsymbol{s}, \tag{4}$$

where $\boldsymbol{I}_p$ is identity matrix of dimension p.

The non-centralized moments of X can be computed from

$$E(X^k) = k!\boldsymbol{\alpha}(-\boldsymbol{S})^{-(k+1)}\boldsymbol{s}, \tag{5}$$

$k = 1, 2, ...$

For a comprehensive review of matrix-exponential distributions and their sub-class of phase-type distributions, we refer to Bladt and Nielsen (2017). Asmussen and Bladt (1997) showed that the Laplace-Stieltjes transform given by Eq. 4 is rational, and it can be written as

$$\mathscr{L}_X(t) = \frac{b_p t^{p-1} + \cdots + b_2 t + b_1}{t^p + a_p t^{p-1} + \cdots + a_2 t + a_1}, \tag{6}$$

where $a_1, ..., a_p, b_1, ..., b_p$ are real numbers. The Eq. 6 holds true when there is no point mass at zero. The representation $(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$ is then given by,

$$\boldsymbol{\alpha} = (b_1, b_2, ..., b_p) \ \ , \boldsymbol{S} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 & \cdots & -a_p \end{bmatrix} \ \ , \ \boldsymbol{s} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{7}$$

The matrix $\boldsymbol{S}$ is called the companion matrix.

The aim of the present paper is to evaluate the reliability function of a system under δ-shock and mixed shock models when the times between successive shocks follow matrix-exponential distribution. The class of matrix-exponential distributions, as a generalization of the phase-type distributions, is useful as it includes well-known distributions such as exponential, Erlang, and hyper-exponential which are suitable for modeling interarrival times between successive shocks. There are various advantages of using matrix-exponential distributions. They allow us to express distributions and moments in a matrix form as presented by the Eqs. 3-5. We obtain matrix-based representations for the LSTs of the system lifetime under both models when the sequence of intershock times $X_1, X_2, ...$ follows $\mathrm{ME}_p(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$ (see Theorems 1 and 2). As it will be shown in the next sections, in general, the lifetime of the system under δ-shock and mixed shock models does not have matrix-exponential distribution when the interarrival times between shocks have matrix-exponential distribution. That is, the LST of the system's lifetime cannot be written in the form of Eq. 6. However, the corresponding LST can be well approximated by a rational LST which admits representation (6). Thus, we approximate the survival function of the system by $-\boldsymbol{\alpha}\exp(\boldsymbol{S}t)\boldsymbol{S}^{-1}\boldsymbol{s}$ after finding the coefficients in Eq. 6 and then using them in Eq. 7.

The remainder of the paper is organized as follows. In Section 2 , we study the δ-shock model and obtain the LST of the system's lifetime. Section 3 contains the results for the mixed shock model. In Section 4, we study the reliability of a two-component repairable system using the mixed shock model. Section 5 concludes the paper with some remarks and possible future studies.

2 δ Shock Model

Recall that according to the δ-shock model, the system fails when the time between two successive shocks falls below a fixed threshold δ. Then, the system's lifetime is defined by

$$T_\delta = \sum_{i=1}^{M} X_i,$$

where $\{M = m\}$ iff $\{X_1 > \delta, ..., X_{m-1} > \delta, X_m \le \delta\}$. We aim to derive the distribution and hence the survival function of T_δ when $X_i \sim \mathrm{ME}_p(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s}), i = 1, 2, ...$ To this end, we obtain the Laplace-Stieltjes transform (LST) of the lifetime random variable T_δ.

In the following Lemma, we first compute the conditional LST of X given $\{X > \delta\}$ and $\{X \le \delta\}$ which will be useful in the sequel. Throughout the paper, the notation $\otimes$ denotes the Kronecker product.

Lemma 1 *Let $X \sim ME_p(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$. Then, for $\delta > 0$,*

$$L_1(t) = E(e^{-tX} \mid X > \delta) = \frac{\boldsymbol{\alpha}\left(-\left(-t \otimes \boldsymbol{I}_p + \boldsymbol{S}\right)\right)^{-1} \exp((-t \otimes \boldsymbol{I}_p + \boldsymbol{S})\delta)\boldsymbol{s}}{-\boldsymbol{\alpha} \exp(\boldsymbol{S}\delta)\boldsymbol{S}^{-1}\boldsymbol{s}}, \tag{8}$$

and

$$\begin{aligned} L_2(t) &= E(e^{-tX} \mid X \le \delta) \\ &= \frac{\left[\boldsymbol{\alpha}\left(t\boldsymbol{I}_p - \boldsymbol{S}\right)^{-1}\boldsymbol{s} - \boldsymbol{\alpha}\left(-\left(-t \otimes \boldsymbol{I}_p + \boldsymbol{S}\right)\right)^{-1} \exp((-t \otimes \boldsymbol{I}_p + \boldsymbol{S})\delta)\boldsymbol{s}\right]}{1 + \boldsymbol{\alpha} \exp(\boldsymbol{S}\delta)\boldsymbol{S}^{-1}\boldsymbol{s}} \end{aligned} \tag{9}$$

Proof Because the pdf corresponding to the conditional random variable $(X \mid X > \delta)$ is $f^*(x) = \frac{f(x)}{1-F(\delta)}, x > \delta$, we have

$$\begin{aligned} L_1(t) &= E(e^{-tX} \mid X > \delta) = \frac{1}{1 - F(\delta)} \int_\delta^\infty e^{-tx} f(x)dx \\ &= \frac{1}{-\boldsymbol{\alpha} \exp(\boldsymbol{S}\delta)\boldsymbol{S}^{-1}\boldsymbol{s}} \int_\delta^\infty e^{-tx}\boldsymbol{\alpha} \exp(\boldsymbol{S}x)\boldsymbol{s}dx. \end{aligned} \tag{10}$$

Manifestly,

$$\begin{aligned} \int_\delta^\infty e^{-tx}\boldsymbol{\alpha} \exp(\boldsymbol{S}x)\boldsymbol{s}dx &= \int_\delta^\infty \boldsymbol{\alpha}(\exp(-tx) \otimes \exp(\boldsymbol{S}x))\boldsymbol{s}dx \\ &= \int_\delta^\infty \boldsymbol{\alpha} \exp((-t \otimes \boldsymbol{I}_p + \boldsymbol{S})x)\boldsymbol{s}dx \\ &= \boldsymbol{\alpha}\left(-\left(-t \otimes \boldsymbol{I}_p + \boldsymbol{S}\right)\right)^{-1} \exp((-t \otimes \boldsymbol{I}_p + \boldsymbol{S})\delta)\boldsymbol{s}. \end{aligned} \tag{11}$$

Substituting (11) in (10), we obtain (8). The proof of Eq. 9 follows from

$$E(e^{-tX}) = E(e^{-tX} \mid X \le \delta)P\{X \le \delta\} + E(e^{-tX} \mid X > \delta)P\{X > \delta\}$$

which yields

$$L_2(t) = E(e^{-tX} \mid X \le \delta) = \frac{1}{P\{X \le \delta\}}\left[E(e^{-tX}) - L_1(t)P\{X > \delta\}\right].$$

□

We are now ready to obtain the LST of the random variable T_δ. The result is given in the following Theorem.

Theorem 1 *Let* $X_1, X_2, \ldots \sim ME_p(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$. *Then, for* $\delta > 0$, *the Laplace-Stieltjes transform of the system's lifetime is*

$$\mathscr{L}_{T_\delta}(t) = \frac{\boldsymbol{\alpha}\left(t\boldsymbol{I}_p - \boldsymbol{S}\right)^{-1}\boldsymbol{s} - \boldsymbol{\alpha}\left(-\left(-t\otimes\boldsymbol{I}_p + \boldsymbol{S}\right)\right)^{-1}\exp((-t\otimes\boldsymbol{I}_p + \boldsymbol{S})\delta)\boldsymbol{s}}{1 - \boldsymbol{\alpha}\left(-\left(-t\otimes\boldsymbol{I}_p + \boldsymbol{S}\right)\right)^{-1}\exp((-t\otimes\boldsymbol{I}_p + \boldsymbol{S})\delta)\boldsymbol{s}}. \tag{12}$$

Proof By conditioning on the random variable M,

$$\begin{aligned}
\mathscr{L}_{T_\delta}(t) &= E(e^{-tT_\delta}) = \sum_{m=1}^{\infty} E(e^{-t\sum_{i=1}^{m} X_i} \mid M = m)P\{M = m\} \\
&= \sum_{m=1}^{\infty}\left[E(e^{-tX} \mid X > \delta)\right]^{m-1} E(e^{-tX} \mid X \le \delta)P\{M = m\} \\
&= \frac{E(e^{-tX} \mid X \le \delta)}{E(e^{-tX} \mid X > \delta)}\sum_{m=1}^{\infty}\left[E(e^{-tX} \mid X > \delta)\right]^{m} P\{M = m\} \\
&= \frac{L_2(t)}{L_1(t)}\varphi_M(L_1(t)),
\end{aligned}$$

where $\varphi_M(z) = \frac{zP\{X\le\delta\}}{1-zP\{X>\delta\}}$ is the probability generating function of M. The proof now follows from Lemma 1. □

In the following, using Lemma 1 and Theorem 1, we compute the LST of the system's lifetime when the intershock times follow exponential and Erlang distributions.

Example 1 Let $f(x) = \lambda e^{-\lambda x}$, $x \ge 0$. Then, $\boldsymbol{\alpha} = \lambda$, $\boldsymbol{S} = -\lambda$, $\boldsymbol{s} = 1$, and

$$L_1(t) = \frac{\lambda}{\lambda + t}e^{-\delta t},$$

and

$$\mathscr{L}_{T_\delta}(t) = \frac{\lambda e^{\delta(\lambda+t)} - \lambda}{(\lambda + t)e^{\delta(\lambda+t)} - \lambda}. \tag{13}$$

As it is clear from Eq. 13, the LST of the system's lifetime is not in the form of Eq. 6. That is, the numerator and denominator are not polynomial functions. Therefore, the system's lifetime does not have matrix-exponential distribution. However, for sufficiently large q, using the Taylor expansion $\sum_{i=0}^{q-1}\frac{[\delta t]^i}{i!}$ for the term $e^{\delta t}$ the LST given by Eq. 13 can be approximated by the rational LST

$$\tilde{\mathscr{L}}_{T_\delta}(t) = \frac{b_q t^{q-1} + \cdots + b_2 t + b_1}{t^q + a_q t^{q-1} + \cdots + a_2 t + a_1}, \tag{14}$$

where

$$a_1 = \frac{(q-1)!}{\delta^{q-1}}\lambda e^{-\delta\lambda}(e^{\delta\lambda} - 1), \tag{15}$$

$$a_m = \frac{(q-1)!}{\delta^{q-1}}\left[\frac{\delta^{m-2}}{(m-2)!} + \frac{\lambda\delta^{m-1}}{(m-1)!}\right], \tag{16}$$

for $m = 2, ..., q$, and

$$b_1 = \frac{(q-1)!}{\delta^{q-1}} \lambda e^{-\delta\lambda}(e^{\delta\lambda} - 1), \tag{17}$$

$$b_m = \frac{(q-1)!\lambda}{(m-1)!\delta^{q-m}}, \tag{18}$$

for $m = 2, ..., q$. Thus, using the coefficients (15)-(18) in (7), the survival function of the system can be approximated by

$$\tilde{P}\{T_\delta > t\} = -\boldsymbol{\alpha} \exp(\boldsymbol{S}t)\boldsymbol{S}^{-1}\boldsymbol{s}, \tag{19}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{S}$ depend on Eqs. 15-18.

Remark 1 For the case when $f(x) = \lambda e^{-\lambda x}, x \geq 0$, an explicit exact expression for the survival function of T_δ is available and is given by (see, e.g. (Eryilmaz 2012))

$$P\{T_\delta > t\} = e^{-\lambda t} \sum_{n=0}^{\left[\frac{t}{\delta}\right]} \frac{(\lambda(t - n\delta))^n}{n!}. \tag{20}$$

Example 2 Let X_is have Erlang distribution with common pdf $f(x) = \lambda^2 x e^{-\lambda x}, x \geq 0$. Then, $X_i \sim \mathrm{ME}_2(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$ with $\boldsymbol{\alpha} = (\lambda^2, 0)$,

$$\boldsymbol{S} = \begin{bmatrix} 0 & 1 \\ -\lambda^2 & -2\lambda \end{bmatrix} \text{ and } \boldsymbol{s} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

From Theorem 1,

$$\mathscr{L}_{T_\delta}(t) = \frac{\lambda^2 e^{\delta(\lambda+t)} - \lambda^2 - \lambda^2\delta(\lambda + t)}{(\lambda + t)^2 e^{\delta(\lambda+t)} - \lambda^2 - \lambda^2\delta(\lambda + t)}. \tag{21}$$

For sufficiently large q, using the Taylor expansion for the term $e^{\delta t}$, the LST given by Eq. 21 can be approximated by the rational function

$$\tilde{\mathscr{L}}_{T_\delta}(t) = \frac{b_{q+1}t^q + b_q t^{q-1} + \cdots + b_2 t + b_1}{t^{q+1} + a_{q+1}t^q + \cdots + a_2 t + a_1}, \tag{22}$$

where

$$a_1 = \frac{(q-1)!}{\delta^{q-1}} \lambda^2 e^{-\delta\lambda}(e^{\delta\lambda} - 1 - \lambda\delta), \tag{23}$$

$$a_2 = \frac{(q-1)!}{\delta^{q-1}} \lambda e^{-\delta\lambda}(\lambda\delta e^{\delta\lambda} + 2e^{\delta\lambda} - \lambda\delta), \tag{24}$$

$$a_m = \frac{(q-1)!}{\delta^{q-1}} \left[\frac{\delta^{m-3}}{(m-3)!} + \frac{2\lambda\delta^{m-2}}{(m-2)!} + \frac{\lambda^2\delta^{m-1}}{(m-1)!} \right], \tag{25}$$

for $m = 3, ..., q$, and $a_{q+1} = 2\lambda + \frac{q-1}{\delta}$,

$$b_1 = \frac{(q-1)!}{\delta^{q-1}} \lambda^2 e^{-\delta\lambda}(e^{\delta\lambda} - 1 - \lambda\delta), \tag{26}$$

$$b_2 = \frac{(q-1)!}{\delta^{q-2}} \lambda^2 e^{-\delta\lambda}(e^{\delta\lambda} - 1), \tag{27}$$

$$b_m = \frac{(q-1)!\lambda^2}{(m-1)!\delta^{q-m}}, \tag{28}$$

for $m = 3, ..., q$ and $b_{q+1} = 0$.

Remark 2 If the intershock times $X_1, X_2, ...$ have matrix-exponential distribution of degree p, then the distribution of the system's lifetime can be approximated by a matrix-exponential distribution of degree $p+q-1$ for sufficiently large q.

An approximate value of the survival function $P\{T_\delta > t\}$ can be computed using the coefficients (23)–(28) in (7) and $\tilde{P}\{T_\delta > t\} = 1 + \boldsymbol{\alpha}\exp(\boldsymbol{S}x)\boldsymbol{S}^{-1}\boldsymbol{s}$. To observe the performance of the approximation $\tilde{P}\{T_\delta > t\}$ that is based on Eq. 22, some numerical results are presented and compared with the simulated values of $P\{T_\delta > t\}$ when the intershock times have Erlang distribution with mean $\frac{2}{\lambda}$. The simulation results are based on one million replications. With the simulation study, we aimed to observe the effect of the increase in q on the approximation. Therefore, the approximated values of the survival function have been presented not only for small values of q but also for its larger values. From Table 1, it is observed that the proposed approximation performs quite well even for small values of q. Thus, to avoid the computing time delay, the use of small values of q seems suitable and efficient for a good approximation. Because the dimension of $\boldsymbol{S}$ increases as q increases, approximate values of the survival function can not be computed for some values of the parameter λ and δ when q is large. However, it does not pose a problem since the approximations are satisfactory for small values of q.

We have illustrated the approximation of the Laplace-Stieltjes transform of the system's lifetime by matrix-exponential distributions when intershock times have exponential and Erlang distributions. In the sequel, we generalize the method for obtaining an approximation for the case when intershock times have a matrix-exponential distribution of degree p. That is, we obtain an approximation for the LST given by Eq. 12. To this end, we follow the inversion formula for the Laplace transform of rational functions. Let $\mathscr{L}(t) = \int_0^\infty e^{-tx} f(x)dx$, and

$$\mathscr{L}(t) = \frac{b_p t^{p-1} + \cdots + b_2 t + b_1}{t^p + a_p t^{p-1} + \cdots + a_2 t + a_1}.$$

If the roots $r_1, r_2, ..., r_p$ of $t^p + a_p t^{p-1} + \cdots + a_2 t + a_1 = 0$ are all distinct, then

$$f(x) = \sum_{i=1}^{p} c_i e^{r_i x}, \tag{29}$$

where

$$c_i = \frac{b_p r_i^{p-1} + \cdots + b_2 r_i + b_1}{p r_i^{p-1} + a_p (p-1) r_i^{p-2} + \cdots + a_2},$$

$i = 1, 2, ..., p$ (see, e.g. (Longman and Sharir 1971))

Lemma 2 *Let $X \sim ME_p(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$ with Laplace-Stieltjes transform given by Eq. 6. Then, the Laplace-Stieltjes transform of the system's lifetime is*

$$\mathscr{L}_{T_\delta}(t) = \frac{\sum_{i=1}^{p} \frac{c_i}{t-r_i} e^{\delta t} - \sum_{i=1}^{p} \frac{c_i}{t-r_i} e^{\delta r_i}}{e^{\delta t} - \sum_{i=1}^{p} \frac{c_i}{t-r_i} e^{\delta r_i}}, \tag{30}$$

where $r_1, r_2, ..., r_p$ are distinct roots of the equation $t^p + a_p t^{p-1} + \cdots + a_2 t + a_1 = 0$, and

$$c_i = \frac{b_p r_i^{p-1} + \cdots + b_2 r_i + b_1}{p r_i^{p-1} + a_p (p-1) r_i^{p-2} + \cdots + a_2}, \tag{31}$$

Table 1 Approximation of $P\left(T_\delta > t\right)$

t	p	$\lambda = 1, \delta = 2$	$\lambda = 2, \delta = 2$	$\lambda = 5, \delta = 2$	$\lambda = 1, \delta = 1$	$\lambda = 2, \delta = 1$	$\lambda = 5, \delta = 1$
0.25	3	0.9738	0.9100	0.6446	0.9752	0.9121	0.6452
	4	0.9735	0.9098	0.6446	0.9738	0.9102	0.6447
	5	0.9735	0.9098	0.6446	0.9736	0.9099	0.6447
	6	0.9735	0.9098	0.6446	0.9735	0.9098	0.6446
	7	0.9735	0.9098	0.6446	0.9735	0.9098	0.6446
	8	0.9735	0.9098	0.6446	0.9735	0.9098	0.6446
	9	0.9735	0.9098	0.6446	0.9735	0.9098	0.6446
	10	0.9735	0.9098	0.6446	0.9735	0.9098	0.6446
	11	0.9735	0.9098	0.6446	0.9735	0.9098	0.6446
	12	0.9735	0.9098	0.6446	0.9735	0.9098	0.6446
	13	0.9735	0.9098	0.6446	0.9735	0.9098	0.6446
	14	0.9735	0.9098	0.6446	0.9735	0.9098	0.6446
	15	0.9735	0.9098	0.6446	0.9735	0.9098	0.6446
	16	0.9735	0.9098	0.6446	0.9735	0.9098	0.6446
	17	0.9735	0.9098	0.6446	0.9735	0.9098	0.6446
	18	0.9735	0.9098	0.6446			
	19	0.9735	0.9098	0.6446			
	20	0.9735	0.9098	0.6446			
	21	0.9735	0.9098	0.6446			
	22	0.9735	0.9098				
	Simulated value	0.9732	0.9093	0.6444	0.9732	0.9096	0.6447

Table 1 (continued)

t	p	$\lambda = 1, \delta = 2$	$\lambda = 2, \delta = 2$	$\lambda = 5, \delta = 2$	$\lambda = 1, \delta = 1$	$\lambda = 2, \delta = 1$	$\lambda = 5, \delta = 1$
0.5	3	0.9121	0.7368	0.2873	0.9214	0.7504	0.2901
	4	0.9102	0.7360	0.2873	0.9139	0.7411	0.2844
	5	0.9099	0.7358	0.2873	0.9114	0.7379	0.2878
	6	0.9098	0.7358	0.2873	0.9104	0.7366	0.2875
	7	0.9098	0.7358	0.2873	0.9101	0.7361	0.2874
	8	0.9098	0.7358	0.2873	0.9099	0.7359	0.2873
	9	0.9098	0.7358	0.2873	0.9098	0.7358	0.2873
	10	0.9098	0.7358	0.2873	0.9098	0.7358	0.2873
	11	0.9098	0.7358	0.2873	0.9098	0.7358	0.2873
	12	0.9098	0.7358	0.2873	0.9098	0.7358	0.2873
	13	0.9098	0.7358	0.2873	0.9098	0.7358	0.2873
	14	0.9098	0.7358	0.2873	0.9098	0.7358	0.2873
	15	0.9098	0.7358	0.2873	0.9098	0.7358	0.2873
	16	0.9098	0.7358	0.2873	0.9098	0.7358	0.2873
	17	0.9098	0.7358	0.2873	0.9098	0.7358	0.2873
	18	0.9098	0.7358	0.2873			
	19	0.9098	0.7358	0.2873			
	20	0.9098	0.7358	0.2873			
	21	0.9098	0.7358	0.2873			
	22	0.9098	0.7358	0.2873			
	Simulated value	0.9099	0.7362	0.2871	0.9100	0.7359	0.2877

Table 1 (continued)

t	p	$\lambda = 1, \delta = 2$	$\lambda = 2, \delta = 2$	$\lambda = 5, \delta = 2$	$\lambda = 1, \delta = 1$	$\lambda = 2, \delta = 1$	$\lambda = 5, \delta = 1$
0.75	3	0.8336	0.5608	0.1117	0.8602	0.5962	0.1174
	4	0.8286	0.5587	0.1117	0.8438	0.5785	0.1153
	5	0.8272	0.5581	0.1117	0.8362	0.5698	0.1140
	6	0.8268	0.5579	0.1117	0.8323	0.5651	0.1132
	7	0.8267	0.5579	0.1117	0.8301	0.5623	0.1127
	8	0.8267	0.5578	0.1117	0.8288	0.5607	0.1124
	9	0.8266	0.5578	0.1117	0.8280	0.5597	0.1121
	10	0.8266	0.5578	0.1117	0.8276	0.5591	0.1120
	11	0.8266	0.5578	0.1117	0.8272	0.5586	0.1119
	12	0.8266	0.5578	0.1117	0.8270	0.5584	0.1118
	13	0.8266	0.5578	0.1117	0.8269	0.5582	0.1118
	14	0.8266	0.5578	0.1117	0.8268	0.5581	0.1118
	15	0.8266	0.5578	0.1117	0.8268	0.5580	0.1118
	16	0.8266	0.5578	0.1117	0.8267	0.5579	0.1117
	17	0.8266	0.5578	0.1117	0.8267	0.5579	0.1117
	18	0.8266	0.5578	0.1117			
	19	0.8266	0.5578	0.1117			
	20	0.8266	0.5578	0.1117			
	21	0.8266	0.5578	0.1117			
	22	0.8266	0.5578	0.1117			
	Simulated value	0.8269	0.5576	0.1120	0.8264	0.5584	0.1118

$i = 1, 2, ..., p$.

Proof Recall from the proof of Theorem 1 that

$$\mathscr{L}_{T_\delta}(t) = \frac{L_2(t)}{L_1(t)}\varphi_M(L_1(t)), \tag{32}$$

where $\varphi_M(z) = \frac{zP\{X\le\delta\}}{1-zP\{X>\delta\}}$. Using Eq. 29,

$$L_1(t) = \frac{1}{1-F(\delta)}\int_\delta^\infty e^{-tx}f(x)dx = \frac{1}{1-F(\delta)}\sum_{i=1}^{p}\frac{c_i}{t-r_i}e^{-\delta(t-r_i)}, \tag{33}$$

and

$$\begin{aligned} L_2(t) &= \frac{1}{P\{X\le\delta\}}\left[E(e^{-tX}) - L_1(t)P\{X>\delta\}\right] \\ &= \frac{1}{F(\delta)}\left[\sum_{i=1}^{p}\frac{c_i}{t-r_i} - \sum_{i=1}^{p}\frac{c_i}{t-r_i}e^{-\delta(t-r_i)}\right]. \end{aligned} \tag{34}$$

The required result is obtained by substituting (33) and (34) in Eq. 32. □

The following result is readily obtained using $\sum_{i=0}^{q-1}\frac{[\delta t]^i}{i!}$ for the term $e^{\delta t}$ in Eq. 30.

Theorem 2 *Let* X *~*$ME_p(\boldsymbol{\alpha}, \mathbf{S}, \mathbf{s})$ *with Laplace-Stieltjes transform given by Eq. 6. Then, for sufficiently large* q*, the Laplace-Stieltjes transform of the system's lifetime can be approximated by*

$$\tilde{\mathscr{L}}_{T_\delta}(t) = \frac{\sum_{i=1}^{p}\sum_{j=0}^{q-1}\frac{c_i}{t-r_i}\frac{(\delta t)^j}{j!} - \sum_{i=1}^{p}\frac{c_i}{t-r_i}e^{\delta r_i}}{\sum_{j=0}^{q-1}\frac{(\delta t)^j}{j!} - \sum_{i=1}^{p}\frac{c_i}{t-r_i}e^{\delta r_i}}, \tag{35}$$

where $r_1, ..., r_p$ *and* $c_1, ..., c_p$ *are defined by Lemma 5.*

To sum up, the method for approximating $\mathscr{L}_{T_\delta}(t)$ by the rational function $\tilde{\mathscr{L}}_{T_\delta}(t)$ for given intershock time distribution which has matrix-exponential distribution of degree p and Laplace-Stieltjes transform (6) involves two main steps. First, the roots $r_1, r_2, ..., r_p$ of the equation $t^p + a_p t^{p-1} + \cdots + a_2 t + a_1 = 0$ are obtained. Then, using these roots in Eq. 31, the coefficients $c_1, ..., c_p$ are computed. Substituting $r_1, ..., r_p$ and $c_1, ..., c_p$ in Eq. 35, the approximation is obtained.

3 Mixed Shock Model

According to the mixed model, the system fails if either the time between two consecutive shocks falls below a fixed threshold δ or the magnitude of a single shock exceeds another critical level γ, for $\delta, \gamma > 0$. Let X_n and Y_n denote respectively the time interval between

$(n-1)$-th shock and nth shock, and the magnitude of the nth shock. Then the lifetime of the system is defined by the random variable

$$T_{\delta,\gamma}=\sum_{i=1}^{N}X_i,$$

where $\{N=n\}$ iff $\{X_1>\delta, Y_1\leq\gamma, ..., X_{n-1}>\delta, Y_{n-1}\leq\gamma, (X_n\leq\delta\ \vee\ Y_n>\gamma)\}$ for $n=1,2,\ldots$.

Theorem 3 *Let* $X_1, X_2, \ldots \sim ME_p(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$. *Then, for* $\delta,\gamma>0$, *the Laplace-Stieltjes transform of the system's lifetime is*

$$\mathscr{L}_{T_{\delta,\gamma}}(t)=\frac{\boldsymbol{\alpha}\left(t\boldsymbol{I}_p-\boldsymbol{S}\right)^{-1}\boldsymbol{s}-r_2\boldsymbol{\alpha}\left(-\left(-t\otimes\boldsymbol{I}_p+\boldsymbol{S}\right)\right)^{-1}\exp((-t\otimes\boldsymbol{I}_p+\boldsymbol{S})\delta)\boldsymbol{s}}{1-r_2\boldsymbol{\alpha}\left(-\left(-t\otimes\boldsymbol{I}_p+\boldsymbol{S}\right)\right)^{-1}\exp((-t\otimes\boldsymbol{I}_p+\boldsymbol{S})\delta)\boldsymbol{s}},\quad(36)$$

where $r_2=P\{Y\leq\gamma\}$.

Proof By conditioning on the random variable N, we obtain

$$\mathscr{L}_{T_{\delta,\gamma}}(t)=\frac{E(e^{-tX}\mid X\leq\delta \text{ or } Y>\gamma)}{E(e^{-tX}\mid X>\delta, Y\leq\gamma)}\phi_N(E(e^{-tX}\mid X>\delta, Y\leq\gamma)),$$

where $\phi_N(z)=\frac{zP\{X\leq\delta\ \vee\ Y>\gamma\}}{1-zP\{X>\delta, Y\leq\gamma\}}$ is the probability generating function of N. Because X_is and Y_is are independent,

$$\begin{aligned}P\{X\leq\delta\ \vee\ Y>\gamma\}&=P\{X\leq\delta\}P\{Y\leq\gamma\}+P\{Y>\gamma\}\\&=1-(1-r_1)r_2,\end{aligned}$$

where $r_1=P\{X\leq\delta\}$, and

$$E(e^{-tX}\mid X>\delta, Y\leq\gamma)=E(e^{-tX}\mid X>\delta)=L_1(t).$$

On the other hand,

$$\begin{aligned}E(e^{-tX}\mid X\leq\delta\ \vee\ Y>\gamma)&=\frac{E(e^{-tX})-L_1(t)P\{X>\delta\}P\{Y\leq\gamma\}}{P\{X\leq\delta\}P\{Y\leq\gamma\}+P\{Y>\gamma\}}\\&=\frac{E(e^{-tX})-(1-r_1)r_2L_1(t)}{1-(1-r_1)r_2}.\end{aligned}$$

Thus,

$$\mathscr{L}_{T_{\delta,\gamma}}(t)=\frac{E(e^{-tX})-(1-r_1)r_2L_1(t)}{1-(1-r_1)r_2L_1(t)}.\quad(37)$$

The proof now follows by substituting (4) and (8) in Eq. 37. □

Example 3 Let X_is have Erlang distribution with common pdf $f(x)=\lambda^2xe^{-\lambda x}, x\geq 0$. Then, from Theorem 2,

$$\mathscr{L}_{T_{\delta,\gamma}}(t)=\frac{\lambda^2(1+\delta\lambda)e^{\delta t}-(1-r_1)r_2\lambda^2(1+\delta(\lambda+t))}{(\lambda+t)^2(1+\delta\lambda)e^{\delta t}-(1-r_1)r_2\lambda^2(1+\delta(\lambda+t))}.\quad(38)$$

For sufficiently large q, using the Taylor expansion for the term $e^{\delta t}$, the LST given by Eq. 38 can be approximated by the rational function

$$\tilde{\mathscr{L}}_{T_{\delta,\gamma}}(t) = \frac{b_{q+1}t^q + b_q t^{q-1} + \cdots + b_2 t + b_1}{t^{q+1} + a_{q+1}t^q + \cdots + a_2 t + a_1}, \tag{39}$$

where

$$a_1 = \frac{(q-1)!\lambda^2(1-(1-r_1)r_2)}{\delta^{q-1}}, \tag{40}$$

$$a_2 = \frac{(q-1)!\left[\lambda^2\delta(1+\delta\lambda) - (1-r_1)r_2\lambda^2\delta + 2\lambda(1+\delta\lambda)\right]}{(1+\delta\lambda)\delta^{q-1}}, \tag{41}$$

$$a_m = \frac{(q-1)!}{\delta^{q-1}}\left[\frac{\delta^{m-3}}{(m-3)!} + \frac{2\lambda\delta^{m-2}}{(m-2)!} + \frac{\lambda^2\delta^{m-1}}{(m-1)!}\right], \tag{42}$$

for $m = 3, ..., q$, and $a_{q+1} = 2\lambda + \frac{q-1}{\delta}$,

$$b_1 = \frac{(q-1)!\lambda^2(1-(1-r_1)r_2)}{\delta^{q-1}}, \tag{43}$$

$$b_2 = \frac{(q-1)!\lambda^2\delta\left[1+\delta\lambda-(1-r_1)r_2\right]}{(1+\delta\lambda)\delta^{q-1}}, \tag{44}$$

$$b_m = \frac{(q-1)!\lambda^2}{(m-1)!\delta^{q-m}}, \tag{45}$$

for $m = 3, ..., q$ and $b_{q+1} = 0$.

Corollary 1 *(a) If $r_2 = 1$, then the mixed shock model is same with the δ -shock model, and hence (38) is reduced to* Eq. 12 *when $r_2 = 1$.*

(b) If $\delta = 0$ and $r_1 = 0$, then the mixed shock model is same with the extreme shock model. In this case, the LST of the system's lifetime is

$$\begin{aligned}\mathscr{L}_{T_{0,\gamma}}(t) &= \frac{(1-r_2)E(e^{-tX})}{1-r_2E(e^{-tX})} \\ &= \frac{(1-r_2)b_p t^{p-1} + \cdots + (1-r_2)b_2 t + (1-r_2)b_1}{t^p + (a_p - r_2 b_p)t^{p-1} + \cdots + (a_2 - r_2 b_2)t + a_1 - r_2 b_1}.\end{aligned}$$

Therefore, the system's lifetime has matrix-exponential distribution with representation $(\boldsymbol{\alpha}_\gamma, \boldsymbol{S}_\gamma, \boldsymbol{s}_\gamma)$, where

$$\boldsymbol{\alpha}_\gamma = ((1-r_2)b_1, ..., (1-r_2)b_p)$$

$$\boldsymbol{S}_\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -(a_1 - r_2 b_1) & -(a_2 - r_2 b_2) & -(a_3 - r_2 b_3) & -(a_4 - r_2 b_4) & \cdots & -(a_p - r_2 b_p) \end{bmatrix},$$

and $\boldsymbol{s}_\gamma = (0, \ldots, 0, 1)$.

4 Application

Consider a system consisting of two identical components. At time $t = 0$, both of the two components are new, and component 1 starts working first, while component 2 is in a cold standby state. The standby component is put into operation when the active component (component 1) fails, and a repair action is immediately taken for component 1. Assume that the repair time for a failed component is fixed as δ, and the component is as good as new after repair. A damage of random size occurs upon the failure of a component. Let Y_{i+1} denote the size of the damage for the component who lives X_i, $i = 0, 1, 2, \ldots$. If the magnitude of the damage is above γ, then the unit cannot be repaired and hence the system fails after the failure of the standby component. Such a repairable system fails either if one of X_is is less than or equal to δ or the size of the damage for a failed component is above γ. Thus, the lifetime of the system is represented by

$$T = \sum_{i=0}^{N} X_i,$$

where the random variable N is defined by

$$\{N = n\} \text{ iff } \{X_1 > \delta, Y_1 \le \gamma, \ldots, X_{n-1} > \delta, Y_{n-1} \le \gamma, (Y_n > \gamma \ \vee \ X_n \le \delta, Y_n \le \gamma)\}.$$

Figure 1 presents two possible realizations of the system. In Fig. 1a, the system failure occurs due to a critical damage which makes the component nonrepairable. In this case, $N = 3$ and $Y_3 \ge \gamma$. In Fig. 1b, the system fails since the repair time exceeds the lifetime of the active unit. That is, $N = 3$ and $X_3 \le \delta$.

The LST of the system's lifetime is

$$\mathscr{L}_T(t) = E(e^{-tX})\mathscr{L}_{T_{\delta,\gamma}}(t). \tag{46}$$

If $X_1, X_2, \ldots \sim \mathrm{ME}_p(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$, then, using Eq. 36 we obtain

$$\mathscr{L}_T(t) = \boldsymbol{\alpha}\left(t\boldsymbol{I}_p - \boldsymbol{S}\right)^{-1}\boldsymbol{s} \times \left[\frac{\boldsymbol{\alpha}\left(t\boldsymbol{I}_p - \boldsymbol{S}\right)^{-1}\boldsymbol{s} - r_2\boldsymbol{\alpha}\left(-\left(-t\otimes\boldsymbol{I}_p + \boldsymbol{S}\right)\right)^{-1}\exp((-t\otimes\boldsymbol{I}_p + \boldsymbol{S})\delta)\boldsymbol{s}}{1 - r_2\boldsymbol{\alpha}\left(-\left(-t\otimes\boldsymbol{I}_p + \boldsymbol{S}\right)\right)^{-1}\exp((-t\otimes\boldsymbol{I}_p + \boldsymbol{S})\delta)\boldsymbol{s}}\right].$$

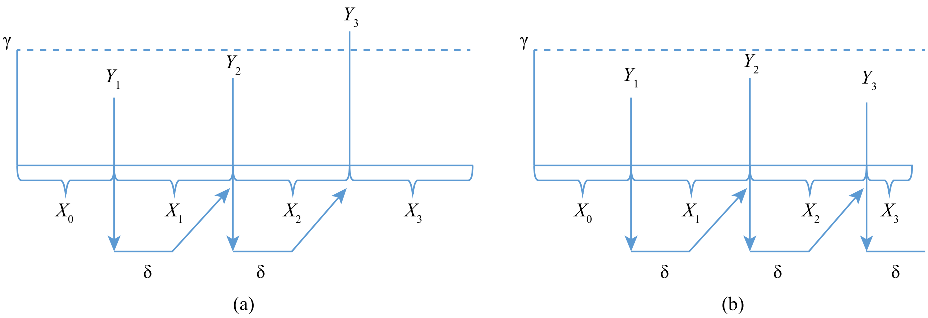

Fig. 1 Two possible realizations of the repairable system

Table 2 Approximation of $P(T > t)$ for r_2=0.75 and γ=0.75

t	p	$\lambda = 1, \delta = 2$	$\lambda = 2, \delta = 2$	$\lambda = 5, \delta = 2$	$\lambda = 1, \delta = 1$	$\lambda = 2, \delta = 1$	$\lambda = 5, \delta = 1$
0.25	3	0.9998	0.9982	0.9617	0.9998	0.9982	0.9617
	4	0.9998	0.9982	0.9617	0.9998	0.9982	0.9617
	5	0.9998	0.9982	0.9617	0.9998	0.9982	0.9617
	6	0.9998	0.9982	0.9617	0.9998	0.9982	0.9617
	7	0.9998	0.9982	0.9617	0.9998	0.9982	0.9617
	8	0.9998	0.9982	0.9617	0.9998	0.9982	0.9617
	9	0.9998	0.9982	0.9617	0.9998	0.9982	0.9617
	10	0.9998	0.9982	0.9617	0.9998	0.9982	0.9617
	11	0.9998	0.9982	0.9617	0.9998	0.9982	0.9617
	12	0.9998	0.9982	0.9617	0.9998	0.9982	0.9617
	13	0.9998	0.9982	0.9617	0.9998	0.9982	0.9617
	14	0.9998	0.9982	0.9617	0.9998	0.9982	0.9617
	15	0.9998	0.9982	0.9617	0.9998	0.9982	0.9617
	16	0.9998	0.9982	0.9617	0.9998	0.9982	0.9617
	17	0.9998	0.9982	0.9617			
	18	0.9998	0.9982	0.9617			
	19	0.9998	0.9982	0.9617			
	20	0.9998	0.9982	0.9617			
	21	0.9998	0.9982	0.9617			
	Simulated value	0.9998	0.9983	0.9615	0.9998	0.9982	0.9616

Table 2 (continued)

t	p	$\lambda = 1, \delta = 2$	$\lambda = 2, \delta = 2$	$\lambda = 5, \delta = 2$	$\lambda = 1, \delta = 1$	$\lambda = 2, \delta = 1$	$\lambda = 5, \delta = 1$
0.5	3	0.9982	0.9810	0.7575	0.9983	0.9814	0.7579
	4	0.9982	0.9810	0.7575	0.9982	0.9811	0.7576
	5	0.9982	0.9810	0.7575	0.9982	0.9810	0.7576
	6	0.9982	0.9810	0.7575	0.9982	0.9810	0.7575
	7	0.9982	0.9810	0.7575	0.9982	0.9810	0.7575
	8	0.9982	0.9810	0.7575	0.9982	0.9810	0.7575
	9	0.9982	0.9810	0.7575	0.9982	0.9810	0.7575
	10	0.9982	0.9810	0.7575	0.9982	0.9810	0.7575
	11	0.9982	0.9810	0.7575	0.9982	0.9810	0.7575
	12	0.9982	0.9810	0.7575	0.9982	0.9810	0.7575
	13	0.9982	0.9810	0.7575	0.9982	0.9810	0.7575
	14	0.9982	0.9810	0.7575	0.9982	0.9810	0.7575
	15	0.9982	0.9810	0.7575	0.9982	0.9810	0.7575
	16	0.9982	0.9810	0.7575	0.9982	0.9810	0.7575
	17	0.9982	0.9810	0.7575			
	18	0.9982	0.9810	0.7575			
	19	0.9982	0.9810	0.7575			
	20	0.9982	0.9810	0.7575			
	21	0.9982	0.9810				
	Simulated value	0.9982	0.9811	0.7575	0.9982	0.9811	0.7569

Table 2 (continued)

t	p	$\lambda = 1, \delta = 2$	$\lambda = 2, \delta = 2$	$\lambda = 5, \delta = 2$	$\lambda = 1, \delta = 1$	$\lambda = 2, \delta = 1$	$\lambda = 5, \delta = 1$
0.75	3	0.9928	0.9345	0.4837	0.9933	0.9368	0.4853
	4	0.9927	0.9343	0.4837	0.9929	0.9353	0.4844
	5	0.9927	0.9343	0.4837	0.9928	0.9347	0.4841
	6	0.9927	0.9343	0.4837	0.9927	0.9345	0.4839
	7	0.9927	0.9343	0.4837	0.9927	0.9344	0.4838
	8	0.9927	0.9343	0.4837	0.9927	0.9344	0.4838
	9	0.9927	0.9343	0.4837	0.9927	0.9343	0.4837
	10	0.9927	0.9343	0.4837	0.9927	0.9343	0.4837
	11	0.9927	0.9343	0.4837	0.9927	0.9343	0.4837
	12	0.9927	0.9343	0.4837	0.9927	0.9343	0.4837
	13	0.9927	0.9343	0.4837	0.9927	0.9343	0.4837
	14	0.9927	0.9343	0.4837	0.9927	0.9343	0.4837
	15	0.9927	0.9343	0.4837	0.9927	0.9343	0.4837
	16	0.9927	0.9343	0.4837	0.9927	0.9343	0.4837
	17	0.9927	0.9343	0.4837			
	18	0.9927	0.9343	0.4837			
	19	0.9927	0.9343	0.4837			
	20	0.9927	0.9343	0.4837			
	21	0.9927	0.9343				
	Simulated value	0.9927	0.9342	0.4835	0.9927	0.9347	0.4832

For an illustration, let $f(x) = \lambda^2 x e^{-\lambda x}$, $x \geq 0$. Then, using the approximation given by Eq. 39 for $\mathscr{L}_{T_{\delta,\gamma}}(t)$ in Eq. 46, the LST of the system's lifetime T can be approximated by

$$\tilde{\mathscr{L}}_T(t) = \frac{\lambda^2 \sum\limits_{i=0}^{q} b_{i+1} t^i}{t^{q+3} + (2\lambda + a_{q+1}) t^{q+2} + \sum\limits_{i=1}^{q+1} (\lambda^2 a_{i+1} + 2\lambda a_i + a_{i-1}) t^i + \lambda^2 a_1}, \tag{47}$$

where $a_0 = 0$, $a_{q+2} = 1$ and $a_1, ..., a_{q+1}, b_1, ..., b_{q+1}$ are given by the Eqs. 40-45.

In Table 2, we present approximate values of $P\{T > t\}$ based on Eq. 47 for selected values of the parameters. Simulated values of $P\{T > t\}$ with one million replications are also included in the Table 2.

5 Summary and Conclusions

In this paper, we have studied δ-shock model, and mixed shock model which combines δ-shock model by the extreme shock model. The novelty of the paper lies in the consideration of the class of matrix-exponential distributions for modeling intershock times. Although the lifetime of the system under δ-shock or mixed shock model does not have matrix-exponential distribution, as illustrated throughout the paper, it has been well-approximated by matrix-exponential distribution.

As a future work, these models can be generalized by defining a more general waiting time random variable for the failure of the system. For example, the occurrence of k consecutive intershock times which are all less than δ, or the occurrence of k consecutive critical shocks which are above γ (see, e.g. (Eryilmaz 2012), (Koutras and Eryilmaz 2017)) might be considered.

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